Introduction to

REINFORCED CONCRETE DESIGN

By

HALE SUTHERLAND *and* RAYMOND C. REESE

AN INTRODUCTION TO REINFORCED CONCRETE DESIGN

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Introduction to REINFORCED CONCRETE DESIGN *by*

HALE SUTHERLAND, PROFESSOR OF CIVIL ENGINEERING, LEHIGH UNIVERSITY, *and* RAYMOND C. REESE, CONSULTING STRUCTURAL ENGINEER · *With chapters on Concrete by* INGE LYSE, PROFESSOR OF REINFORCED CONCRETE AND MASONRY BRIDGES, NORWEGIAN INSTITUTE OF TECHNOLOGY, TRONDHEIM, NORWAY · *SECOND EDITION based on the first edition by Hale Sutherland and the late Walter W. Clifford*

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PREFACE TO SECOND EDITION

The first edition of this book appeared in 1926 with Walter W. Clifford, consulting engineer of Boston, Massachusetts, as co-author. Mr. Clifford died within a few months of its publication. The scope and purpose of this book, determined with his cooperation, remain essentially unchanged in this edition and accordingly the original preface again appears.

Two notable advances have been made in the realm of reinforced concrete in the last sixteen years. The concrete of 1942 is very much superior to that of 1926 in every way and especially in possible definiteness and uniformity of quality. Striking advance has also been made in the methods of analysis of highly indeterminate structures of the sort common in this field and much more exact determination of stresses is possible than formerly. These changes have led to the appearance of new design specifications setting large increases in working stresses. A new edition has therefore become imperative to demonstrate design under these new rules.

In recent years plastic flow and shrinkage of concrete have received much study, opening up debate which still continues, and leading to revolutionary change in column design. These topics receive considerable attention in this edition.

There has been increasing dissatisfaction with the basis of the common theory of reinforced concrete design which at the best is of comparative rather than absolute significance. New theories setting forth ultimate strengths for direct design by use of a safety factor are coming to the fore and may some day be the rule. These are here briefly outlined and the application to arches set forth.

Increased attention has been given here to practical considerations and to the design problem as it presents itself to the practitioner. It is hoped the student will emerge from the study of this book with a very practical viewpoint. Several of the common problems in practice, but uncommon in books, have been given somewhat extended consideration — such matters as holes in slabs, and beam stems, unsymmetrical beams and torsion in spandrels. Rigid frame analysis and the short cut methods of the office have been given much space. Many of these are matters beyond the scope of the undergraduate classroom but of active interest in the design office.

Again the authors are happy to acknowledge indebtedness to many friends for criticism and counsel. In particular thanks are due to Professor J. R. Shank of Ohio State University for much painstaking consideration of certain sections of the manuscript.

H. S.
R. C. R.

December, 1942

PREFACE TO FIRST EDITION

The purpose of the authors has been to present the fundamentals of reinforced concrete design as simply and completely as possible. The method of the transformed section, more familiar in European than in American texts, is used for the development of the theory as it is believed to be by far the clearest and most logical approach. It has the great advantage that instead of leaving the student with a mass of formulas which are often difficult to visualize, it impresses on his mind the basic concepts of the subject and frees him from dependence on texts and equations.

The usual formulas are presented as the basis of diagrams and tables, indispensable as time-savers in practice. The computations that illustrate the application of the theory are arranged systematically in the form usual in office work, with parallel comments in the text. This manner of presentation enables the reader to grasp the problem as a logical whole and gives the student a clear idea of the proper manner of presenting design calculations and results. It is hoped that this arrangement will free the instructor from the drudgery of detailed presentation of designs and enable him to devote the class hour to general discussion of the important features.

The computations cover a wide range of construction: retaining walls, slab and beam bridges, floors, columns and footings for buildings, and the hingeless arch. It is hoped that the discussion paralleling these examples will serve to make plain many matters not usually explained in textbooks.

Enough is included about the modern theories of concrete, form-work, drawing and detailing to give a good background of knowledge in matters where real proficiency can come only with experience.

By combining the viewpoint of the teacher and the practicing engineer the authors have endeavored to direct the work of the student to practical ends with no sacrifice of theory. While the book is primarily for the student of engineering, it is believed that it will prove useful to the practitioner by reason of its compact and complete presentation of specific problems with discussion of the reasons for the various operations. Unless he is a specialist in this field he will find particularly useful the articles dealing with the analysis of rigid frames by the slope deflection method and those treating of arch design.

It is assumed that the reader is conversant with the principles of applied mechanics and understands the elements of design in steel and wood, such knowledge being almost a necessity as a preliminary to the study of reinforced concrete. However, for the sake of completeness and for an aid to rapid review, these fundamental principles are outlined briefly and simply in the text.

The authors wish to express their appreciation to the many friends who have aided their work. It has been their intention to give full credit in the text to all to whom they stand indebted for material and for ideas. With the passage of time a great deal of fundamental information has become common property and the sources are too often not recorded. It is hoped that no borrowings have been inadvertently and wrongfully assumed to belong in that class.

H. S.
W. W. C.

Boston, Massachusetts
August, 1926

A HINT TO THE STUDENT

The engineer thinks in pictures at all stages of the analysis and design of structures. It is a practice the student should carefully cultivate.

As a basis for work in reinforced concrete design there are two fundamental pictures to be fixed in mind: that of a free rigid body at rest acted on by a system of coplanar forces, conforming to the conditions $\Sigma X = 0$, $\Sigma Y = 0$, and $\Sigma M = 0$; that of the free body at rest formed by isolating a portion of a reinforced concrete member or structure for purpose of analysis. Form the habit of expressing the problems of design simply and clearly in terms of these basic diagrams.

The process of studying this or any other structural book consists in thoroughly understanding and committing to memory with great exactness a short series of abstract laws and building up in mind a series of definite pictures of the force systems, the members, the structures, and so on, which furnish concrete expressions of these laws.

Read through any article of this or any textbook for the first time with the purpose of seeing the general outlines of the picture there presented. Do not try to fill in the details word by word at first. If the meaning of a sentence is not clear, pass on to the next. The explanation of the difficulty may be there. After the outline is seen, perhaps dimly, read through the article again with more attention to detail. Use scratch paper and pencil liberally. Make many sketches. No problem can be understood until all the elements are clearly placed. A book with empty margins has probably never been properly studied. Here should show neat notes and sketches in amplification and explanation. When a statement or equation seems obscure determine what would be a clearer statement and compare with that given.

In reinforced concrete design the student must learn to distinguish two kinds of precision: that pertaining to form of computations, clarity of argument, neatness of workmanship — exceedingly important matters never to be slighted; and that relating to the significant figures expressing mathematical operations. A definite hint as to the latter comes with comparing the table of reinforcement areas on page 9, two-figure precision, and the corresponding data in any structural steel handbook. The attempt to produce finished computations of the mathematical precision possible and often demanded in steel design is unwise in reinforced concrete design since the repetition of

operations demanded for this result is time-consuming beyond all practical possibilities and the results are meaningless on account of the nature of the materials used and the uncertain action of continuous structures. Only experience and observation can teach a designer what is meaningful, truthful precision.

To obtain wide understanding in any field it is necessary to consult more than one authority. The student is advised to make free use of the several excellent American textbooks on reinforced concrete. Perhaps two in particular are to be noted: "Principles of Reinforced Concrete Construction," by F. E. Turneaure and E. R. Maurer (John Wiley & Sons, New York) which contains much useful information on tests which have been made to verify theoretical reasoning, and "Theory and Practice of Reinforced Concrete," by C. W. Dunham (McGraw-Hill Book Company, New York) which expresses the viewpoint of an engineer with long experience in the exacting field of municipal and state work.

Every student using this book will require for reference a copy of the Report of the Joint Committee (see page 2), "Recommended Practice and Standard Specifications for Concrete and Reinforced Concrete" (1940), which may be obtained from any of the cooperating societies. It would be well also to have on hand for comparison the American Concrete Institute's "Building Regulations for Reinforced Concrete" (1941) and also the War Production Board's "National Emergency Specifications for the Design of Reinforced Concrete Buildings" (1942), a revision of the A. C. I. code in the interest of economy of strategic materials. These publications may be obtained from the A. C. I. and from the local offices of the Portland Cement Association.

Lastly, do not fail to read carefully Professor George F. Swain's little book on "How to Study" (McGraw-Hill Book Company).

H. S.

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CHAPTER I

INTRODUCTION

1–1. The design of reinforced concrete structures involves two major problems: first, the determination of the type and general features of the structure required for the purpose in hand; second, the detailed proportioning of the various members, such as slabs, beams, columns, and footings, which make up the whole. For example, the engineer who is planning a reinforced concrete factory must study the requirements of the manufacturing process to be housed therein and lay out a building whose arrangement as regards floor plan, column spacing, story height, lighting, elevator service, and so on, makes possible the utmost efficiency of production. The factory must be fitted to the manufacturing process. Having settled the general layout, the engineer next proportions the reinforced concrete skeleton and records this design in the structural drawings. It is evident that these two major problems are closely interrelated; that decisions as to details of arrangement must constantly be based upon knowledge of the possibilities, limitations, and economical use of the structural materials. Furthermore, the designer is responsible not only for the adequacy and strength of the structure but also for its durability, economy, and good appearance.

It is not within the scope of this book to consider the first of these major problems of design nor to do more than introduce the reader to the elements of the second. Experience in active practice is necessary to give the knowledge and judgment necessary for the successful planning of structures since that requires familiarity with construction methods and with the costs of labor, of material, and of finished structures in whole and in detail. This elementary text is limited to a brief outline of the methods of making strong and durable concrete and to a somewhat more thorough study of the application of the principles of theoretical mechanics to the proportioning of structural members, in conformity to the general usage of modern practice.

1–2. Joint Committee. In the United States modern practice has been standardized to conform fairly closely with the recommendations made by a " Joint Committee on Standard Specifications for Concrete and Reinforced Concrete," composed of representatives from six

national engineering groups.* The current report is dated 1940 and replaces that issued in 1924. It was presented in modified form as a progress report in 1937. The dates of the Joint Committee reports (1916, 1924, 1940) are indications of the rapidity with which advancing knowledge and practice have compelled radical changes in design and construction procedures. It seems certain that within a comparatively few years the present theory of stress analysis which has held from the beginning of the use of reinforced concrete will be replaced by one now much used in research work which enables more accurate estimate of potential strength. (See page 74.)

American practice was also much influenced by the "Building Regulations for Reinforced Concrete," issued as a standard building code by the American Concrete Institute in 1928 and 1941. Many cities have adopted these regulations, in whole or part, as their own municipal code. The designer must conform his designs to the regulations in legal force for the community where his structure is to stand.

The student should obtain copies of the 1940 J.C. and 1941 A.C.I. codes and refer to them constantly in reading this text.

1-3. Concrete and Reinforced Concrete. Concrete is artificial stone made by cementing together into a solid mass a mixture of inert material such as sand and broken stone, gravel or other aggregate. The cementing material almost universally used for reinforced concrete work is portland cement, the only exception of note being the alumina cements. These cements are extremely fine powders, made from definite but differing proportions of argillaceous and calcareous materials, which, when wet with the proper amount of water, become chemically active and harden.

Concrete is easily given any desired shape by pouring the wet mixture of materials into suitable forms in which the mass hardens. When the various ingredients are properly proportioned and mixed together the resulting product is hard, durable, strong in compression and shear, very weak in tension, brittle, and, when not reinforced, adapted for use only in relatively massive members subject to compression. In combination with steel rods properly placed to resist the tensile stresses, concrete may be used for all types of structural members. This reinforcement is made possible by the adhesion of the concrete to the steel which prevents slipping between the two materials and forces the member to act as a unit as it deforms under load.

Experience has shown that, generally speaking, steel embedded suffi-

* American Society of Civil Engineers, American Society for Testing Materials, Portland Cement Association, American Concrete Institute, American Railway Engineering Association, and American Institute of Architects.

ciently in concrete is fully protected against corrosion and against fire. The required depth of protective concrete covering varies with the shape of the piece, the aggregate, and the intensity of the exposure.

1-4. Historical Note. There are in existence today examples of concrete construction dating back to Roman times and even earlier. The cement used by these early builders was not a true cement but a mixture of hydrated lime and volcanic ash, a product known today as slag or Puzzolan cement. The first true hydraulic cementing material, that is, one that hardens under water, was made about 1756 by the English engineer, John Smeaton, as a result of his searches for a proper binding material for building the third Eddystone Lighthouse. This product is known today as hydraulic lime. Another Englishman, James Parker, in 1796, made the first natural cement by calcining and grinding an argillaceous limestone. In 1824 Joseph Aspdin of Leeds patented portland cement, a much superior product, though crude judged by the more refined products of today. The name portland was chosen on account of the resemblance of the hardened cement to the building stone quarried on the Isle of Portland. The industry did not begin to develop actively either in England or on the Continent until about the middle of the last century.

In the United States natural cement was first made in 1818 by Canvas White and portland cement in 1872 by David O. Saylor. The manufacture of portland cement lagged behind that of its lower priced rival until the modern method of manufacture (burning the cement clinker in rotary kilns) was introduced in 1892. Quickly the production of portland cement mounted until now it ranks as one of the ten leading industries, an increase that tells eloquently of the increase in reinforced concrete construction.

In 1908, Bied in France and Spackman in the United States took out patents covering a high-alumina cement that surpasses ordinary portland cement in several important respects. The development of the new product took place, however, in France where it has been manufactured in increasingly large quantities. In recent years advances in cement technology have made it possible to secure greatly improved portland cements.

The beginnings of reinforced concrete go back to 1850 when the Frenchman, Lambot, constructed a small boat of that material. In England, W. B. Wilkinson patented a true reinforced concrete floor slab in 1854. Seven years later Francois Coignet published his statement of the principles of the new construction. In the same year, 1861, Joseph Monier, a Parisian gardener, used metal frames as reinforcement for garden tubs and pots, and before 1870 had taken out a series of

patents. There was comparatively little construction, however, until the German engineers, Wayss and Bauschinger, investigated and reported on the Monier system in 1887. From that time the use of reinforced concrete spread rapidly, the greatest developments in theory and practice being made by Austrian engineers. Melan's system, employing structural steel shapes as reinforcement, was developed in the early 90's, at the same time as that of Hennebique, whose methods, of all the pioneers, probably most nearly resemble those of today.

In the United States the pioneer was W. E. Ward, who built a reinforced concrete house in Port Chester, New York, in 1872. Thaddeus Hyatt published the results of tests on various types of beams in 1877. About the same time E. L. Ransome and his coworkers were beginning their work on the Pacific Coast, erecting several notable buildings in California in the two following decades. The Melan system was introduced into this country from Europe in 1894. Edwin Thacher began his distinguished career as a bridge builder with a Melan type of arch in 1896.

During all this period structures of reinforced concrete had been modeled largely on those of the more familiar wood and steel. In 1906 Mr. C. A. P. Turner of Minneapolis devised the girderless or flat slab type of floor, the mushroom floor, as he termed it. This innovation marked a great step forward in utilizing the materials in the most advantageous and economical manner, recognizing to the full the monolithic character of the structure. At this date the extensive use of reinforced concrete was in full swing, a use that has increased tremendously and still increases from year to year.

In so new and rapidly developing a field as that of reinforced concrete it was inevitable that construction should often be in advance of theory. This was notably true of the flat slab floor which is still designed largely by rule-of-thumb methods. For the most part, however, the fundamental principles may be considered as definitely known and agreed upon, having proved themselves by a long series of satisfactory structures which in many cases have endured extremely large overloading with few signs of distress. However, there are still many details to be determined and the status of the theory is far less clearly settled than is that of steel design.

Recent developments have been in the field of improved techniques in mixing and placing, permitting higher unit stresses, and in the methods of rigid frame analysis, making it practicable to take increased advantage of the continuity of action inevitable in monolithic structures.

CHAPTER II

CONCRETE MATERIALS

2–1. Concrete is "a compound of gravel, broken rock or other aggregate, bound together by means of hydraulic cement, coal tar, asphaltum, or other cementing materials. Generally when a qualifying term is not used portland cement concrete is understood."* In order to secure satisfactory concrete it is usually necessary to separate the aggregates into two portions by size; hence the Joint Committee definition: "a mixture of portland cement, fine aggregate, coarse aggregate and water." The 1940 report deals only with portland cement concrete.

2–2. Reinforced Concrete. Plain concrete is brittle and weak in tension and so is suitable only for relatively massive members subject to compression. Combination structural members made of concrete reinforced by steel bars, placed so as to carry the tensile stresses, are sturdy and reliable. The name reinforced concrete should not be applied to a combination piece of steel and concrete unless both materials assist in carrying the load and the whole acts as a unit. Of the three fundamental types of structural members, beams, columns, and ties, only beams and columns can ever be said to be of reinforced concrete.

The advantages of reinforced concrete as a structural material are evident. Each element makes up for the deficiencies of the other; the steel supplies the tensile strength and toughness and the concrete supplies the compressive strength besides protecting the steel from corrosion and from fire.

2–3. Portland Cement. The usual description of portland cement is "the product obtained by finely pulverizing clinker produced by calcining to incipient fusion an intimate and properly proportioned mixture of argillaceous and calcareous materials with no additions subsequent to calcination except water and calcined or uncalcined gypsum." It differs from natural cement ("the finely pulverized product resulting from the calcination of an argillaceous limestone at a temperature sufficient only to drive off the carbonic acid gas") in being slower setting, much stronger, more uniform and reliable.

* Definition adopted in 1923 by the American Concrete Institute.

Until lately portland cement has been considered a standard article of commerce, largely because all brands can be depended upon to satisfy the standard tests.* It has been found, however, that the quality and the appearance of the concrete produced with different approved brands vary, sometimes rather widely. On all work of importance it is essential to make careful tests of the cement and of the concrete made with it. The selection of the type and brand of cement is just as important, and probably more so, than the selection of the proper kind and gradation of aggregates.

Various modifications of portland cement are often used for special purposes, such as low heat cements for large dams and sulfate-resisting cements for certain exposures.

Cement is generally shipped in bags of standard weight of 94 lb; four bags constitute a barrel. For large operations bulk shipments are common, either in ordinary freight cars or in special tank cars, loaded and unloaded pneumatically.

2-4. High Early Strength Portland Cement. High early strength portland is a true portland cement, complying with the standard tests in practically every respect. The proportioning of materials and the refinements of manufacture (principally more careful burning and finer grinding) make the strength of the concrete at early ages very much higher than for the same mixture with ordinary portland cement. For later periods the difference is much less, with the ultimate strength only slightly, if any, higher. The present specifications of the American Society for Testing Materials† require that briquette tensile strengths at 1 and 3 days be the same as those for ordinary portland cement at 7 and 28 days.

Since high early strength cements are more expensive than ordinary portlands and since high early strength concrete can be made with very rich mixes of ordinary portland cements, the economic advantage of high strength at an early age must justify the added cost of the cement. The time element thus becomes a criterion for the selection of both the kind and amount of cement used in concrete.

2-5. Alumina Cements. High-alumina cements are made by reducing to a powder a fused mixture of bauxite (aluminum ore) and limestone. Concrete made with these cements sets in about the same time as portland cement concrete and then proceeds to harden and gain in strength

* "Standard Specifications and Tests for Portland Cement" (serial designation C9-38), issued by the American Society for Testing Materials and adopted as standard by the U. S. Government, the American Engineering Standards Committee, etc.

† Standard Specifications for High-Early-Strength Portland Cement, A.S.T.M. designation C74-39.

very rapidly, attaining in 24 hours approximately 75 per cent of its 28-day strength. This rapid gain in strength is accompanied by a considerable development of heat — sufficient to protect the mass from freezing until high strength is attained under weather conditions which would entirely prevent portland cement from setting or hardening. Another important advantage, that which led to the development of this cement in France, is that concrete made with alumina cement apparently resists the action of sea water and alkalis which often disintegrate portland cement concrete.

It is probable that the limited supplies of raw material suitable for making high-alumina cements will always keep their cost in North America far above that of portland cement. Consequently, they will be used only for a limited class of work where their high strength and quick hardening justify the increased expenditure. At present there is no reason to believe they will ever replace portland cement for ordinary construction.

2–6. Aggregate. Aggregate is used in concrete primarily for the purpose of reducing the amount of cement paste and thereby decreasing the cost of the final product. Aggregate is also necessary since it reduces the volume changes (p. 43) of concrete and makes it a more durable product than would the neat cement paste alone. Being inert material, the aggregate serves as a filler in the concrete and consequently has relatively slight effect upon the quality. This effect is sufficient, however, to require aggregate of uniform quality and size gradation in order to secure uniform concrete with constant cement content. Uniform size gradation is facilitated by using aggregate in two different size ranges, fine and coarse, and combining them in the desired proportions. Fine aggregate consists of particles less than about ¼ in. in diameter; the coarse aggregate of particles larger than ¼ in. The size and grading of an aggregate are studied by means of standard sieves made of wire cloth, of which the small sizes (No. 4 and finer) are designated by the number of openings per linear inch and the larger sizes by dimensions of openings. Natural sand and finely crushed rock are the most common types of fine aggregates, and gravel and crushed rock are the most common coarse aggregates.

Aggregates are generally required to be clean, hard, strong, and durable. Although these properties are desirable some tolerances can often be permitted and still concrete of high quality will be obtained. The cement paste, the binding part of the concrete, primarily determines its quality, so that under many conditions of exposure low grade aggregate will give a strong and durable concrete if incorporated in a high grade paste. For severe exposure high grade aggregate must be used.

Since it is the object of the engineer to obtain a hardened concrete of a given quality his attention should be focused principally upon this requirement and not diverted to consideration of the characteristics of the ingredients beyond the point of useful return. The method to use for obtaining the desired quality of the concrete is determined by the economic factor. Those materials and methods of production which insure the given quality of the concrete at the lowest cost are naturally the most desirable. For the greatest assurance of economy it is necessary to secure reliable data on the quality of the concrete produced with different materials as well as by different methods of manufacture. With this knowledge the selection of the proper aggregates becomes a simple matter. The past has probably seen an overemphasis placed upon knowledge of the ingredients of the concrete. In many instances a standard of quality has been established which has added to the cost more than could possibly be justified by the improved quality of the finished product. The future will no doubt place the emphasis upon the final product, the concrete.

2-7. Water. Mixing water for concrete may be taken without hesitation from any supply suitable for drinking. Waters containing small quantities of impurities may also serve without appreciably reducing the quality. A few waters are highly injurious, the most common being acid water containing even small amounts of tannic and humic acids, waters from paint factories, and those containing high percentages of common salt. Sugar and alcoholic beverages are highly injurious impurities in mixing water.

2-8. Admixtures. There are at present a great number of different concrete admixtures on the market. These consist of powdered material to be added to the concrete during manufacture to improve its final quality. Most of these are more or less inert and so their effect upon concrete quality is indirect through improving workability. Important improvement in workability and quality of concrete may also be made by increasing cement proportion.

2-9. Reinforcement. The reinforcement for concrete usually consists of steel rods, round or square, sometimes made up in the form of wire fabric for use in slabs. For columns and arches the reinforcement often consists of built-up members of structural steel shapes. Though the mills roll a great variety of sizes of plain round or square bars, the Simplified Practice Division of the Bureau of Commerce, in conjunction with the mills, has standardized the following sizes of deformed reinforcing bars* and none others should ever be called for:

* As a war efficiency measure the rolling of the ½-in. square has been discontinued (June, 1942).

	Weight in Pounds per Foot	*Area in Square Inches*
¼ in. round	0.167	0.05
⅜ in. round	0.376	0.11
½ in. round	0.668	0.20
½ in. square	0.850	0.25
⅝ in. round	1.043	0.31
¾ in. round	1.502	0.44
⅞ in. round	2.044	0.60
1 in. round	2.670	0.79
1 in. square	3.400	1.00
1⅛ in. square	4.303	1.27
1¼ in. square	5.313	1.56

In European practice, plain bars are commonly used. In the United States preference is given to deformed bars that are rolled with small projections to engage the concrete and prevent slipping between the two materials. Many styles of such rods are made. Several mills, though adhering to the weights and areas in the above table, furnish bars in round or oval equivalents, as this simplifies the rolling. Square twisted bars are sometimes used.

The Joint Committee specifications provide for three grades of bars rolled from billet steel (structural, intermediate, and hard), for bars rolled from old steel rails or steel axles, and for cold-drawn wire.*

Cold-drawn wire, used chiefly in the form of wire mesh for the reinforcement of slabs and pavements, has its elastic limit considerably raised by cold-drawing. This permits working stresses considerably above those allowed for steel that has not been cold-worked. This device of cold-working for the purpose of raising the elastic limit, and so the permissible working stresses with resulting smaller steel quantities, is used in new variations from time to time. For example, a recent development is the Isteg bar which consists of two plain round mild steel bars helically twisted around each other while cold, with the ends securely anchored to prevent shortening of bars during the process. A variant material is named Torstahl. The use of such patented commercially developed products with increased unit stresses usually requires special affirmative action by the code-enforcing authority in any community.

* Reinforcing bars are customarily sold at base price plus extras. Base price is the cost of ¾″ round rods and larger per 100 lb. Size extras are as follows:

⅝″ ϕ	$ .10 per cwt
½″ ϕ or ½″ □	.20 per cwt
⅜″ ϕ	.40 per cwt
¼″ ϕ	1.00 per cwt

Bending extras are 25¢ to 35¢ per cwt for column and beam rods and 75¢ to 90¢ per cwt for stirrups and ties. Extras for designing, detailing, etc., average from 10¢ to 25¢ per cwt.

CHAPTER III

DESIGNING CONCRETE MIXES

3–1. All reinforced concrete design proceeds on the assumption that the concrete used in the structure will be of definite strength and uniform quality. The desired standards in these matters will be set forth in the design specifications and obviously it is of the utmost importance that the concrete actually measures up to these standards. Advance in knowledge of concrete has been great in recent years and today it is possible to study the available materials and design a mix with considerable accuracy to attain certain specified qualities. Fortunately the desirable qualities of concrete, such as strength, imperviousness, durability, weathering and wear resistance, occur simultaneously so that concrete of high compressive strength has all the other needed qualities also in high degree. This simplifies the problem of efficient concrete manufacture.

It is not the purpose of this text to render its readers competent concrete technicians; that is a competency which can be gained only in the laboratory and to which a book is only a guide. However, sufficient discussion is here given to enable the reader to understand the principles of modern concrete making, a discussion which should proceed hand in hand with application of the principles in the laboratory. To supply in some small part the definiteness which can come only with actual handling of the materials, the reader's attention is now directed to the following table which lists the quantities of materials used in the making of a representative concrete mix.

The example chosen is a portland cement, stone aggregate concrete, designed to have a compressive strength of 3000 psi in standard 6 by 12 in. test cylinders at 28 days. In this particular instance the material quantities were those given in the table on the following page.

The proportions employed in the example are typical only and would show considerable variation from those of other mixes using different materials or designed for other strengths. Accordingly, although it is useful to note certain relationships of material proportions here used it should be remembered that they are only approximate and representative. On the job bulk volume measurement is frequently used in mixing concrete, whereas modern batching plants usually operate on a weight basis. Accordingly beneath the table two sets of proportions are given.

MATERIAL REQUIRED FOR 1 CU. YD. OF CONCRETE: $f'_c = 3000$ psi

Material	*Bulk Volume*	*Absolute Volume*	*Weight*
Cement	$5\frac{1}{2}$ sacks = 5.5 cf	$\dfrac{5.5\times 94}{3.1\times 62.5} = 2.67$ cf	$5.5\times 94 = 517$ lb
Water	$6\frac{3}{4}$ gal per sack $\dfrac{6.75\times 5.5}{7.5} = 4.95$	4.95	$4.95\times 62.5 = 309$
Fine aggregate (sand)	$2\frac{1}{4}$ cf per sack $2.25\times 5.5 = 12.38$	30% voids $12.38\times 0.70 = 8.66$	$8.66\times 2.65\times 62.5 = 1435$
Coarse aggregate	$3\frac{3}{4}$ cf per sack $3.75\times 5.5 = 20.63$	50% voids $20.63\times 0.50 = 10.31$	$10.31\times 2.65\times 62.5 = 1710$
Totals	43.46 cf	26.59 cf	3971 lb

Proportions by bulk volume:	Cement	Sand	Stone
	1 :	$2\frac{1}{4}$:	$3\frac{3}{4}$

Proportions by weight:	Cement	Water	Sand	Stone
	13.0%	7.8%	36.2%	43.0%
Or roughly	$\frac{1}{8}$	$\frac{1}{13}$	$\frac{3}{8}$	$\frac{3}{7}$

In explanation of certain of the figures above these facts are to be noted. The specific gravity of cement is about 3.10 and that of both fine and coarse aggregate about 2.65; loose sand is about 70 per cent solid material, that is, it has about 30 per cent voids; stone is about 50 per cent solid material, about 50 per cent voids; a cubic foot of fresh water weighs 62.5 lb; a sack of cement is accounted a cubic foot and weighs 94 lb.

Note that the final volume of the concrete is the sum of the absolute volumes of material; this indicates that the cement paste fills the voids in the sand and this mixture in turn fills the voids in the stone, with a resultant wedging apart of the particles so that the final volume exceeds that of the loose stone measured in bulk.

Note that a cubic yard of concrete weighs about 2 tons and that the unit weight per cubic foot in this case is 147 lb. It has been pointed out that reinforced concrete is usually taken as weighing 150 pcf.

3–2. Theory of Proportioning. Concrete proportioning today proceeds on the basis of the following known facts:

1. With the same materials, the concrete having the highest concentration of cement in the cement-water paste which forms the matrix has the highest strength. This concentration may run as low as one sack to 9 or 10 gal of water, or as high as 1 sack to 4 or 5 gal. Lower ratios would be decidedly deficient in strength; higher ratios would be too rich, expensive, and likely to craze.

2. Within the limits of workability and of getting all the particles coated with cement paste the quantity of fine and coarse aggregate added does not affect the strength. As aggregate is cheaper than cement paste it is economical to use as much aggregate as possible. The limit is set by workability.

3. Increasing the amount of aggregate increases the stiffness of the concrete and so makes it more difficult and expensive to mix and place properly in the forms. This fact must be kept in mind when estimating the economy to be gained by adding aggregate in order to save cement.

4. Variations in aggregates are so numerous (notably the size and grading of particles, the amount of contained free water, their ability to absorb moisture out of the mix) that it is impossible to establish any exact proportions for any and all cases. Tests and analyses of the materials must be made for each individual case.

Designing a concrete mix consists (*a*) in selecting a water-cement ratio which will produce the desired strength and durability, and (*b*) in selecting a combination of aggregates which will give the desired workability.

(*a*) *Strength.* The ultimate compressive strength desired in standard 6 by 12 in. cylinders tested at 28 days may be decided solely from economic studies (which should involve not only the unit cost of the concrete but also such matters as the cost of additional formwork required for weaker concrete, extra size of structure to obtain the same available clear space with larger members, changes in reinforcing steel, differences in placing costs, etc.), or the strength may be selected as a measure of watertightness, durability, etc., or for the purpose of obtaining smaller members when space and clearances are at a premium. Common strength values are about as follows:

Strength (psi)	*Structural Use*	*Degree of Exposure*
1500	Mass concrete and filling.	
2000	Many parts of buildings.	Protected members with moderate exposure.
2500	Buildings and bridges. Widely used by conservative designers.	Severe exposure in massive members or moderate exposure in thin members.
3000	About as commonly used as 2500 psi and for the same purposes.	Severe exposure in thin members.
3750 5000 6000	Used when space and weight are very important. Imperative that inspection be very rigid with these strengths.	For extreme exposure in thin members.

Approximate values of the water-cement ratios giving these strengths are given on page 14.

(*b*) *Workability.* Decision as to necessary workability rests upon consideration of the space between the forms into which the concrete is to be poured, the clearances around the reinforcing steel, conduits, etc., the method of placing, whether by buggies, by spouts, or by pumping, and particularly the method of compacting, whether by tamping, spading, or vibrating. The field method of gaging workability is the slump test. This consists in noting the amount of slump in inches of a 12-in. truncated cone after the enclosing metal form is lifted. Common slump values in inches to give required consistencies are as follows:

	Buggied	*Vibrated*
Massive sections, pavements, floors on ground	1 to 3 in.	1 to 2 in.
Heavy slabs, beams, and walls	3 to 6 in.	2 to 4 in.
Thin walls and columns, ordinary slabs and beams	4 to 8 in.	2 to 6 in.
Heavy mass concrete, such as dams or heavy walls	3 in.	1 in.
Thin, confined horizontal sections	8 in.	4 to 6 in.
Mortar for floor finish	2 in.	

These examples are from practice:

For a foundation wall 24 in. thick, 18 to 20 ft high, 3000 psi concrete made with sand and crushed limestone aggregates was used. The concrete was put in place by buggies and compacted by vibrating. Slump 1¾ in.

For several reinforced concrete balcony girders, spans 70 to 90 ft, widths 3 to 4 ft, depths 9 to 15 ft, reinforced with 6 and 7 layers of 1¼-in. bars, 3000 psi concrete made with sand was used. No. 6 stone (passing a ½-in. mesh sieve and retained on a ⅜-in.) was the only aggregate up to a level above the top of the reinforcing steel. The concrete was vibrated. Slump 2 in.

For thin joist sections (rib slab construction) for a school house floor, 3000 psi concrete was used, compacted by vibration. Slump 4 in.

The desirable proportions of aggregates are usually determined in advance of pouring concrete on the job by making a series of trial mixes. A batch of convenient size for laboratory study results when the cement content is made one-tenth of a sack; such a batch is readily mixed by hand, there is sufficient material to make it representative, and with ten as a multiplier it is a simple matter to convert proportions to a single-sack basis.

In making these trial batches the aggregates are surface dried so that correction for moisture is avoided. A paste is made of cement and water in the ratio selected to give the desired strength. Aggregates are added to the paste until the batch has the desired consistency. In the table below are given approximate values of the water-cement ratio for various strengths and approximate proportions of cement-fine aggregate-coarse aggregate for various consistencies as measured by the slump.

Strength (psi)	Water-Cement Ratio (gal/sack)	Cement-Sand-Stone Proportions Slumps		
		½″–1″	3″–4″	5″–7″
1500	8.5	1–3.5–5.5	1–3.25–5	1–3–5
2000	8	1–3.25–5	1–3–4.5	1–3–4
2500	7.25	1–3–4	1–2.5–3.75	1–2.25–3.5
3000	6.5	1–2.5–3.5	1–2.25–3.25	1–2–3
3750	5.5	1–2–3	1–1.75–2.5	1–1.5–2

It is to be noted that a low percentage of fines permits a higher yield but that too low a percentage will impair workability. The combination of fine and coarse aggregates which gives maximum weight of mixture will generally be very close to the combination which requires the least amount of cement paste for concrete of a given workability. Various trial batches may also be prepared for the purpose of comparing different types of aggregates which may be available locally. As explained in the following article, it is possible to avoid making very many trial batches for comparison.

3–3. Lyse's Simplified Method of Design. A simplification of the trial method was developed by Inge Lyse at the Fritz Engineering Laboratory, Lehigh University. It is based on the following approximate relationships:

(*a*) For a given type and gradation of aggregates and to maintain the same slump, the total water required per cubic yard of concrete is very nearly constant regardless of the leanness or richness of the mix (Fig. 3–1).

(*b*) For ordinary concrete mixes the strength increases in direct proportion to the increase in the concentration of the cement particles in the mixing water (Fig. 3–2).

From (*a*) it follows that if the absolute volume of water per cubic yard is constant the absolute volume of solids per cubic yard must also be constant. To vary from a lean to a rich mix (using the same materials and same slump) requires adding a certain absolute volume of cement and deducting the same absolute volume of aggregate. If proportions are on a weight basis this means deducting $2.65 \div 3.10 = 0.85$ lb of combined aggregate for every pound of cement added.

Since the strength is proportional to the cement-water ratio, and the total water per cubic yard is constant (*a* and *b* above), it follows that:

(*c*) For a given type and gradation of aggregates and to maintain the same slump, the strength is directly proportional to the amount of

cement per cubic yard of concrete. Expressed as a formula,

$$f'_c = M + Pc \tag{3-1}$$

where f'_c is the ultimate compressive strength in pounds per square inch. M and P are constants depending upon the materials and conditions of testing: c is the cement content in sacks or pounds. (P for sacks is 94 times P for pounds.)

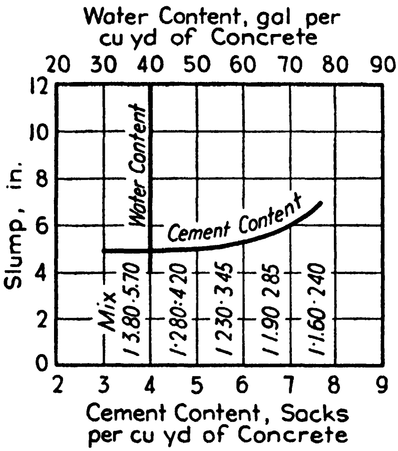

FIG. 3-1

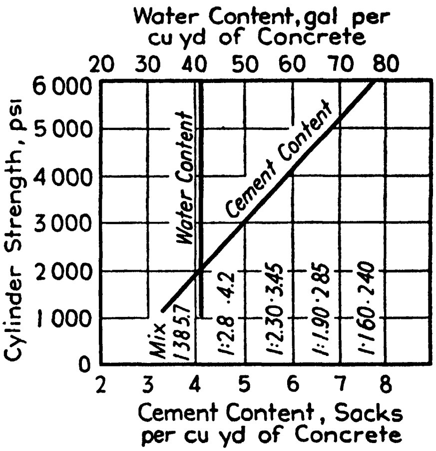

FIG. 3-2

A couple of preliminary tests with the given materials and conditions are sufficient to establish values for M and P. Then for any desired strength solve the equation

$$c = \frac{f'_c - M}{P} \tag{3-2}$$

This gives directly the cement content for any desired strength. Then for any cement added to or deducted from one of the test values an equal absolute volume (or 85 per cent as much weight) of combined aggregate is deducted or added, thus maintaining the same absolute volume of solid material in a cubic yard of concrete.

In brief, the strength-to-cement ratio is practically constant and field values are prorated between a few test values previously established.

3-4. Moisture, Bulking, Absorption. Laboratory volume measurements are made "dry and rodded," that is, the dried materials are tamped in the measuring box in a standard manner. When moisture is added to dry sand the mixture fluffs up or bulks. Water to the extent of 6 per cent by weight will increase the bulk volume as much as 20 to 25 per cent. The finer the material the greater the bulking will

be. Further additions of water tend to flood the sand and decrease the bulking. When completely inundated the volume is approximately the same as measured dry and loose.

If job measurements are on a volume basis corrections must be made for this bulking by adding sand and, since bulking indicates the presence of water in the sand, by using less mixing water.

The volume of coarse aggregate is but little affected this way by water.

Some aggregates are highly absorptive. When placed in the mix they absorb and withdraw from action considerable quantities of mixing water. Extra water must be added to make up for this absorption. The amount of water taken up by job-delivered aggregates in 30 min is often taken as a criterion.

3–5. Job Conditions. The bulk of concrete at present is job-mixed. Aggregates are delivered in bulk, and usually stored in stock piles on the ground. Cement is delivered in sacks, stored in sheds, and protected from the weather. The aggregates absorb ground water, are soaked when it rains, and dry out again under the hot sun. On a job of considerable size aggregates may originate in different sand banks or stone quarries, though this is avoided as much as possible. More exact and careful methods of proportioning are being devised and today materials are weighed or carefully measured instead of being determined as "so many wheelbarrows full." Often materials are delivered already batched from the dealer's yard where they have been stored dry in bins. In many localities the concrete is delivered already mixed in agitator trucks. All this is sufficient indication that concrete is manufactured of quite variable ingredients and, since they are not chemically combined in the mixing, that, therefore, absolute uniformity is hardly obtainable. It is surprising that on the same job tests will often fall within a limit of, say, 10 per cent either way. The designer must not expect too close agreement with his specified strength. One rule has been: At least 90 per cent of the test specimens shall exceed the specified strength and the balance shall not fall more than 10 per cent short of requirements.

3–6. Former Theories of Proportioning. Until within a few years the only accepted principles governing the proportioning of concrete were three:

1. For any given combination of aggregates, strength, low permeability, and durability increase with increased proportions of cement, the consistency remaining the same.

2. For any given aggregates, the proportion of cement being fixed and consistency remaining constant, maximum strength, low permeability, and durability are obtained with the combination of ingredients giving the densest mixture.

3. The quality of the concrete is best when only sufficient water to insure proper placing is used, because an excess of water results in a weak concrete.

Although not generally realized, these principles are in harmony with our present conception of concrete. They all imply an emphasis on the concentration of cement particles in the paste, the paste being the quality-giving part of the concrete.

Four general methods of design were developed, each aiming to determine the proportions of the several ingredients that result in a mass containing the maximum amount of solid matter per unit volume. These methods were:

Method of Void Determination,
Method of Arbitrary Proportions,
Method of Mechanical Analysis (Fuller's Ideal Curve, Fineness Modulus, Surface Area),
Method of Trial Mixes.

The basis for the method of *void determination* was the theory that for maximum density the voids between the particles of coarse aggregate should be completely filled by the fine aggregate, and all remaining voids should be filled by the cement paste, which, if it is to perform its function as a glue binding the whole mass together, must coat every particle.

In the method of *arbitrary proportions* use was made of the fact that most coarse aggregates have approximately 50 per cent voids. This leads to the simple 1 : 2 ratio of fine and coarse aggregates which is so commonly used. The amount of cement was determined by the ratio of cement to aggregate judged necessary for the production of concrete of a given quality. Thus we have the 1 : 1½ : 3 (i.e., 1 part cement, 1½ parts fine and 3 parts coarse aggregate by loose volume), 1 : 2 : 4, 1 : 3 : 6, etc., mixes which have been used extensively in concrete construction.

The methods of *mechanical analysis* make use of artificial gradation of fine and coarse aggregate. The Fuller curve* for maximum density of aggregate (Fig. 3–3) has been given much attention and aggregate of this gradation has been used on a number of jobs. The expense of separating the aggregate into groups of different sizes and recombining these sizes in accordance with the ideal curve often will more than offset the advantage of the improved gradation. This method is illustrated in Fig. 3–3 which shows the sieve analysis of three aggregates and the

* In 1907 William B. Fuller and Sanford E. Thompson made public the grading curve for granular material combined so as to make the resulting mixture the densest possible (Transactions A.S.C.E., Vol. LIX, 1907).

analysis of their combination in stated proportions and gives a curve approximating the ideal curve as specified by Fuller.

Among methods of *mechanical analysis* the *fineness modulus* is still often used in determining and comparing the qualities of aggregates. This modulus is ¹⁄₁₀₀ of the sum of the percentages of the material coarser than the openings of each of the following standard series of

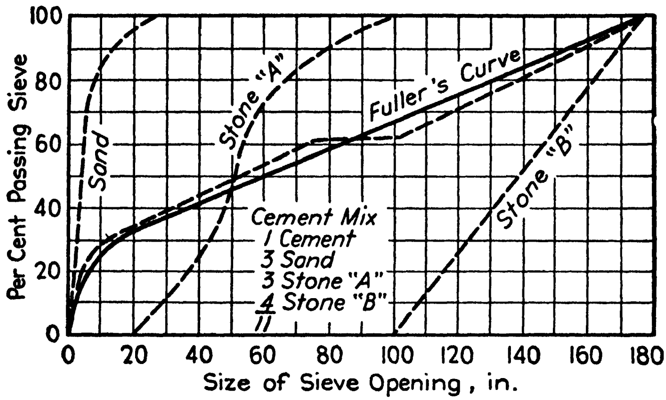

FIG. 3-3. Fuller's grading curve (from Mills's *Materials of Construction*).

sieves: 100, 50, 30, 16, 8, 4, ⅜, ¾, 1½. Each sieve in this series has an opening twice the width of the preceding one. The method of calculating the fineness modulus is illustrated in the following table.

TYPICAL SIEVE ANALYSES OF AGGREGATES

Aggregate	100	50	30	16	8	4	⅜	¾	1½	Fineness Modulus	Range in Size
Sand	100	90	70	55	35	20	0	0	0	3.70	0–⅜
Sand	100	85	65	40	20	0	0	0	0	3.10	0–4
Sand	95	75	60	30	0	0	0	0	0	2.60	0–8
Screenings	85	80	75	35	25	0	0	0	0	3.00	0–4
Stone	100	100	100	100	100	100	100	40	0	7.40	⅜–1½
Pebbles	100	100	100	100	100	100	70	30	0	7.00	4–1½
Pebbles	100	100	100	100	100	100	45	15	0	6.60	4–1½

The fineness modulus increases with the coarseness of the aggregate and the same fineness modulus may be secured from an infinite number of different aggregates.

The *surface area* method of mechanical analysis is much the same as the fineness modulus method, except that the calculated area of the particles of aggregate, or a factor related thereto, is used as the basis of design. It is subject to the same limitations as the fineness modulus.

The method of *trial mixes* consists simply in weighing a series of aggregate combinations of given volume and determining which gave the heaviest, and so densest, mix. This method survived the changed conceptions of concrete which came with Abrams' investigations and today the most efficient way of applying our modern understanding to the proportioning of concrete is by a method of trial mixes, the old criterion of density being replaced by one of economy.

3-7. Modern Theories of Proportioning. Our present-day conceptions of concrete began with the announcement of the water-cement ratio theory by Professor Duff A. Abrams of Lewis Institute in 1918. He stated this theory thus: "With given concrete and materials and conditions of test, the quantity of mixing water used determines the strength of the concrete so long as the mix is of workable plasticity." Today we emphasize the concentration of cement particles in the paste, the cement-water ratio, rather than the quantity of water. Professor Abrams also devised the fineness modulus with relation to aggregate gradation and the slump test for workability determination.

3-8. Summary of Concrete Mixes. The proper design of a concrete mix can be viewed from two angles: (1) that of the designing engineer and specification writer and (2) that of the testing laboratory, contractor, and field inspector.

1. As far as the designer is concerned, he is mainly interested in obtaining concrete of proper strength and workability. He usually achieves this by specifying that the concrete shall be proportioned by the controlled mix process, to develop a definite strength in 28 days when tested in standard 6 by 12 in. cylinders; that the water-cement ratio shall not exceed a definite number of gallons per sack of cement; that the resulting concrete shall contain not less than a stated number of sacks of cement per cubic yard; and that the slump (measure of workability) shall not exceed certain limits.

Such limitations will pretty well determine the kind, character, and strength of concrete which will be obtained.

2. From the laboratory angle the problem is to take available materials and determine what relative proportions will produce the above-described strength, durability, and workability at minimum total expense. From previous experience and by reference to tables and diagrams, for example pages 14 and 32, it is possible to arrive at one or two trial mixes which should come reasonably close to the requirements. Tests can be run on a few trial mixes and rapid adjustment can be made by Lyse's method, because, as shown by equation 3-1, for given materials the strength relationship is linear and values can be prorated as soon as a couple of points are established.

It is possible to obtain a desired strength and workability with numerous combinations of the materials, but (since cement is relatively expensive) the placing of a stiff mix is more costly than a better lubricated one, the relative cost of fine and coarse aggregates varies in different localities, and, finally, as the "yield" differs with varying mixes, a study of all the factors involved is required in order to produce the desired qualities at a minimum of expense.

Example 3–1. A trial batch of concrete of the desired placeability had the contents: fine aggregate, 9.0 lb; coarse aggregate, 19.0 lb; cement, 6.0 lb; water, 1200 cc (1 cu in. = 16.4 cc); specific gravity of aggregates, 2.65, of cement, 3.10.

(*a*) What was the theoretical unit weight of the concrete in place? (*b*) Give the weight of each ingredient per cubic yard of concrete.

Solution.

(*a*) Aggregate 28.0 lb $= \dfrac{28}{2.65 \times 62.4} = 0.170$ cf

Cement 6.0 $= \dfrac{6}{3.10 \times 62.4} = 0.031$

Water (1200 cc) 2.64 $= \dfrac{2.64}{62.4} = 0.042$

Totals 36.64 lb 0.243 cf

Weight per cubic foot $= \dfrac{36.64 \text{ lb}}{0.243 \text{ cf}} = 151$ pcf

(*b*) Number of batches per cubic yard = 27/0.243 = 111.0

Fine aggregate	1000 lb
Coarse aggregate	2110 lb
Cement	667 lb
Water	294 lb

Example 3–2. A trial batch gave the following information on the ingredients of the concrete:

Net water requirement	287 lb per cu yd of concrete
Sand-coarse ratio	1 : 2 by volume
Unit weight of sand	115 pcf
Unit weight of coarse	105 pcf
Specific gravities as in Ex. 3–1	
Absorption of aggregates	1 per cent by weight of dry material

(*a*) Give the weights of the ingredients for a net cement-water ratio of 1.0 by weight. (*b*) For a cement-water ratio of 2.0 by weight.

Solution.

(*a*) Cement content = 287 lb × 1.0 = 287 lb

$$\text{Paste volume} = \text{cement} + \text{water} = \frac{287}{62.4 \times 3.10} + \frac{287}{62.4}$$

$$= 1.48 + 4.60 \text{ cf} = 6.08 \text{ cf}$$

Aggregate content = (27.0 − 6.08) = 20.92 cf

Weight of total aggregate = (20.92 × 165) = 3450 lb

Since there are two volumes of coarse aggregate for each volume of sand:

$$\text{Sand} = \frac{115}{325} \times 3450 = 1220 \text{ lb} \quad \text{Coarse aggregate} = \frac{210}{325} \times 3450 = 2230 \text{ lb}$$

Water absorbed by aggregates = 0.01 × 3450 = 34.50 lb
Total water = 287 + 34.5 = 321.5 lb

(*b*) Cement content = 287 × 2.0 = 574 lb
Aggregate change = 0.85 × 287 = 244 lb
Total aggregate = 3450 − 244 = 3206 lb

$$\text{Sand} = \frac{115}{325} \times 3206 = 1135 \text{ lb} \quad \text{Coarse aggregate} = \frac{210}{325} \times 3206 = 2070 \text{ lb}$$

Total water = 287 + (0.01 × 3206) = 319 lb

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CHAPTER IV

MANUFACTURE OF CONCRETE

4–1. Concrete differs from other structural materials which come to the job as finished products in that it is generally manufactured where it is used. Good quality is the first essential of concrete for the permanence and solidity of the structure in which it is placed and accordingly its manufacture places a heavy responsibility upon the engineer. Structural steel is a standardized article of commerce, made under rigid supervision, and it can be bought in the open market with confidence that it will pass the rigid requirements of the American Society for Testing Materials. On all important work involving large tonnage, however, the engineer provides for careful inspection and tests of the steel. How much more essential is it that the engineer supervise with care the manufacture of the concrete and hold the contractor rigidly to the best methods of modern workmanship to insure that the structural concrete be of the requisite strength and quality!

The preceding chapter outlines the best methods of proportioning concrete; the present chapter is concerned with the best methods of the actual manufacturing process itself.

4–2. Measurement of Materials. The accurate measurement of all ingredients is very essential for the production of a uniform concrete. It is especially important that the ingredients for the cementing paste, the cement, and the water be measured accurately. With a given placeability of the concrete the materials must be kept constant and even slight variations in quantities may be detrimental to the method of placing used.

Volumetric measurements have been used extensively in the past but the requirement for higher uniformity has led to the development of weighing devices which are coming into use increasingly. If volumetric measurements are used, corrections for bulking caused by the moisture in the fine aggregates have to be made. The weighing devices eliminate the bulking correction but adjustment for the amount of water carried in the aggregates must be made. The cement is conveniently measured in sacks or by direct weighing. The water is conveniently measured by volume or by weight. The water content must be corrected for free water carried by moist aggregates and for water absorbed by dry aggregates.

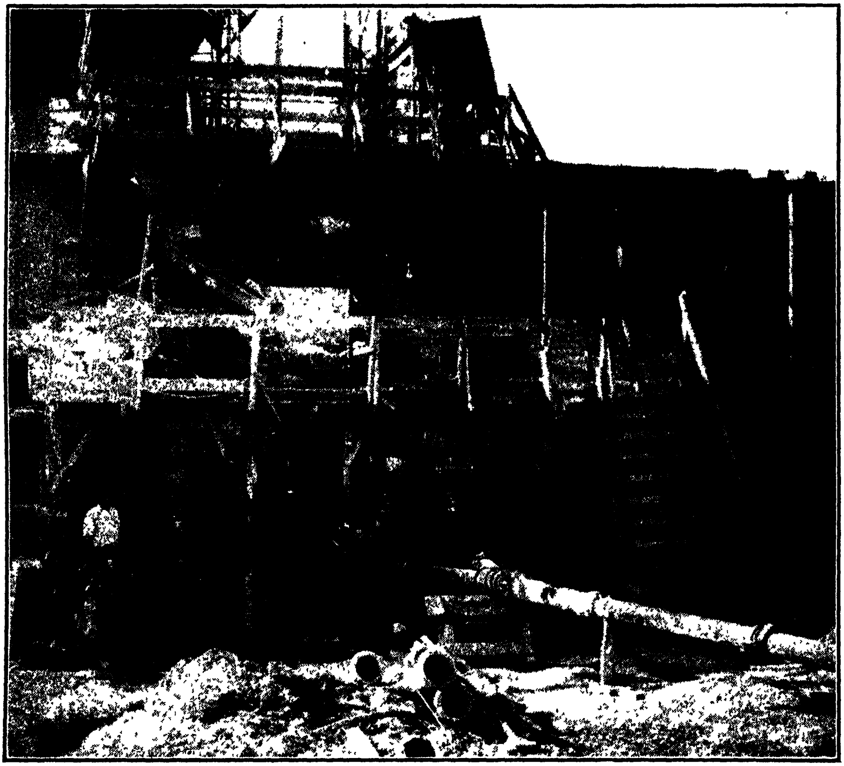
4-3. Mixing Concrete. In order to produce a homogeneous mass of concrete the ingredients have to be thoroughly mixed before being placed in the forms. The function of the concrete mixer is to produce this homogeneity, and the length of time of mixing depends upon the efficiency of the mixer and the characteristics of the mix. The drier and harsher the mix the longer will be the mixing time needed to give uniformity. Beyond the production of a uniform concrete the length of time of mixing is immaterial. One-minute mixing is generally sufficient in modern mixers for concrete such as is ordinarily used in building construction. Two-minute mixing is favored in many localities.

4-4. Ready-Mixed Concrete. In most cities mixing plants have been established for the commercial manufacture and sale of concrete according to the buyers' specifications. In some of these plants both proportioning and mixing are accomplished at the yard and the mix is shipped to the job in special agitator trucks designed to prevent segregation of the mixed material; in others, the dry ingredients and water are carefully proportioned into special trucks and the mixing takes place en route. Some of these trucks are arranged so that the water is dumped into the batch and the mixing started by the driver who plans this so that the contents will be properly mixed by the time of arrival on the job. Accurate weighing devices have usually been installed by these mixing plants and laboratory control methods have been introduced so that the seller is prepared to deliver concrete of any desired strength economically proportioned. In using ready-mixed concrete some care is necessary to see that the material has not been mixed too long, as might ensue in case of a long haul or delay in dumping at delivery point; and also to be sure that segregation of material has not occurred.

Another variation in the commercial production of concrete is the delivery of "batched" ingredients, the dry cement and carefully proportioned aggregates, ready for delivery into the mixing hopper at the job. This saves the installation of material bins and weighing devices at the job site. The major difficulty to be guarded against here is the loss of dry cement in transit.

4-5. Placing Concrete. The concrete may either be deposited directly or transported by different means from the mixer to the forms. The main problem in transportation is the prevention of segregation, the separation of the larger, heavier pieces from the bulk of the mass, which results in honeycombed spots in the finished concrete where these large pieces appear loosely cemented together, unsurrounded by mortar. On small jobs and for short hauls the concrete is carried from the mixer to the forms in wheelbarrows and buggies. For longer distances chutes may be used down which the concrete slides by force of gravity. It is

essential that the motion should be fast enough to keep the chutes clean and not so fast that segregation will occur. The mixing operation should be so timed that a nearly continuous flow of concrete is obtained at the discharge end of the chute. The use of chutes, singly or in series, longer than about 300 ft is inadvisable on account of the difficulty in preventing segregation. The use of conveyors for transporting the concrete from the mixing plant to the forms has been found very successful and so has placing from buckets by means of derricks. Concrete pumps have also proved successful for transport of concrete from mixer to form. Fig. 4-1 shows the use of a concrete pump on a reinforced concrete job.



Courtesy Hausman Steel Co., Toledo, Ohio

FIG. 4-1. Pumping plant for concrete showing twin mixers, twin agitators and Siamese connection to feed line, as well as stand-by equipment. Approximately 60,000 cu yd of concrete were pumped distances up to 2000 ft.

Often a certain amount of water will accumulate on the top of the concrete during the placing operation of even a well-designed mix.

This excess water carries with it some of the finer particles of the cement and, if it is allowed to remain when placing is stopped, the drying of the water will leave a layer of slimy scum, called laitance, which will prevent the bonding of the next layer of concrete to that below the scum. The realization that the quality of the concrete depends upon the concentration of the cement in the paste is leading increasingly to the use of stiff mixes where the cement paste contains a relatively small amount of water and so is of high binding quality.

A mix may be stiff because the proportion of paste to aggregate is small. Under old methods of placing this would result in poor concrete but the modern high frequency vibrator has made possible the successful use of leaner mixes than was practicable with the older methods of hand spading and tamping. The concrete vibrator has therefore gained much ground during recent years and a large proportion of present-day concrete is vibrated into place.

The economic relation between the cost of placing and the cost of the cementing paste is a very important one. The less cement paste of a given quality used in the concrete, the stiffer is the mix and the greater the amount of mechanical work required. The method of placing to be employed on any concrete job is therefore primarily an economic problem. The important item is to produce a homogeneous concrete of a given quality at a minimum cost. Comparatively stiff concrete mixes, that is, mixes of low paste content, can be placed by the vibrators so that an important saving in cost may result from the lower cement content for concrete of a given quality.

When concrete is placed under water provision must be made for protection against washing the cement out of the concrete. Consequently concrete should not be allowed to drop freely through water but should be deposited by means of tremie pipe, drop-bottom buckets, or other devices suitable for preventing segregation.

4-6. Placing Reinforcement. It is of the greatest importance that reinforcing bars be accurately placed and held securely so that they will not be dislodged by the necessarily rough treatment incident to the placing of the concrete. An error of 1 in. in the vertical position of slab steel may easily double the stresses and result in serious cracking. Small errors in the location of beam reinforcement may cause large increases over the calculated stresses. Welding of the reinforcement into units before placing in the form will go far to insure correct location of the bars.

Conservative engineers are specifying increasingly the use of some approved type of bar support, several varieties of which are on the market. The best of these devices hold the steel securely at the proper distance from the forms with proper spacing and clearances. A com-

mon method of supporting slab steel is to place the bars on precast concrete blocks of the right height. These blocks are not so good as some of the patented supports but are satisfactory for much work. Beam steel may be supported from the bottom of the forms or it may be hung in the stirrups by means of light bars, called loop bars, which run under the hooks of the stirrups and are supported from the slab forms.

A detail in placing that is too often neglected is the provision against settlement of the concrete away from horizontal bars held rigidly in place. In deep members where the consistency tends toward fluidity, considerable settlement may take place in the few minutes immediately after placing. Unless provided for, this will leave air or water cavities under the horizontal bars. Either the concrete should be carefully spaded to insure filling these cavities or the rigid support should be slightly released to allow the bars to settle naturally into the concrete.

Vertical steel in columns and walls usually requires temporary support at the top to keep it in correct alignment during placing. In the design of such vertical steel consideration should be paid to its need for stiffness in supporting itself before and during the placing of the concrete.

Where it is possible reinforcement is usually made up into units before being placed in the forms. This can almost always be done with column steel and usually with beam steel. Where a large amount of negative reinforcement (that is, reinforcement extending over the supports in the top of continuous beams) interlaces with column hooping it must be placed piece by piece. Slab steel, except wire fabric, can seldom be handled by units. Loose negative reinforcement, that is, straight top bars not connected with those in the bottom of the slab, must often be placed during placing of the concrete. This should be done only under careful and experienced supervision. Such top bars can usually be made up into units with the spacer bars (often called temperature reinforcement). This is advisable because a unit is less likely to be pushed too deeply into the concrete than are separate rods.

In and near cities and for large jobs in almost any location the bending of the reinforcement is usually done in the yard or shop of the reinforcing contractor and the steel is delivered on the job in bundles all bent and tagged. When the bending of the steel is done on the job it should be done in a field shop and not in place.

4–7. Curing Concrete. During the hardening process the cement in the concrete reacts chemically with a part of the mixing water and forms a paste which solidifies and binds the aggregates into a rocklike mass. This action proceeds very rapidly at first and then continues more slowly for a considerable time under favorable moisture and temperature con-

ditions. The problem of the curing is to provide for favorable moisture and temperature conditions and thus prevent premature cessation of the hardening process. The chemical reactions of the cement and water will stop soon after the temporary excess of mixing water has been allowed to evaporate, and the quality of the paste will remain close to what it was at the beginning of the curing period. If the hydration is allowed to continue under favorable curing conditions the quality of the concrete will increase over a very long time. Fig. 4-2 gives an illustration of the increase in strength with length of moist curing at a temperature of 70°F. The temperature of curing is very important since the chemical reaction in the paste proceeds at a more rapid rate under high temperatures than under low temperatures. An excellent illustration of the effect of the curing temperature on the strength of the concrete is presented in Fig. 4-3.

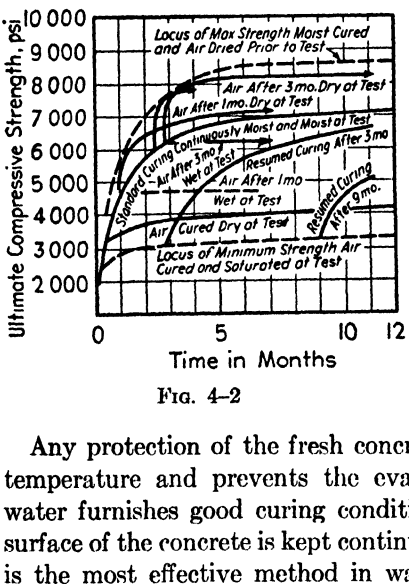


FIG. 4-2

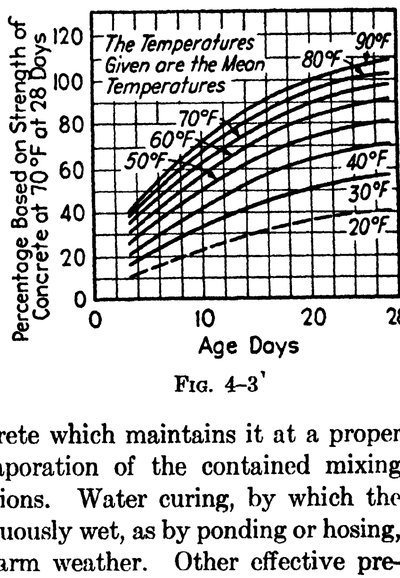


FIG. 4-3'

Any protection of the fresh concrete which maintains it at a proper temperature and prevents the evaporation of the contained mixing water furnishes good curing conditions. Water curing, by which the surface of the concrete is kept continuously wet, as by ponding or hosing, is the most effective method in warm weather. Other effective preventives of evaporation are the application of various coatings such as linseed oil or bituminous material, or the covering with some tight material like Sisalkraft paper. Forms are often left on for this reason.

It should be kept in mind that high quality concrete may be obtained either by using a high concentration of cement in the paste with little or no moist curing, or by increasing the efficiency of a paste of a lower cement concentration by an increased length of curing. Consequently the problem of curing, as well as that of the most advantageous method of curing, is primarily an economic one. The method of production

which gives the desired quality of the concrete at the lowest cost is the one to choose.

4-8. Surface Finish. In much construction the concrete is left with the surface finish imparted to it by the forms, the unfinished surface being satisfactory, either in itself or because it is covered by other materials which form the exposed surface of the construction. Often exposed concrete surfaces are required to present a specified uniform texture and require special treatment after forms are removed. Information concerning surface finishing is found in the A.C.I. Building Code and the J.C. Report.

The upper surfaces of slabs in bridges and buildings require special working in way of leveling and preparation in order to provide a proper wearing surface for the traffic carried. This finishing has of necessity to be done at a proper interval, neither too short nor too long, after initial set of the concrete and, accordingly, it is a troublesome and expensive part of concrete manufacture. These difficulties have been much lessened by a recent development, the removal of excess mixing water immediately after the leveling and rough smoothing by the use of rubber vacuum mats connected to a suction pump.* By this process the early gain of concrete strength has been much accelerated, the finishing time has been greatly advanced, and better surfaces have been obtained. The process is also used with vertical surfaces.

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* This is a patented process controlled by Vacuum Concrete Inc., Philadelphia, Pennsylvania.

CHAPTER V

PROPERTIES OF CONCRETE

5-1. The value of well-made concrete as a structural material is due chiefly to its relatively high compressive and shear strengths, to its durability and good weathering qualities, to its fire-resisting properties and the consequent protection it gives to embedded steel against corrosion and fire, and to its low permeability. Its value as an insulator against heat and sound is sometimes important. Other properties are often of moment to the designer who must understand in all situations the behavior of the material he uses for his structures. Chief of these other properties, which often may prove a hazard if neglected, are its weakness in tension, its elastic and plastic action under load, and the volumetric changes which the material suffers with variation of water content and with temperature changes. Fortunately several of these properties are interdependent and a knowledge of the one element, compressive strength, will give adequate information about these related matters.

Recent investigations have contributed to the establishment of the relations between the ingredients in the concrete mix and the strength and other qualities of the resulting product. It has already been pointed out that, since the cement is the only chemically active ingredient in the concrete and since the cement reacts with water, the relation of cement to water must necessarily be the major determining factor for the characteristics of the concrete. In the following articles the different properties of concrete are discussed. Special emphasis has been given to the compressive strength because to a large extent this element reflects the other strengths and also indicates other characteristics such as modulus of elasticity, permeability, durability, fireproofing, etc.

High compressive strength usually indicates high modulus of elasticity, low plastic flow, low leakage, high durability, and high fire resistance. However, there is one quality which, unfortunately, does not follow the same general law: volume changes due to variation in moisture content. The volume changes are generally greatest when the cement concentration is greatest; that is, this undesirable property is most troublesome when all other desirable qualities are realized. Although the *quantity* of cement paste in the concrete within reasonable limits has little effect upon the strength, it has an appreciable effect upon

such qualities as permeability, plastic flow, and volume changes. Thus the compressive strength becomes the controlling factor for most of the significant qualities of the concrete. A detailed discussion of the laws by which it is governed is presented in the following articles.

5-2. Compressive Strength: Laws. The primary function of the cement paste in determining the quality of concrete was realized as early as 1897 by R. Feret[1]* who found that the strength of cement mortars is determined by the amount of cement per combined unit volume of water and voids in the mortar. The reason for this is that the cement should, as much as possible, fill the entire space not occupied by sand. Since the mortar volume consists of sand, cement, water, and voids, this is equivalent to saying that mortar strength depends on the amount of cement per unit of paste in the mortars. His experimental evaluation of the relationship was

$$f'_c = K\left(\frac{c}{1 - V_s}\right)^2 \qquad [5\text{-}1]$$

where f'_c is ultimate unit compressive strength

K is a constant depending upon the materials and conditions of test

c is proportional amount of cement

V_s is absolute volume (that is, proportion of volume occupied by solid matter) of sand.

In 1914 Professor M. O. Withey[2] of the University of Wisconsin published results which showed a straight line relation between the strength of concrete and its cement-void ratio, another expression for the concentration of cement particles in the paste. Professor Withey's relation may be expressed by the following equation:

$$f'_c = A + B \cdot \frac{c}{v} \qquad [5\text{-}2]$$

where A and B are constants depending upon the materials and conditions of test and v is the proportional amount of voids (water + air).

In 1918 Professor Abrams[3] of Lewis Institute brought forward his water-cement ratio theory which states that the strength of concrete of a workable consistency is given by the equation

$$f'_c = \frac{C}{D^{w/c}} \qquad [5\text{-}3]$$

where C and D are constants and w and c are the respective volumes of

* Superior numbers refer to items in the bibliography at the end of the chapter.

water and cement. For average conditions Abrams gave C a value of 14,000 and D a value of 7. The vast amount of research which followed the development of the water-cement ratio theory emphasized the water content more than the actual quality of the cementing paste, and engineers lost sight of the close agreement between Feret's law and the water-cement ratio relation. At the University of Illinois Professors A. N. Talbot and F. E. Richart[4] published results in 1923 which showed a very good agreement with Feret's law. Their equation for average conditions was

$$f'_c = 32{,}000\left(\frac{c}{c+v}\right)^{2.5} \qquad [5\text{–}4]$$

Except for the exponent 2.5 instead of 2.0, this equation is identical with Feret's equation.

The importance of the quality of the cementing paste regained much ground as a result of the publications of F. R. McMillan[5] and Professor Slater.[6] The quality of the paste was further emphasized by R. L. Bertin[7] in 1930. He found that Abrams water-cement ratio curve became very nearly a straight line when the strength of the concrete was plotted against the specific gravity of the paste. His equation was

$$f'_c = Fg_p - G \qquad [5\text{–}5]$$

where F and G are constants and g_p is the specific gravity of the paste. A more simple expression for the quality of the paste in the concrete was presented by Inge Lyse[8] in 1931. It was shown that both the water-cement ratio and the cement-space ratio curves became very nearly straight lines when the quality of the paste was expressed by the cement-water ratio by weight. Fig. 5–1 shows compressive strengths for the water-cement ratio by volume and for the cement-water ratio by weight. The cement-water ratio relation is given by the equation

$$f'_c = M + N(c/w) \qquad [5\text{–}6]$$

where M and N are constants, and c and w are relative weights of cement and water. This straight line relation holds fairly well within a range which covers nearly all practical concrete mixes. Very lean as well as very rich mixes generally do not follow the straight line law, but the relationship between the quality of the paste and the strength of the concrete is not dependent upon a straight line function. The general law for the strength of the concrete may therefore be expressed as follows: *For cement contents above a minimum required for giving binding strength to the concrete, the strength of the concrete increases with the increase in the concentration of cement particles in the water of the paste.*

These six equations, the principal attempts of western engineers to state precisely the law of concrete strength, have the common quality of recognizing the cement paste as the quality-giving ingredient of the concrete, although this fact was not always evident to their authors nor is it evident at all except upon close study. The quality of the paste is expressed in terms which vary with the equation: the cement-space ratio, the cement-voids ratio, the water-cement ratio, the specific gravity of the paste and the cement-water ratio.

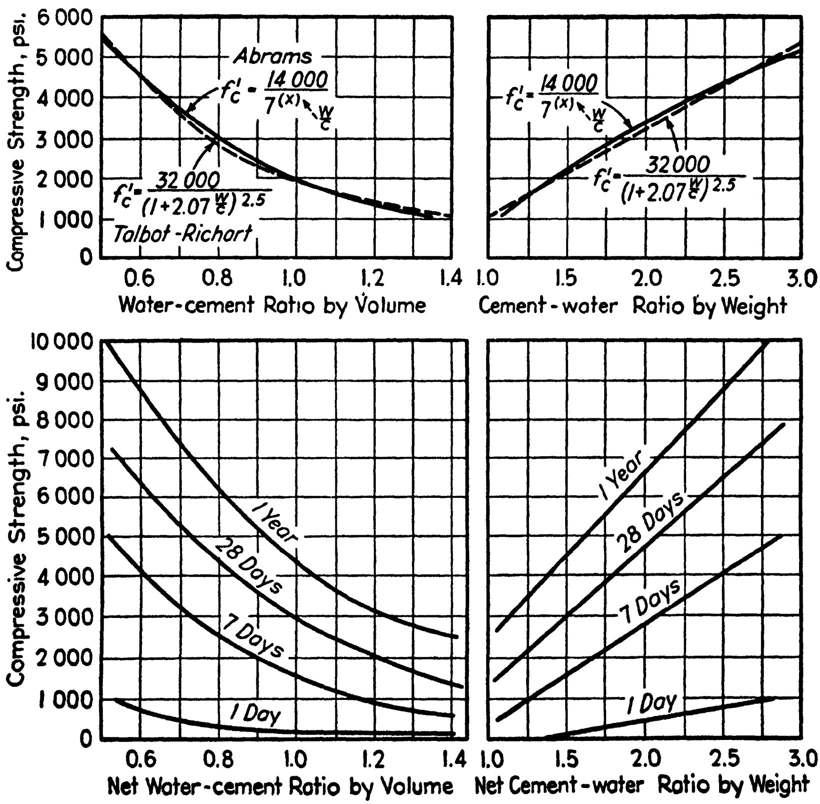

FIG. 5-1

For properly produced concrete the amount of air voids is negligible so that the voids are equal to the water content for all practical purposes. The water content is measured directly for all concrete mixes but the voids have to be determined by calculations from indirect measurements. Consequently water content is far more convenient than void content to use on actual construction jobs. The quality of the cementing paste

is therefore expressed most simply by the cement-water ratio or water-cement ratio.

Experiments[8] have shown that for a given type and gradation of aggregates the water requirement for a given placeability of the concrete is nearly the same for both lean and rich mixes. Making use of this approximately constant water content for different mixes, the strength equation becomes

$$f'_c = M + \frac{N}{w}c = M + Pc \qquad [5\text{-}7]$$

This constant water content theory forms the basis for the " simplified method of design " which has already been presented on page 14 and also gives the foundation for a rational study of the economy of concrete mixes.

5-3. Tensile and Flexural Strengths. Concrete is weak in tension as compared to compression and in general design the use of the tensile strength of the concrete is not permitted. For all that, a knowledge of

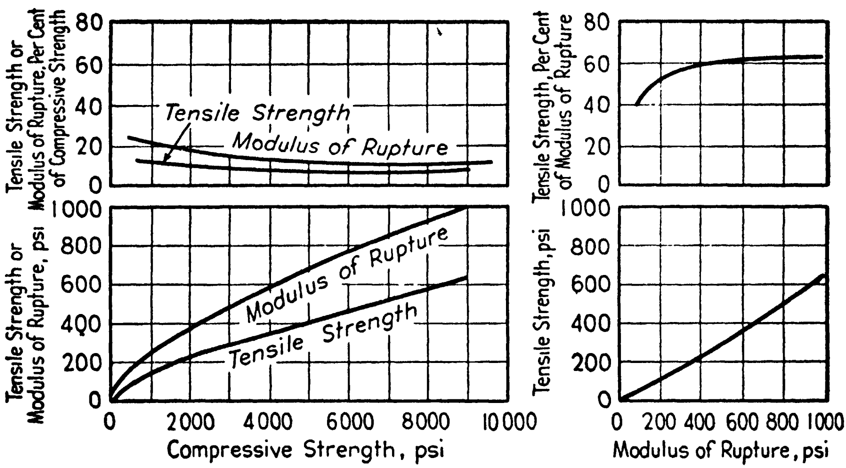

FIG. 5-2 (from Gonnerman and Shuman).

the tensile strength of the concrete is essential for the rational proportioning of massive unreinforced structures, such as gravity dams, and for the design of steel reinforcing to prevent cracking from shrinkage and from temperature changes. The tensile and flexural* strengths of concrete have not been studied as fully as has the compressive strength but the data available[9] indicate (see Fig. 5-2) that the same factors

* In this article flexural strength refers to the flexural strength of a plain, unreinforced concrete member undergoing transverse loading and failing by rupture of the extreme fibers in tension. The apparent flexural strength is about double the direct tensile because it is a computed value based on an assumed linear stress distribution.

which determine the compressive strength also determine to a large extent the tensile and flexural strengths. Attention should be called to the fact that the characteristics of the aggregates have a greater effect upon the tensile and flexural strengths than upon the compressive strength, owing to the greater demand upon the bond between paste and aggregates. Fig. 5-2 presents a typical illustration of the general relationship between compressive, tensile, and flexural strengths. Roughly, the ratio between tensile and compressive strengths is one-tenth to one-fifteenth, and between flexural and compressive strengths one-fifth to one-seventh. The effect of the characteristics of the aggregates on the tensile and flexural strengths has not been determined definitely for all cases as yet but the papers referred to in the bibliography give considerable information on the question.

5-4. Shear Strength. Test results reported by Professors Talbot and Spofford indicate that the shear strength of concrete is at least 50 per cent of the compressive strength. In this article shear strength refers to resistance to sliding such as is afforded by a rivet or pin or by a metal plate during the punching of a hole. Concrete is rarely used in a manner that would subject it to this kind of shear. In fact it is difficult to test concrete in shear because of the necessity of wide bearing surfaces to prevent crushing of the specimen during loading. Consequently we are not vitally interested in the shear strength of concrete.

When the term shear is used in connection with diagonal tension in beams and slabs it is used with the wrong meaning; the usage arises from the fact that the diagonal tension intensity in the section below the neutral axis is equal to that of the horizontal shear and acts in a direction at 45° with the horizontal plane. Consequently the diagonal tension is computed by means of a formula for the shearing stress, but the failure to be guarded against is purely one of tension and not shear.

5-5. Bond Strength. But for the adhesion of the concrete to the steel it would be impossible to reinforce concrete effectively with steel rods so that there would be no slipping between the two materials as the combined member deforms under load. In design care must be taken that there is no excessive tendency for the steel to slip from the grip of the surrounding concrete since in general a small movement will result directly or indirectly in the destruction of or serious damage to the piece. This bond, or resistance to sliding, is of two kinds: an adhesion between the two materials and a sliding resistance that develops after the adhesion is broken and movement begins. Tests made by Professor Abrams at the Structural Materials Research Laboratory[10] by pulling out ordinary plain round rods from 8 by 8 in. concrete cylinders of different ages, where the only resistance to pull was the force

developed by bond on the surface of the rod, showed that there was no slip until the bond stress reached an average value of 10 to 15 per cent of the compressive strength of the concrete, and that the maximum bond resistance, reached when the slip was about 0.01 in., equaled approximately 24 per cent of the concrete strength. Earlier tests made at the University of Illinois[11] showed that square bars give results about 75 per cent of those obtained with plain round bars. The same series of tests proved that deformed bars begin to slip at about the same bond stress as plain rounds and that the resistance to sliding offered by the bearing of the projecting lugs on the concrete is considerably larger than that for the plain bars but does not become effective until a considerable slip has occurred.

Recent investigations[12] have indicated a definite increase in bond strength of deformed bars over plain bars at both initial and final slip. The standard codes for reinforced concrete recognize the efficiency of deformed bars and permit greater bond stress than for plain bars.

5-6. Compressive Strength. The laws of strength for concrete have already been outlined together with the methods for obtaining any desired strength. Although no concrete under good engineering control is at present proportioned for strength by the old rules which related cement to total aggregate volume, there is still sufficient use made of the method to make it worth while to note the expectancies. The minimum ratio of cement to aggregate is still often specified for various situations. The following table is from the 1916 J.C. Report:

COMPRESSIVE STRENGTHS OF DIFFERENT MIXTURES OF CONCRETE

In pounds per square inch at an age of 28 days, testing cylinders 8 in. in diameter and 16 in. long, made, stored, and tested under laboratory conditions.

Aggregate	1 : 3*	1 : 4½*	1 : 6*	1 : 7½*	1 : 9*
Granite, trap rock	3300	2800	2200	1800	1400
Gravel, hard limestone and hard sandstone	3000	2500	2000	1600	1300
Soft limestone and sandstone	2200	1800	1500	1200	1000
Cinders	800	700	600	500	400

* Combined volume fine and coarse aggregate measured separately.

With modern scientific methods of design and control greater strengths can be obtained for these cement concentrations than indicated by this table. With advance in concrete technique it has become the practice to use stronger concretes in construction than formerly. The table below summarizes roughly both present-day practice and that of

16 years ago when this text appeared in its first edition. Wide variations from these averages will be met.

Construction	Unit Strength of Concrete Used	
	1925	1940
Reinforced columns	3000–3300	3000–6000
Highway slabs	2500–3000	3000–5000
Reinforced concrete slabs, beams, bridges, arches, and ordinary watertight work	2000–2200	2500–3750
Foundation walls, plain concrete, retaining walls, piers, abutments, machine foundations	1600–1800	2000–2500
Unimportant mass work	1300–1400	1500–2000

5–7. Elastic Properties of Concrete and Steel. Concrete is not, strictly speaking, an elastic material. Strain increases faster than stress from the very first and permanent deformation occurs under low stress. On the average the stress-strain curve in compression may be taken as approximately a parabola with the vertex at the point of ultimate strength and the axis vertical, stress being plotted vertically and strain horizontally. Within the range of the usual working stresses this curve does not deviate greatly from a straight line and it is universally the custom to assume that the modulus of elasticity is constant for working stress conditions. The values of the modulus given by different authorities differ greatly, 3,000,000 psi being perhaps the most accepted average value for ordinary 3000 psi building concrete at 28 days. As the concrete ages it gets harder and stiffer and the modulus increases. The modulus of elasticity for any concrete increases as the concrete increases in strength with age. Differences in strength between concretes of widely varying mixtures or materials are not always accompanied by commensurate differences in modulus of elasticity. The quantity of paste, quality of paste, and the type of aggregate are important factors. The relationship between the modulus and the strength is nearly a straight line for ordinary concrete mixes, as shown in Fig. 5-3. The recommendation of the American Concrete Institute for the design modulus corresponds to a direct ratio between modulus and strength,

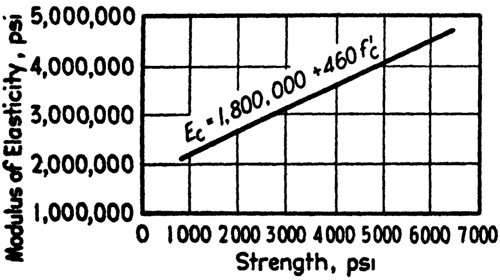

Fig. 5-3

$E_c = 1000\, f'_c$, a recommendation which does not seem to be well supported by experimental data. Alumina cement concrete has a considerably higher modulus than portland cement concrete of the same proportions.

When concrete is subjected to a continuously sustained load it continues to deform, or flow plastically, for a long time after the first application of the load. The modulus of elasticity of concrete determined by the usual relatively rapid application of load by a testing machine is affected to a large extent by this phenomenon of plastic flow; the more leisurely the investigator in his procedure the larger the measured deformations and the smaller the reported value of the modulus. See Fig. 5-4.

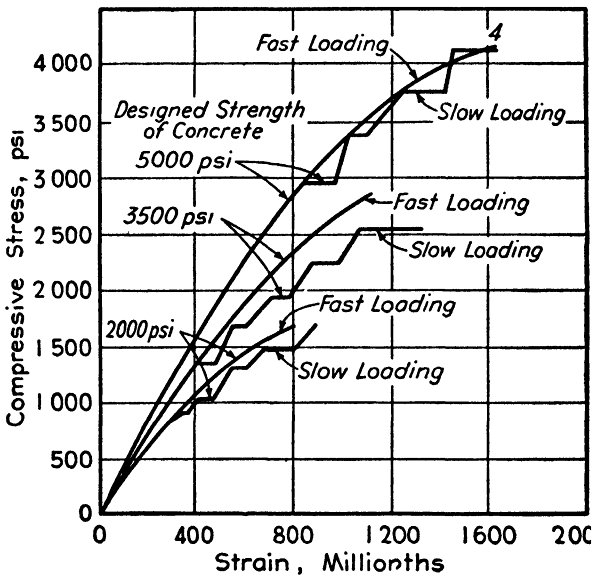

FIG. 5-4

Strictly speaking, there is no elastic limit for concrete. The term is often used inexactly to indicate the limit of stress that may be applied repeatedly without causing increase in permanent deformation. This limit, which is generally called fatigue limit, varies between 50 and 60 per cent of the ultimate; loads beyond this limit and below the static ultimate when applied repeatedly cause continually increasing deformation and, finally, rupture.

The modulus of elasticity of steel is usually taken as 30,000,000 psi for all grades of steel in all computations of reinforced concrete. This value is slightly higher than the average given by tests.

5-8. Poisson's Ratio. Poisson's ratio, the ratio between lateral and longitudinal deformation under direct stress, is made use of in the structural analysis of reinforced concrete slabs, arch dams, and other statically indeterminate structures. Experimental results have shown that this ratio does not vary much for different grades of concrete and can generally be considered constant. The values most frequently used for concrete are $\frac{1}{6}$ and $\frac{1}{5}$.

For homogeneous materials the following relation exists between the modulus of elasticity in compression (or tension), E, the modulus of elasticity in shear, G, and Poisson's ratio, m.

$$G = \frac{E}{2(1 + m)} \qquad [5\text{-}8]$$

For lack of accurate data the above formula may be used in design. The modulus of elasticity in shear is only used for accurate analysis of slabs, arches, and dams.

5-9. Plastic Flow. It is only comparatively recently that the attention of the engineering profession has been called to the plastic action of concrete under load referred to in the previous discussion of the elasticity of concrete. Experiment has seemed to demonstrate that under sustained constant load the deformation of the concrete increases progressively and that this plastic flow may be of appreciable magnitude and cause large changes of stress from those set up initially. In 1921 F. R. McMillan[13] found that the stress in the longitudinal reinforcement of concrete columns in buildings approaches the yield-point stress of the steel when the structure is subjected to working loads for a long period. Other tests have shown that the stress in the tensile reinforcement of beams is affected by plastic flow. In the design of reinforced concrete columns attention is given to this phenomenon and the design is modified to take care of any excess stresses set up.

Plastic flow of concrete has been found to depend upon such factors as the magnitude of stress, the strength of the concrete, the duration of the loading period, the humidity of the atmosphere, the age of the concrete, the characteristics of the aggregates, and the quantity of cement paste. Extensive investigations, particularly at the University of California, have shown that for concrete subjected to ordinary working loads the ultimate strength is in no way affected by flow. Experimental data indicate that the plastic flow is approximately proportional to the stress; that it proceeds at a relatively high rate at the application of the load and gradually decreases and approaches a constant rate (see Fig. 5-5); that the higher the humidity of the air the less is the flow; that the greater the age of concrete at application of load the less is the

flow; that for stress of a certain percentage of the strength of the concrete the flow is approximately equal for concretes of various strengths; that the flow is approximately proportional to the quantity of cement paste in the concrete, and that the characteristics of the aggregates are very important. Experimental results also have shown that the plastic deformation may be several times the elastic deformation and may continue to increase for several years.

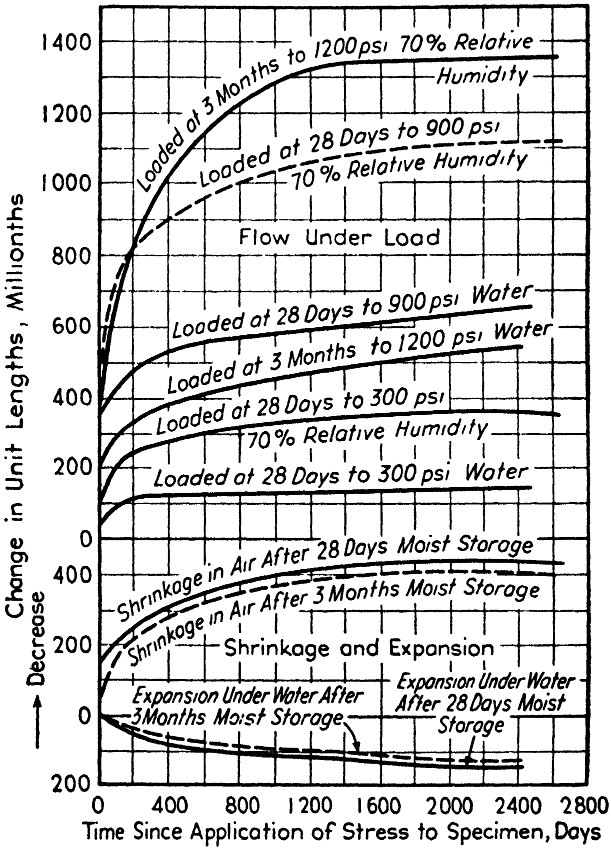

FIG. 5–5 (from Davis's *Plastic Flow of Concrete*).

Structures subjected to rapid repetition of load will also show plastic flow and should be designed to provide for the stress redistribution caused by this additional deformation.

The ordinary theory of the interaction of steel and concrete in structural members does not take account of the flow of loaded concrete and consequently the actual stresses are very different from those directly

computed. For stresses within the elastic limit steel does not flow under load and the continued deformation of the concrete greatly changes the manner of the interaction of the two materials progressively over a considerable period, resulting in lower concrete stresses and higher steel stresses than given by the ordinary theory. The details of this phenomenon are discussed later when study is made of the mechanics of reinforced concrete structural members.

Recently doubt has been cast upon all these theories of plastic flow by Professor G. A. Maney* who presents experimental evidence interpreted to indicate that under working conditions all time-yield changes are due primarily to warping or non-uniform shrinkage. Professor Maney suggests changes in our design theories to accommodate this new interpretation of long-observed phenomena and we must await verdict of the engineering profession upon his conclusions.

5–10. Fatigue of Concrete. Concrete behaves like other structural materials under repeated stress of high magnitude, exhibiting a lower ultimate strength than when loaded directly to failure. Loads causing stress below the fatigue limit may be repeated indefinitely without structural damage. Loads causing stress above this limit will cause failure if repeated enough times; the higher the stress the fewer the repetitions needed. Laboratory experiments have indicated that the fatigue limit for concrete is about 50 to 60 per cent of the static ultimate strength, both in direct compression and in flexure, a limit which will be increased by the reinforcement to a certain undefined extent. It should always be kept in mind that plain and reinforced structures subjected to rapid repetitions of loads should be designed so as to have a proper factor of safety based on the fatigue limit. Loads sustained for a long period of time have an effect similar to that of repeated loading. Tests have shown that when reinforced concrete columns are subjected to 80 per cent or more of their ultimate strength load, they deform and deflect to such an extent that they exceed their limit of usefulness and may be considered failing. For long-sustained loads the actual factor of safety should be based on about 80 per cent instead of 100 per cent of the strength of the column.

Unfortunately the effect of repeated and sustained loadings has not been fully investigated as yet and much research work must be carried out before we shall be able to design our structures with both proper safety and economy.

5–11. Permeability of Concrete. Most concrete structures are required to be more or less impermeable. Experience and laboratory

* "Concrete under Sustained Working Loads," paper presented before the American Society for Testing Materials, June, 1941, "Studies in Engineering — No. 1," Northwestern Technological Institute of Northwestern University, 1941.

investigations have shown that the permeability of concrete is determined by much the same factors of paste composition as is the compressive strength. The concentration of cement in the paste is the most important factor for the production of a watertight concrete. For a given method of curing, the leakage decreases in direct proportion to the increase in strength. In Fig. 5-6 the leakage has been plotted against the compressive strength and the relation is found to follow very

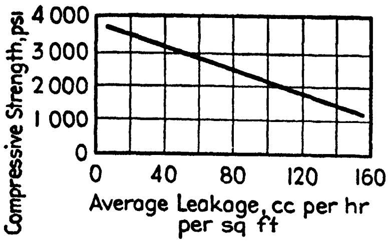

FIG. 5-6

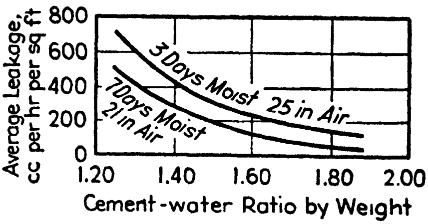

FIG. 5-7

nearly a straight line. Proper curing of the concrete is even more important for low permeability than for strength (Fig. 5-7). The desired low permeability may be secured either by using a high concentration of cement in the paste of a concrete cured only a short time or by curing a concrete of lower cement content for a longer time. By curing is always meant advantageous curing, such as water curing or any other method used for the prevention of the evaporation of the mixing water. The amount of cement paste has been found to affect the permeability of concrete; the greater the amount of paste the larger is the leakage.

Placing is also highly important to success in securing low permeability. A slight defect in the placing of the concrete may have little effect upon its strength but may increase surprisingly its permeability.

Certain powdered admixtures, such as pulverized sand and fine clay, may decrease the permeability of concrete and yet leave the strength practically unaffected. Other admixtures may be detrimental to the watertightness of the concrete. The problem of waterproofing admixtures and surface coatings can only be solved by dependable experimental results. In general the desired watertightness is secured more cheaply by the use of the proper concentration of cement in the paste of the concrete and proper curing than by the use of admixtures or surface coatings.

5-12. Durability of Concrete. Exposed concrete structures must be designed to resist the destructive forces of weathering. Concrete of low permeability will naturally resist these destructive forces much more effectively than will permeable concrete. There is a very close relationship between permeability and durability and the permeability test

is probably one of the best means we have for ascertaining concrete durability. A more direct test is made by repeated freezings and thawings of the test specimens. Results from both permeability and freezing and thawing tests indicate that the factors which determine the strength of concrete also to a large extent determine its durability. For given materials the durability improves in direct relation to the increase in strength. The concentration of cement particles in the paste is therefore a criterion for durability, as shown in Fig. 5-8. This figure shows that initial disintegration decreases with the increase in the cement-water ratio of the paste in the concrete. The effects of such items as brands and types of cement, characteristics of aggregates, method and length of curing and effectiveness of placing are much more important

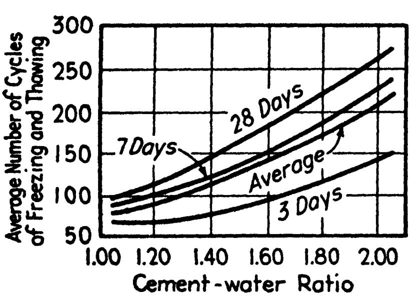

Fig. 5-8

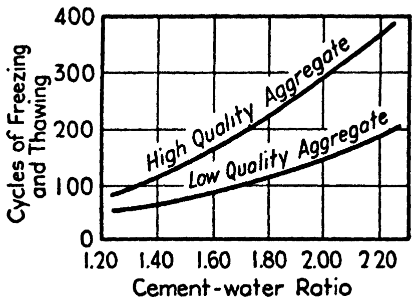

Fig. 5-9

for durability than for strength. Experimental results have indicated that the desired durability of any concrete may be obtained by using high quality concrete (see Fig. 5-8), high quality aggregates in good quality paste, aggregates of lesser quality in high quality paste (Fig. 5-9), or longer curing. The proper selection of materials and of placing and curing methods are important economic problems which can only be solved by a full knowledge of the qualities of the available materials and a thorough study of the price relations.

5-13. Fire Resistance. Recent investigations at the research laboratory of the Portland Cement Association have revealed some very important facts regarding the fire resistance and heat insulation qualities of concrete at high temperatures. The tests were carried out on masonry building units but the deductions are applicable to concrete structures in general. These tests showed that the quality of the paste as measured by cement content is the outstanding criterion of fire resistance of concretes made of any given material. In Fig. 5-10 the fire-resisting values have been plotted against the cement content of the units. It is noted that the fire resistance increases in direct proportion to the

increase in cement content for all the three different aggregates used. This is in perfect harmony with the strength results, indicating that the same factors affect both fire resistance and strength. The effect of the characteristics of the aggregates, however, is seen to be much more prominent for fireproofing than for compressive strength. The artificial lightweight aggregate Haydite and water-cooled slag are particularly well suited for concrete of high fire resistance. Any concrete made from porous or lightweight aggregates would naturally have similar advantages over concrete made from ordinary solid aggregates. Ordinary limestone and gravel aggregate are seen to give considerably lower fire resistance than the porous aggregates.

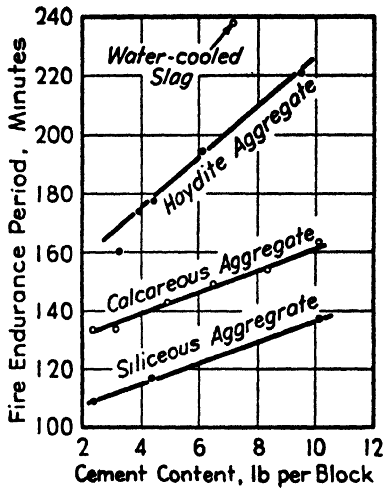

FIG. 5-10

5-14. Volume Changes. Recent extensive experiments at the research laboratories of the Portland Cement Association, the University of California, and at Lehigh University have yielded much information on the subject of volume changes in concrete due to variations in its moisture content. An excellent review of these test results was presented by the late Professor Slater.[6] The outstanding conclusion gained from these tests is that the shrinkage of the concrete increases with an increase in the paste content of the concrete. Rich mixes and mixes of high water content are therefore marked by greater volume change than lean and stiff mixes. The interrelation between volume change and paste content is presented in Fig. 5-11 for concrete specimens cured 7 days in the moist room

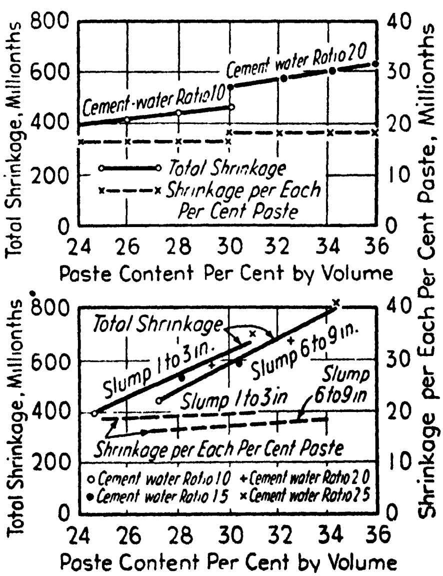

FIG. 5-11

at 70°F, and then in the observation room at a relative humidity of 40 to 60 per cent and a temperature of 80°F until the age of 6 months. These tests show that the amount of paste is the principal factor in volume change of concrete. The shrinkage for a given percentage of paste was practically the same for rich and lean mixes.

The basic principle for making concrete of low volume change is the use of low paste content, which means lean and dry mixes. In order to secure other desirable qualities with low volume changes mechanical placing by vibrating, machine tamping, or high pressure must be used.

5–15. Coefficient of Thermal Expansion for Concrete. Experimental results have shown that the average coefficient of expansion for concrete is about 0.000,005,5 per 1°F. This compares fairly well with the coefficient of 0.000,006,5 per 1°F for steel. The slight difference has been found to cause no difficulty in the bond between steel and concrete. The coefficient for reinforced concrete may therefore be taken as 0.000,006 per 1°F.

A problem closely related to that of volume change with moisture content is the expansion due to the heat development during the hardening of the concrete and the subsequent contraction when the concrete cools off. For ordinary structures the effect of these temperature changes is negligible, but for large masses, such as Boulder Dam, certain provisions have to be made to reduce this effect. Complete water-cooling systems are incorporated in the concrete for the purpose of removing the heat developed so that the entire mass will be uniformly at mean annual temperature when the joints are grouted. This insures no further shrinkage due to temperature equalization and thus prevents opening of joints and formation of cracks. Special low heat-developing cements are also used as an aid in keeping the expansion at a minimum and reducing the amount of subsequent cooling. Future experiences will show if these precautions are necessary and effective.

5–16. Weight of Concrete. Ordinary rock and gravel concrete weighs on the average about 145 pcf in the dry condition. The greater the maximum size of the aggregate the greater is the weight of the concrete because of less paste per unit of volume. In reinforced concrete the unit weight is increased by the steel, which may be assumed to add on an average about 5 pcf, making the unit weight of reinforced concrete about 150 pcf.

Using light-weight aggregates and aerating by means of chemicals which generate gas and cause the concrete to fluff up so that on setting it is full of air bubbles will produce concrete of much less weight than ordinary rock and gravel concrete. The weight per cubic foot of slag concrete is about 130 lb; cinder concrete, about 120 lb; burned clay

concrete, using the commercial Haydite, Pottsco, Lytag, etc., about 100 lb, and aerated concrete as little as 50 lb. The economic significance of lightweight concrete is important.[14]

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A text such as this can give but very limited data on concrete properties. For further information the reader should consult the files of the Proceedings of the American Society for Testing Materials and the American Concrete Institute. The Portland Cement Association, which has offices in the principal cities of the United States, is prepared to answer specific requests for information.

CHAPTER VI

FORMS

6–1. Since concrete is manufactured in a plastic or semi-fluid state, it must of necessity be confined in a mold or form until it has set or hardened sufficiently to hold its shape and, in many instances, support its own weight. Under certain circumstances forms may be omitted. The bottoms of foundations and ground floors and, sometimes, the sides of footings or walls are examples. This ordinarily means that an earth surface acts as a form. Forms represent 15 to 40 per cent of the final cost of a concrete structure. This cost may vary considerably with design, and therefore forms must be very carefully studied by the designer, although their detailed layout is commonly left to the contractor or field engineer.

6–2. Requirements. The first essential of forms is that they must be carefully built to the required dimensions and made of sufficient strength to hold their shape and alignment under the load of the wet concrete and any construction loads which may come upon them. They must also be sufficiently tight to prevent the escape of water, for escaping water carries with it much of the finest and most effective cement. A second essential is that they be designed to facilitate, as much as possible, easy removal.

The requirements of the finished concrete surface must often be considered. Forms for footings or other substructure work may be of the roughest construction. Forms for ordinary building work must be of dressed stock to give a satisfactory appearance. For ornamental work, cornices, etc., the forms must be built with great care and the outlines should be designed with a view to reasonable construction. Offsets in moldings should be $\frac{7}{8}$ in., $1\frac{3}{4}$ in., or some other stock lumber size, no offsets less than $\frac{7}{8}$ in. being practical in ordinary commercial construction. It should be remembered that sharp corners are always liable to spalling when forms are stripped. This is one of the reasons why triangular fillets are usually used at all corners of beams and columns. Construction joints are seldom completely obliterated, so they should be made where they will show the least or they should be located with a view to symmetry.

6–3. Materials. Wood is still the most common form material, spruce and pine being used most. (See Figs. 6–1 to 6–3.) Certain

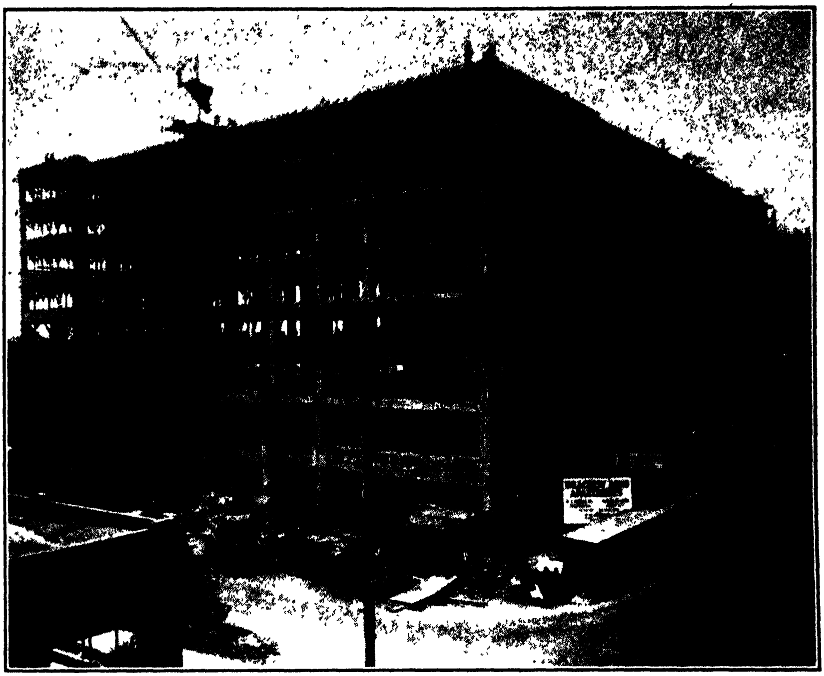
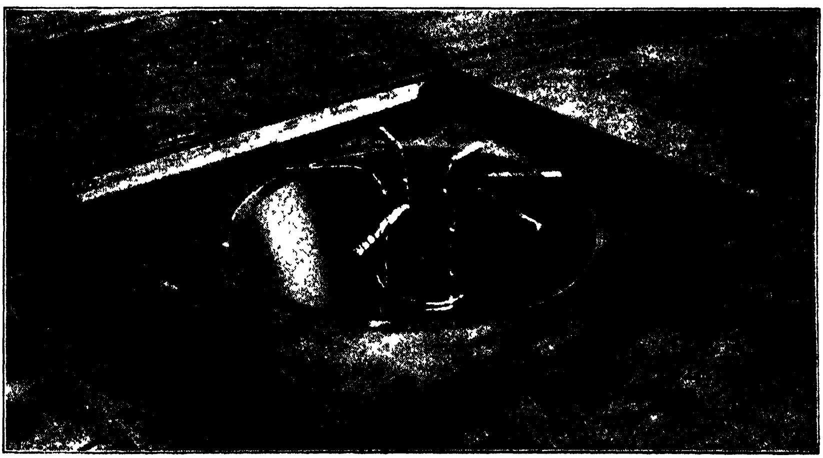


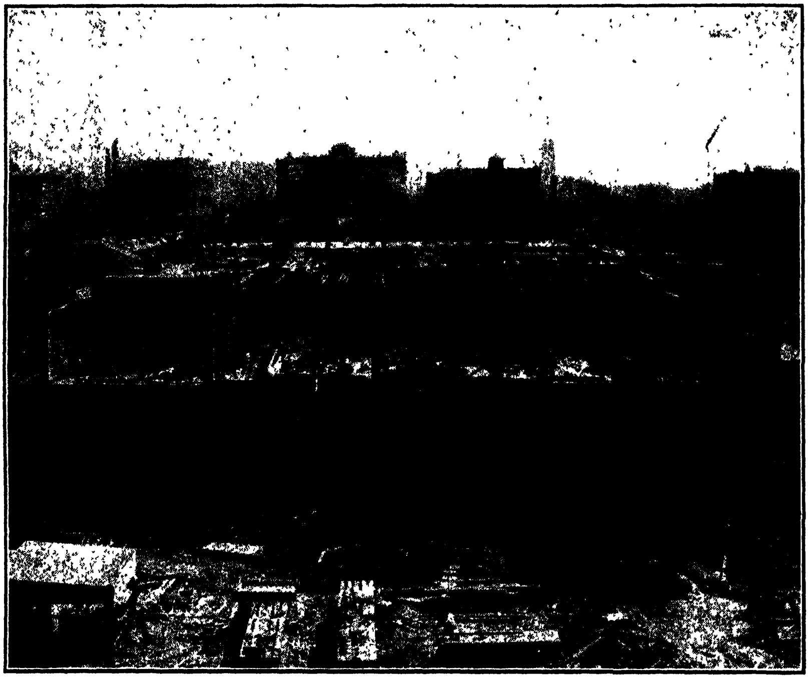
FIG. 6-1. Concrete structural frame showing wood forms with shores in place for four stories. Note brickwork starting up from the fourth floor level, and the hoisting tower in the foreground: also the wood forms in the upper stories.



Courtesy Hausman Steel Co., Toledo, Ohio

FIG. 6-2. In the foreground is the top of a column form like that in Fig. 6-4. Spirals and column verticals are shown, the latter bent out into the roof slab. Deck panels are of plywood.

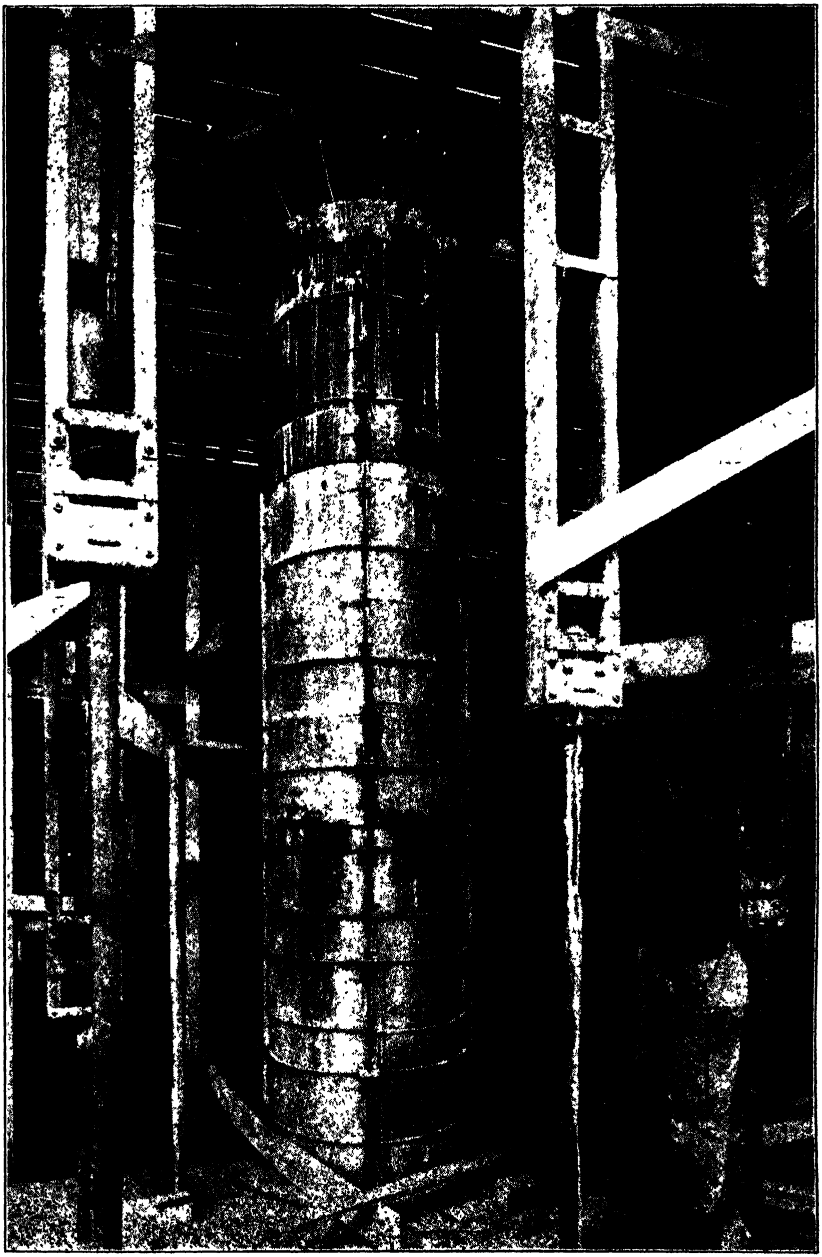
woods, notably hemlock, are unsuitable for nice work because they stain the concrete. Partially seasoned stock is usually used, for fully dried lumber swells too much when wet. For all except substructure work lumber should always be planed on at least one face and one edge, and usually it is dressed on all four sides.



Courtesy Hausman Steel Co., Toledo, Ohio

FIG. 6–3. Wooden forms for beams and columns are shown partially completed for third floor. Steel pan forms are in place for center portion; floor already poured in background. Note form builders' benches in foreground, and shoring still in place under second floor.

Plywood panels for formwork are built up of several thin layers of wood held together by waterproof glue and having protected edges. Sheets are available in widths of 4 to 6 ft, lengths of 8 to 12 ft or more, and thicknesses of ¼, ⅜, ½, ⅝ and ¾ in. These large panels reduce placing costs, eliminate joint marks between boards, and, although somewhat higher in first cost, decrease greatly the expense of finishing and rubbing exposed surfaces. Lenticular patches are easily made in the outer veneer to remedy surface defects. Both sides of the panels are finished and thus permit reversing.



Courtesy Hausman Steel Co., Toledo, Ohio

FIG. 6–4. This illustration shows a steel form for a typical round column as used in flat slab construction. It also shows adjustable shores for the support of the decking. The deck forms here shown are open as cork insulation was applied before pouring concrete.

Panels of highly compressed wood fibers are available in sheets of about the same sizes as plywood. Having a smooth, hard surface on one side they eliminate the grain marks that appear with plywood. The reverse side has a matte surface from the screens on which the panels are made, and can only be used when a slightly textured surface is desired.

Steel forms for concrete have a considerable and perhaps growing use. They are more expensive in first cost than wood, but more substantial for rehandling, so that if steel forms can be used enough times, they are cheaper than wood. Steel gives a smoother concrete surface than wood and does not show board marks, but the joints of the panels show in the finished work. Steel is used generally for concrete chimney forms, for circular columns and flat slab column capitals. It is also used for slab forms both in the shape of domes and pans for ribbed floors and in panels, sometimes reinforced with wood ribs, for plain slabs and walls.

Plaster and glue forms ("staff") are used for ornamental work, usually the province of the architect rather than the engineer. Outlines so elaborate as to require such special forms should be designed in consultation with someone experienced in special formwork.

6-4. Design. In the past concrete forms were frequently built by rule of thumb but experience has shown that careful designing saves money on a big job. In computing the size of form members required for strength wet concrete is assumed to weigh 150 pcf. In addition a live load of about 75 psf is usually allowed for the weight of distributing buggies, runways, equipment, and men working. The lateral thrust of wet concrete is equivalent to that of a liquid weighing 140 pcf. The capacity of the concrete plant must be known, therefore, to design forms, since they must be figured for a head equal to the depth which will be poured during the time of initial set. In the absence of more specific information this is frequently taken as one and one-half hours.

The catalogs of many manufacturers of clamps and ties for formwork contain safe load tables and charts for those who have to make such computations frequently.

It is a growing practice to design members for the use of stock widths of lumber. Current commercial sizes and their properties are given in the table on page 51. These actual sizes vary somewhat in different sections of the country.

The working stresses commonly used for formwork are:

Flexure	1800 psi
Longitudinal shear	150 psi
Columns*	$1200\left(1 - \dfrac{L}{80D}\right)$

where L is the effective length of column in inches and D is the least side in inches. These stresses are for use with pine or fir lumber graded No. 2 or better. When poorer materials are used these stresses should be reduced accordingly.

COMMERCIAL LUMBER SIZES

Nominal Size	*Finished Size — S4S*	*Area — S4S*	*Section Modulus on Edge — S4S*
1 × 4	¾ × 3⅝	2.72	1.64
1 × 6	¾ × 5⅝	4.22	3.96
1 × 8	¾ × 7⅝	5.72	7.26
2 × 4	1⅝ × 3⅝	5.89	3.56
2 × 6	1⅝ × 5⅝	9.14	8.57
2 × 8	1⅝ × 7⅝	12.19	15.23
2 × 10	1⅝ × 9⅝	15.44	24.44
2 × 12	1⅝ × 11⅝	18.69	35.82
3 × 4	2⅝ × 3⅝	9.53	5.76
3 × 6	2⅝ × 5⅝	15.12	13.86
3 × 8	2⅝ × 7⅝	20.62	25.78
3 × 10	2⅝ × 9⅝	26.12	41.36
3 × 12	2⅝ × 11⅝	31.62	60.61
4 × 4	3⅝ × 3⅝	13.14	7.94
4 × 6	3⅝ × 5⅝	20.39	19.12
4 × 8	3⅝ × 7⅝	28.12	35.16
4 × 10	3⅝ × 9⅝	35.62	56.41
4 × 12	3⅝ × 11⅝	43.12	82.66

Economy is gained by designing to permit reuse of forms with a minimum of change. Beams are made to fit forms from floors below even if concrete sizes are not a minimum. When beams must change on upper floors it is economical to reduce the depth, keeping the same stem width. Column forms are usually designed for easy reduction in width.

6–5. Construction. Building a form for concrete is just the opposite of the older carpentry problems, for it is building outside a surface instead of inside. The ordinary concrete form is a surface of boards,

* The Forest Products Laboratory of the U. S. Department of Agriculture recommends an Euler type of formula for slender columns as follows:

$$\frac{P}{A} = \frac{0.274E}{(L/D)^2}$$

when L/D is greater than about 24, taking E = 1,600,000 psi.

plank or steel plate, supported by joists and posts with the necessary braces.

Posts and braces are usually adjusted to length with a pair of wedges. This allows the strain to be taken off for removal.

Several types of adjustable shores on the market consist of a pair of 2 by 4's between which slides an iron post held in any desired position by a suitable clamping device. These shores save cutting lumber and making wedges, permit easy adjustment to exact grade, are particularly useful in reestablishing grade in case of settlement, and aid very materially in stripping forms.

There are three steps in the use and cost of a concrete form: making, erecting, and removing. The making, including material, is the most expensive step; consequently economy is effected by reusing as many times as possible.

To improve surfaces, prevent absorption, and facilitate removal, a coating of mineral oil is used on form surfaces. Column and wall forms should have cleanout doors in the base so that all shavings and other debris in the bottom of the forms may be removed before pouring concrete.

The time that forms must remain in place will vary with the kind of the members, the weather, and the character of the concrete. The minimum time will vary from 48 hr for walls and columns to a longer period for slabs and beams at a standard 70° average setting temperature. This time should be increased if members are carrying load beside their own weight. Lower temperature delays setting. Fig. 6–5 gives an idea of the time of setting at different temperatures compared with the time required at 70°.

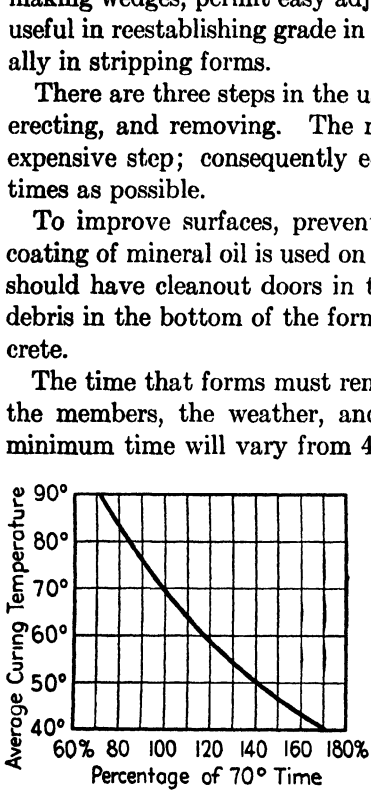


Fig. 6–5

An old rule for ordinary portland cement allowed one day of setting for each foot of clear span of solid slabs and beams. With high early strength cement this time can be cut down to one-third, or one-half of the above. Three-day test cylinders frequently attain strengths higher than the design stresses. The best rule is to break cylinders cured under job conditions and strip forms when the ultimate strength is 50 per cent greater than the design stresses. Care must then be taken not to overload the green slab.

It is sometimes desirable to leave forms in place to prevent premature drying of the concrete.

6-6. Examples. The following examples of form design for the typical building shown in Chapter XVII and detailed in Chapter XXI will illustrate the application of the foregoing suggestions.

Example 6-1. Design foundation wall forms shown in Fig. 6-6.

Solution. The soil is sufficiently cohesive to permit pouring the footing course directly against earth banks without wood forms. A keyway and dowels are provided to anchor the wall above.

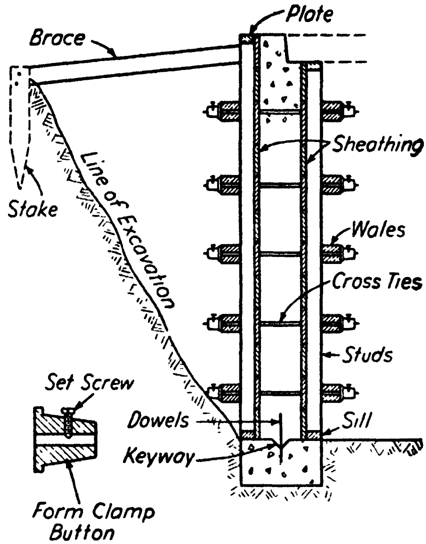


FIG. 6-6

For wall forms use ordinary commercial yellow pine lumber, graded No. 2 and better and surfaced four sides. Although panels of steel and plywood are available this job is rather small for special material. Plywood or pressed wood fiber sheets would give an excellent finish but are expensive unless an unusually well-finished surface is demanded. Set 2 by 4 sill first on top of footing course. Next erect the outside wall form braced to stakes in the bank. Then place the reinforcing steel. Finally erect inside forms and brace securely to the outer ones with cross ties. Pocket inside top of wall to receive the floor slab.

Assume that rate of pouring will be 4 vertical feet per hour and that the maximum head of wet concrete to provide for will be 6 ft. The lateral pressure is computed:

$$6 \text{ ft @ } 140 \text{ pcf} = 840 \text{ psf}$$

Maximum safe span of 1 by 8 square edge sheathing S4S is determined:

$$M = \frac{wL^2}{12} = f_w \frac{bd^2}{6} \qquad 840L^2 = 1800 \times 12 \times \tfrac{3}{4} \times \tfrac{3}{4} \times \tfrac{1}{6} \qquad L = 1.56 \text{ ft or } 18.5 \text{ in.}$$

Common spacings of studs are 16 and 24 in. as they exactly fit standard lengths of boards. Use 16-in. centers for studs.

Maximum span for 2 by 4 studs at 16 in. c to c using $M = wL^2/10$ for semi-continuity:

$$wL^2 \tfrac{12}{10} = f_w S \qquad 840 \times \tfrac{16}{12} \times L^2 \times 1.2 = 1800 \times 3.56 \qquad L = 2.2 \text{ ft}$$

For wales, maximum span of two 2 by 4's at 2.2 ft c to c, using $M = wL^2/12$ for full continuity, assume uniform loading:

$$840 \times 1.1 \times L^2 = 1800 \times 3.56 \qquad L = 2.63 \text{ ft}$$

The cross ties are made of ordinary plain round reinforcing steel. They carry 2.2 × 2.63 sq ft @ 840 psf = 4900 lb. The required area of rod is 4900/25000 or 0.20 sq in., which is exactly furnished by a ½ in. round rod.

A stress of 25,000 psi is permissible for temporary members of this sort. The cross-tie rods are best held by a special button with offset hole and set screw on each end as shown.

Example 6–2. Design forms for typical interior column " B3 " from first to second floor of the building considered in Chapter XVII and elsewhere.

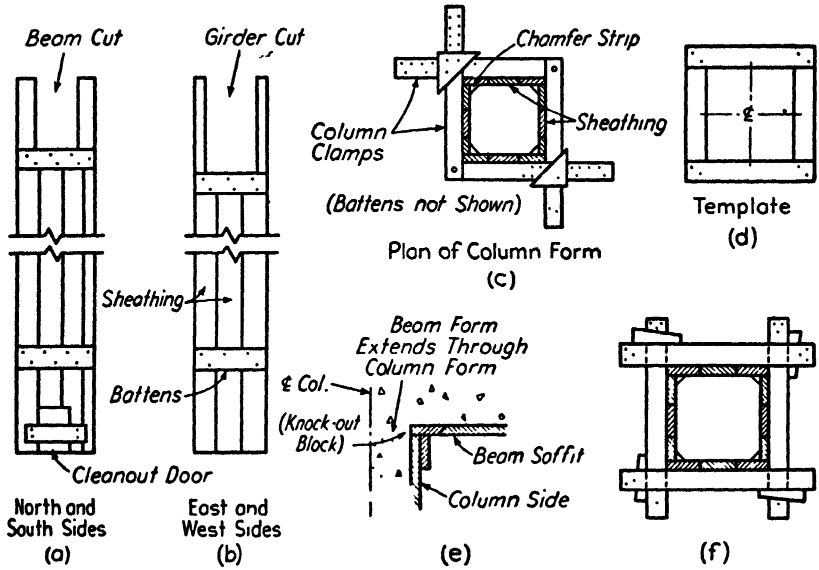


Fig. 6–7

Solution. See Fig. 6–7. Use 1 × 6 square edge No. 2 and better yellow pine S4S vertical sheathing. Some contractors on large work use 2-in. plank S4S for column forms because they wear longer and provide stronger support for the floor forms, but 1-in. material is ordinarily strong enough. Use 1 by 4 rough battens 24 in. c to c with two 6d* common nails at each intersection. Make cutouts on bench for beam forms. Ordinarily the beam form extends through the column forms. Hence the width of cutout equals the beam width plus ⅞ in. each side. Depth of cutout extends down from underside of slab panel to lower side of the 1⅝-in. plank form for the beam bottom, called the beam soffit. Forms for two sides of the column, as, for example, north and south, overlap the other two sides at the corners. Then the east and west

* For these problems in formwork the following table of common nails will be useful (6d is read " six-penny "):

Size	*Length*	*Gage*
6d	2″	11½
8d	2½″	10¼
10d	3″	9
12d	3¼″	9
16d	3½″	8
20d	4″	6

sides would cut between these. Thus the width of two side forms equals the column size plus ⅞ in. each side. The width of the other two side forms is equal to the column size.

Provide ¾-in. chamfer strips in each of four corners. Cut cleanout openings in bottom of two opposite side forms. See Fig. 6–7. Some designers are satisfied with one cleanout door per column, but cleaning is much simpler with two doors.

For assembling use 36 by 36 in. patent adjustable steel column clamps of a type that automatically squares the form and holds it rigidly in place. (" Yokes " of 2 by 4 are sometimes substituted; see Fig. 6–7*f*.) The spacing of these clamps will be closer for the lower half of the column and farther apart at the top where the pressure is less. The spacing can be determined from the stiffness of the sheathing or from the manufacturer's catalog. About six clamps should be used for this column. Make a template of 1 by 4 rough (Fig. 6–7*d*) with inside dimensions 1¾ in. greater each way than concrete column size. These are fastened on top of the rough slab with concrete stub nails — large-diameter hardened nails that can be driven into the concrete — and serve to center and square the forms.

To reduce forms for upper-story columns rip one-half the difference in size off each side of column form with a portable electric handsaw. Use a blade that cuts both wood and metal, thereby making it unnecessary to bother about nails.

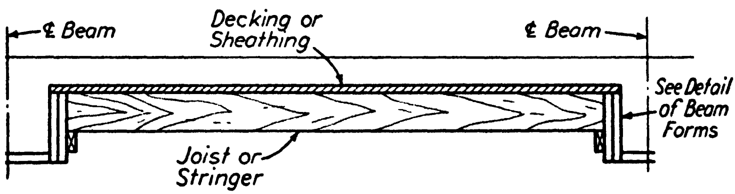


FIG. 6–8

Example 6–3. Design forms for floor slab FS1 of the typical building of Chapter XVII.

Solution. See Fig. 6–8. For appearance use plywood panels. The difference in cost will be well repaid in the cheaper finishing and rubbing. With beams as close together as in this example span from beam side to beam side and eliminate intermediate shoring. As the stringers will probably require closer spacing than the plywood check their capacity first.

Live load of pouring crew	75 psf
4 in. concrete slab	50
Formwork, etc.	10
	135 psf

Try 2 by 8 stringers and compute the maximum spacing from $wsL^2\,{}^{12}\!/_{8} = f_w S$.

$$135s \times 9 \times 9 \times 1.5 = 1800 \times 15.23 \qquad s = 1.67 \text{ ft or } 20 \text{ in.}$$

Use 2 by 8 stringers at 20 in. c to c. They could be checked for shear and deflection in the customary way but it is hardly necessary.

If the decking is made of ½-in. plywood, its flexural stress, using $M = wL^2/12$ for full continuity is

$$135 \times 1.67^2 = 12 \times \tfrac{1}{2} \times \tfrac{1}{2} \times \tfrac{1}{6} \times f_w \qquad f_w = 750 \text{ psi}$$

This is within the allowable for plywood, so the 20-in. spacing will be used.

Example 6-4. Design forms for beam B1 of the typical building of Chapter XVII.

Solution. See Fig. 6-9. Use a 2 by 10 plank soffit of No. 2 and better yellow pine S4S. As this will be supported by both shores and nailing to the beam sides, the selection is not so much a matter of computation as experience. Thinner lumber will sag or warp and produce an unsightly beam bottom. It is well to have excess strength and stiffness in the soffit form to maintain true lines and levels. Note that beam form runs through the column form, as provided for in Ex. 6-2. Provide adjustment blocks at each end to facilitate stripping. Replacing these blocks with longer ones on the floors above takes care of the reduction in column size.

For beam sides use 1 by 8 square edge No. 2 and better yellow pine S4S. Batten together with 1 by 4 rough not over 30 in. c to c. Use two 6d common nails at each intersection. Square-edged stock is better than tongue-and-groove since such a board is more easily replaced if it becomes damaged during reuse. Because of shrinking and swelling, tongue-and-groove stock does not remain any tighter than square-edged. For bench work a boiler plate top is common. The builder is cautioned to use short nails in assembling. If long nails are used they will automatically clinch over and, although they make a fine first job, they will increase exceedingly the work of making repairs. Install ¾ by ¾ in. chamfer strips in the two lower corners. Sometimes the reentrant upper corners are chamfered. This requires considerable expensive work and detracts from rather than adds to the appearance.

Supply continuous ledger (see Fig. 6-9) each side to support floor joists. Space tee-head shores about 3 ft c to c. Cut supporting legs between underside of ledger and top of shore. Ledgers then span 3 ft and carry, at worst, one floor joist in the center. Ledgers are continuous, so compute the maximum flexural stress in a 2 by 4 as $f_w = M/S$ where $M = \dfrac{PL}{4} \times 8/12$; here $P = 135 \times 1.67 \times 9/2 = 1015$ lb.

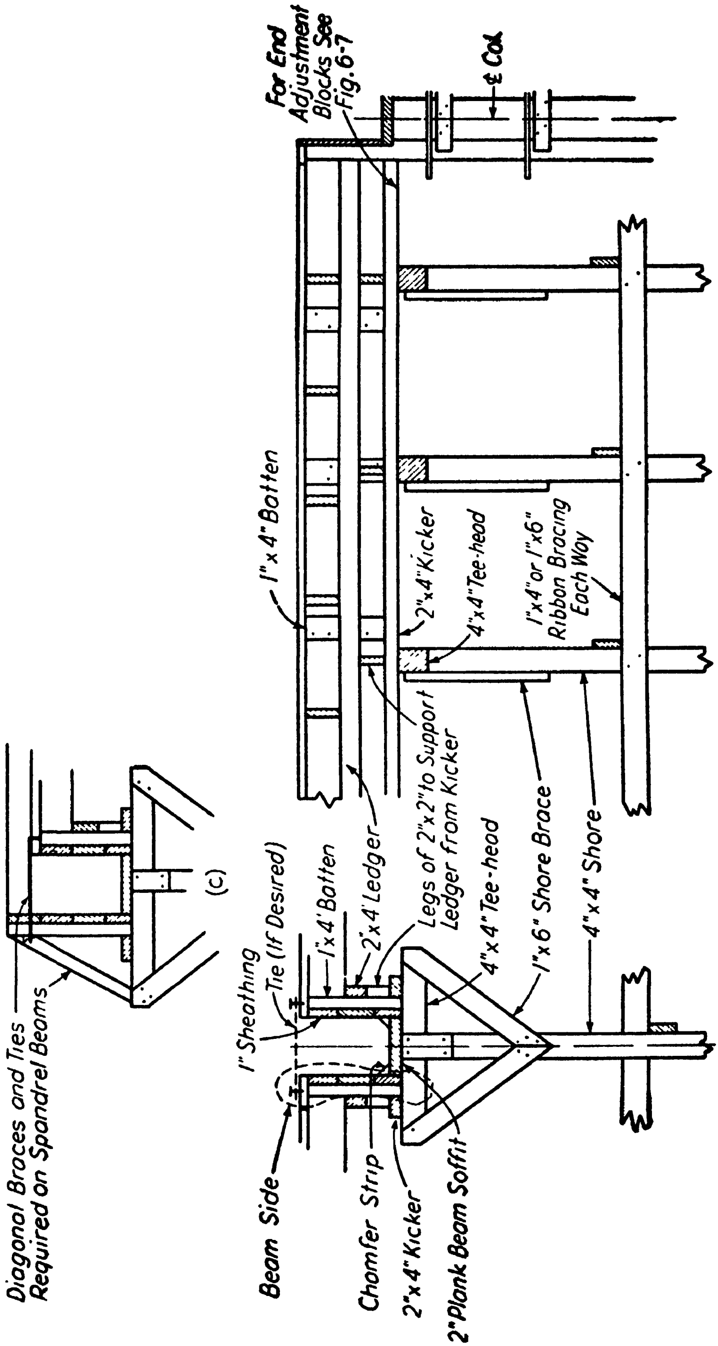
$$f_w = \frac{1015 \times 3 \times 12}{3.56 \times 4} \times \frac{8}{12} = 1700 \text{ psi}$$

This is based on full continuity of the ledger. If there is any doubt about end spans a diagonal 1 by 4 brace can be added.

For legs (see Fig. 6-9) the total load is $3 \times 4.5 \times 135$ or 1820 lb. Spreading this over a 1⅝ by 1⅝ in. leg amounts to about 700 psi, which is well on the safe side.

Furnish kickers (see Fig. 6-9) of either 2 by 4 S4S or 1 by 4 rough to hold beam sides from pushing out. Also fasten sides to bottom with 12d doubleheaded nails to facilitate removing.

For spandrel beams the design will be the same except that the outer side will extend to the top of the slab and will require diagonal bracing down to the tops of the shores to hold it in line. The design of the forms for the girders is similar to that for the beams. The beam forms extend through the girder



For End Adjustment Blocks See Fig. 6-7
℄ Col.
1"x 4" Batten
2"x4" Kicker
4"x4" Tee-head
1"x4" or 1"x6" Ribbon Bracing Each Way
Legs of 2"x2" to Support Ledger from Kicker
1"x6" Shore Brace
4"x4" Shore
(b)
Diagonal Braces and Ties Required on Spandrel Beams
(c)
1" Sheathing
Tie (If Desired)
1'x4' Batten
2"x4' Ledger
4"x4" Tee-head
Beam Side
Chomfer Strip
2"x4" Kicker
2" Plank Beam Soffit
(a)

FIG. 6-9

forms, and cutouts are made at the bench in the side forms of the girders for the beams the same as the cutouts in the tops of the columns.

Strap iron ties may be used across the tops of the beam sides at the level of the bottom of the slab to prevent spreading. With the joists of the floor panels framing in as here, such spreading is unlikely but does sometimes occur.

Compute the load on a shore:

$$\begin{aligned} \text{Slab} &= 10\text{ ft at }135 = 1350\text{ lb per ft} \\ \text{Concrete }10\text{ in.} \times 16\text{ in.} &= 160 \\ \text{Forms, etc.} &= \underline{25} \\ & \ 1535\text{ lb per ft} \times 3\text{ ft shore spacing} = 4600\text{ lb} \end{aligned}$$

This load is about equal to or slightly in excess of that recommended on a single adjustable shore. Manufacturer's data must be consulted for the particular shore selected and the spacing must be varied to suit. This load is only about one-third of the capacity of a solid 4 by 4 shore. For that reason some builders take a chance on 2 by 4 shores. This is too risky, as a misplaced brace would lead to failure. Others obtain special 3 by 3 or 3 by 4 shores which are satisfactory but may not prove more economical.

In erecting provide a sill piece under the shores to distribute the weight over the slab. Also provide ribbon bracing as shown on Fig. 6–9*b* of 1 by 4 rough or 1 by 6 S4S, using one horizontal line each way for each 6 or 7 ft of height. One row should be about 6 ft below the bottom of the slab forms to serve as a support for a platform during stripping. The bottom row of bracing should be 6 ft or more above the floor slab below to permit easy walking through the floor that is occupied by shores.

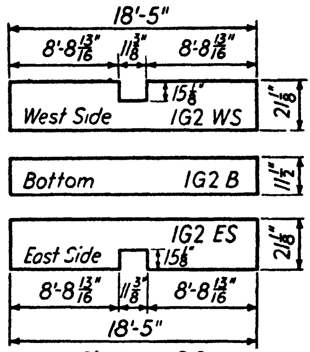
Remarks. The above examples by no means exhaust the possibilities of form design. They are intended only to show the basic principles and methods used in working out forms. Many patented systems of forming are available. Wherever the number of reuses warrants it, metal forms can be constructed. Special forms are available for sewers, culverts, curbs, round columns, ribbed slabs, and flat slabs. Also many types of clamps, ties, and supports are on the market for use with wood or metal forms. The designer should have a fairly good knowledge of form methods so that his selection of members will best suit the forming.

Example 6–5. Prepare shop details of slab, beam, and column forms for the typical building of Chapter XVII, showing the information necessary for shop fabrication.

Solution. For easy identification and erection a marking diagram or key plan is essential. This will be like the design sketch shown in Fig. 17–2, except that on a complicated structure notes, sections, and details will be added for special conditions. Beam sides are best given the mark of the beam on which they are to be used, with a prefix to show on which floor they fit, and a suffix to indicate whether it is the east, west, north, or south side. Beam bottoms are similarly marked except the suffix may be "B" for bottom. Beam cuts and special conditions are economically made on the bench to detail. Column marks are prefixed with a number to indicate the story and suffixed with a letter to indicate the north, south, east, or west face. As these

details are intended only for the shop foreman's use, freehand sketches are the rule. No attempt is made to show the number of boards, battens, etc., as these can all be covered with a series of notes. Schedules will save time for simple framing. Forms should be viewed from the concrete side so the builder will know which is the finished side. The method shown parallels rather closely the recognized methods for shop detailing structural steel members.

Beam Mark	No. of Beams	Form Mark	Description	Width	Thick	Length
B1	5	1B1 NS	North Side	15⅛"	⅞"	18'-6¼"
		1B1 B	Bottom	9⅝"	1⅝"	
		1B1 SS	South Side	15⅛"	⅞"	
B2	8	1B2 NS	North Side	15⅛"	⅞"	19'-0½"
		1B2 B	Bottom	9⅝"	1⅝"	
		1B2 SS	South Side	15⅛"	⅞"	



Notes:
Material: Yellow Pine Graded #2 and Better.
Beam and Column Sides: 1x6 Square Edge S4S.
Beam Bottoms: 2" Plank S4S.
Slab Soffits: ½" Plywood.
Beam Battens: 1"x4" Rough at 2'-6" cc.
Column Battens: 1"x4" Rough at 2'-0" cc.
2 Cleanout Doors per Column
Provide Std Extension and Stripping Piece Each End of Each Beam

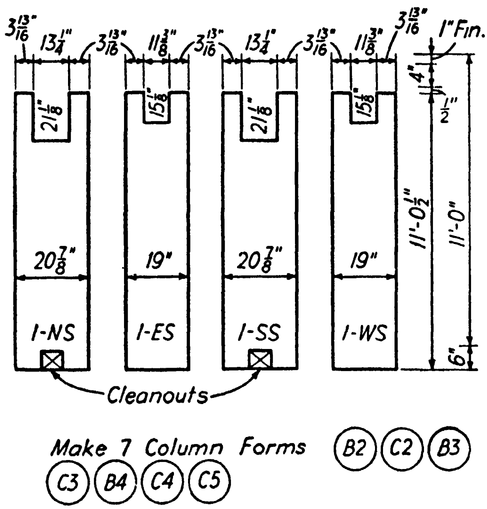


FIG. 6–10

Fig. 6–10 shows the general methods to use. The schedule is very effective for simple beam forms that have no cutouts or special conditions. Forms have been scheduled for first-floor beams " B1 " and " B2 " only. Others would be similar. When girder sides have to be cut out for beams, individual sketches such as those shown for first-floor girder " G2 " are better. For detailing column forms, sketches such as those for basement columns " B2," " C2," etc., are clear. Schedules are sometimes used for girder and column

forms, but unless the carpenters and the detailers have a well-developed understanding of all the conventions employed in scheduling, misunderstandings and costly mistakes may occur.

Problem 6–1. Determine the maximum span of 1 by 6 sheathing S4S to support a reinforced concrete slab 12 in. thick and a construction load of 75 psf which is assumed to include the weight of the forms.

Assume semi-continuity over the stringers, $M = wL^2/10$.

Ans. $L = 2.74$ ft; deflection ¼ in. approximately.

Problem 6–2. Determine size of stringers to support the sheathing of Prob. 6–1 on a span of 6 ft between supports. $M = wL^2/10$.

Ans. 2 by 8 S4S.

Problem 6–3. Determine the total load-carrying capacity of a dressed 4 by 4 of short-leaf yellow pine on an unsupported height of (*a*) 6 ft, (*b*) 8 ft, (*c*) 10 ft.

Ans. (*a*) 11,900 lb, (*b*) 8,200 lb, (*c*) 5250 lb.

Problem 6–4. A retaining wall is to be built 20 ft high, 3 ft thick at the bottom and 1 ft at the top. Assuming an average filling rate of 6 vertical feet per hour and initial set at end of 1½ hours:

(*a*) Determine the maximum span for 1-in. sheathing S4S, $M = wL^2/10$.

Ans. 1.16 ft.

(*b*) Determine the maximum span for 2 by 4 in. vertical studding at 12-in. centers to carry the sheathing of (*a*) using $M = wL^2/10$.

Ans. 1.9 ft.

(*c*) Determine the maximum spacing for pairs of 2 by 6 wales S4S to support the studs of (*b*) at $M = wL^2/10$. Assume uniform loading.

Ans. $L = 3.27$ ft.

(*d*) Select a size of threaded rod wall tie to hold the wales of (*c*), using $f_s = 25{,}000$ psi.

Ans. Area required 0.317 sq. in.; ⅞-in. diameter bolt at root of thread = 0.419 sq. in.

CHAPTER VII

BEAMS

7-1. The ordinary formulas used for steel and timber beams apply only to members of homogeneous material and accordingly are not directly applicable to composite beams of steel and concrete. The special formulas which have been devised for reinforced concrete are numerous and somewhat complicated but simple of solution with the help of the charts and tables in common use. Unfortunately the beginner finds them a serious obstacle as he attempts to get an understanding of the few basic principles commonly assumed in reinforced concrete design and he often falls into the fatal habit of using them blindly. More unfortunately still, he often becomes dependent upon formulas and incapable of solving problems without a list of them at hand. It is to his advantage, therefore, to master the fundamental principles of composite beams before attempting to use, or even to derive, these special formulas with their involved notation, and before attacking problems that bring in the confusing details of actual design. This is easily done since nearly all problems of stress in reinforced concrete members may be solved by the ordinary methods made familiar to the student by his study of structural pieces of homogeneous material, methods and formulas being made applicable by transforming the steel-concrete section into its equivalent in the one material, concrete. Anyone who has mastered the method of the "transformed section" not only understands the derivation and use of the standard reinforced concrete formulas but also is independent of them and their necessary accompaniments, tables and plots, an independence conducive to self-confidence and often very desirable in emergencies. This process is fully illustrated in the succeeding articles. In the examples given it should be noted that all practical considerations, such as choice and spacing of bars, are ignored in order that attention may be centered on the principles involved. Experience has shown that it is important that the student avoid raising such questions during this preliminary study.

Nearly all reinforced concrete design today proceeds on the assumption of elastic action on the part of the concrete as well as of the steel, an assumption which the previous discussion of plastic flow shows to be

incorrect. Besides plastic flow, the increase of deformation in concrete under load, there is another phenomenon which affects beam action, namely, the shrinkage of concrete which accompanies its drying out. The effect of shrinkage on beams stress is to decrease slightly the stress in the steel and to increase rather largely that in the concrete. The effect of plastic flow is the reverse of this and the combined effect is not of great significance. Accordingly the theory of beam action can be formulated and applied on the basis of elastic action of the concrete with accuracy of result sufficient for all practical intent. The designer should always keep in mind, however, that shrinkage and plastic flow are present in uncertain amounts and that any pretense of great precision of computation is simply self-deceit. This does not justify careless design which may easily be dangerous; but, even when compelled to follow code or specification with great exactitude in order to meet competition, the designer should realize that his computed stresses may be considerably in error. Codes and specifications are drawn up to insure safety and, when they are followed, this inaccuracy will not be dangerous.

7-2. Kinds of Reinforced Concrete Members. Every structural member of reinforced concrete belongs to one of three classes: beams,* subject only to bending, caused by loads that act perpendicular to the longitudinal axis, or by applied couples, or by both transverse loading and applied couples; compression members, carrying loads whose lines of action coincide with the longitudinal axis and which cause uniform compression on any section normal to that axis; members subject to both direct compression and bending. Since the concrete of a purely tension member does not assist in carrying the load such a piece cannot logically be said to be one of reinforced concrete.

Concrete members are not commonly used for transmitting torsion. The effect of torsional shear on spandrels is considered later.

7-3. Beams of Homogeneous Material. The relation that exists between the internal fiber stresses of a homogeneous rectangular beam and the external forces may be found in the following manner.† Consider the end-supported beam AB (Figs. 7-1a and b) shown for convenience in a horizontal position, with the known external forces acting perpendicular to the longitudinal axis and in the plane of the vertical axis, thus insuring bending alone, without direct stress or torsion. The

* The strict limitation of the term " beam " adopted in this text is not in accordance with everyday usage which applies the word loosely to any member whose loading is either largely or entirely transverse. The strict usage is desirable for clearness of analysis.

† For a general and rigorous derivation of the common beam theory the reader is referred to the standard texts on strength of materials. The purpose here is to review important principles and emphasize the main points of the argument.

beam as a whole is at rest under the action of the outer forces. Therefore the portion of the beam to the left of the plane section *mn* (Fig. 7-1*c*) is also in equilibrium, the balanced system of forces acting thereon being the external forces applied to that part of the beam and the internal

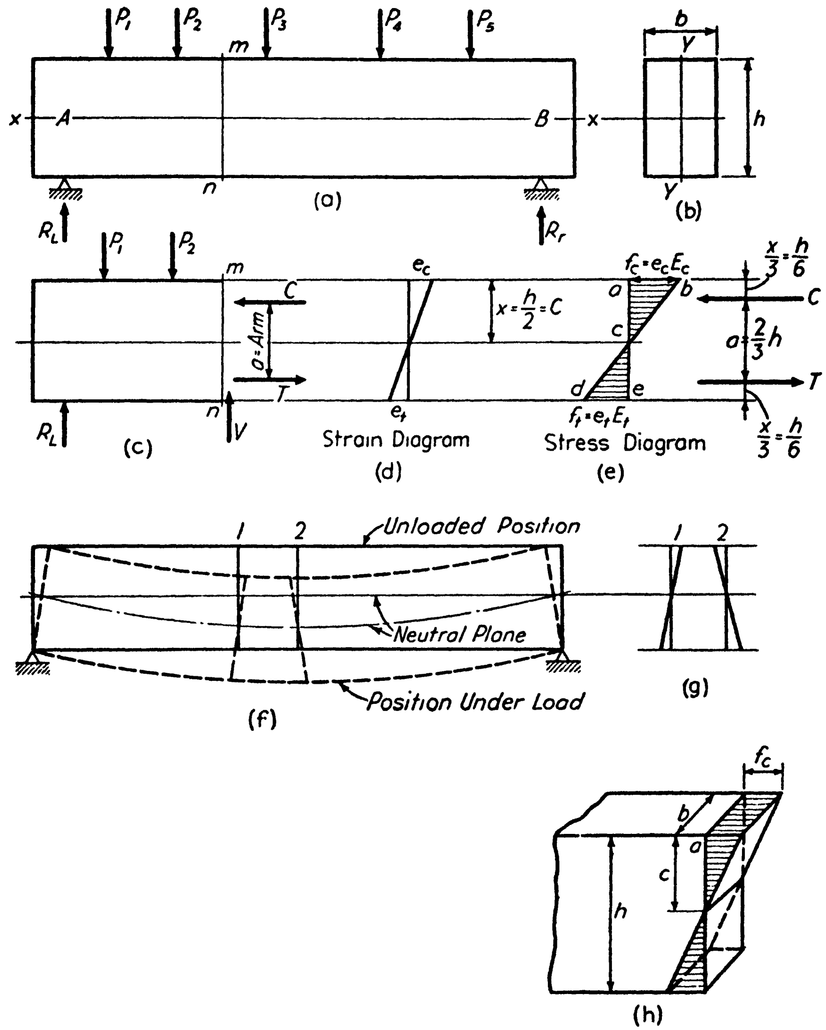


FIG. 7-1

fiber stresses exerted on it by the part of the beam to the right of *mn*. Considering each of these forces to be represented by its resultant, this force system is coplanar, and the conditions of equilibrium for such a system give certain information regarding the unknown fiber stresses.

The condition $\Sigma Y = 0$ shows that the total shearing stress along *mn* (V in Fig. 7–1*c*) equals the resultant of the external forces to the left of the section; that is, the internal shear equals the external shear. The condition $\Sigma M = 0$ (taking moments about any point in the plane *mn*) shows that there must be a counterclockwise moment acting on this part of the beam to counterbalance the clockwise moment of the external forces. Since only the horizontal (or X) components of fiber stress can have moment about a point in plane *mn*, this counterclockwise moment must be that of a compressive and a tensile force, represented by C and T respectively in Fig. 7–1*c*. Since $\Sigma X = 0$ these two forces must be equal ($C = T$) and the moment of the internal fiber stresses is, accordingly, a couple, equal to $C \cdot a$ or $T \cdot a$ where a is the arm of the couple, the distance between the resultant compression and resultant tension. This couple is the *resisting moment* (MR), equal in value and opposite in direction to the external *bending moment* (BM) at the section.

All the information possible having been gained from the principles of statics, recourse next must be had to direct observation of the actual beam under load, the loaded and unloaded positions being represented in Fig. 7–1*f*, with the deflection greatly exaggerated. Careful measurements show that any normal section, such as 1 or 2, is practically a plane section in the loaded as well as in the unloaded beam. Examination of the positions taken by these two planes in the bent beam shows that one horizontal fiber extending from section to section remains unchanged in length and that the fibers above are all shortened and those below lengthened. Therefore the amount of the deformation in any fiber varies directly with its vertical distance from the fiber which remains unchanged in length (Fig. 7–1*g*). This makes possible the drawing of the strain diagram for the fibers at the section *mn* (Fig. 7–1*d*) which will be a continuous straight line crossing the section at an unknown distance (x) from the top.

Experiment has also shown that within the elastic limit of the material there is a fixed relation between strain and stress, that $\dfrac{\text{stress (pounds per square inch)}}{\text{strain (inches per inch)}}$ = a constant named the modulus of elasticity (pounds per square inch) or, in the usual notation, $E = f/e$. It is also true that for timber and steel the modulus of elasticity in compression may be taken as equal to that in tension. It is now possible to draw the stress diagram (Fig. 7–1*e*), each abscissa of the strain curve being multiplied by E to obtain that of the stress diagram, with the result that it also is a straight line (*bcd*). The compressive fiber stress on the section, therefore, is represented by the solid seen in side elevation as *abc* (compare Fig. 7–1*h*) and the tensile force by that projected

as *cde*. Since the total compression equals the total tension, area $abc = dce$. Since the angles at c are equal $ab = de$ and $x = ac = ce = h/2$; or, in words, the maximum unit compressive stress equals the maximum unit tensile stress and the neutral fiber is at mid-depth. This is a very important fact to keep in mind: that the *neutral axis* (which is the trace of the neutral plane with the plane of a right section) *passes through the center of gravity* (*or centroid*) *of the cross section*. The resultant compression and resultant tension evidently act through the centroids of the triangles by which they are respectively represented and the lever arm of the resisting moment couple equals

$$a = \tfrac{2}{3}h$$

The total compression equals the average compressive unit stress multiplied by the area over which the compression acts; thus

$$C = T = \tfrac{1}{2}f \times b \times \tfrac{1}{2}h$$

and the resisting moment equals

$$MR = C \cdot a = T \cdot a = \frac{1}{2}f \times b \times \frac{1}{2}h \times \frac{2h}{3} = \frac{1}{6}fbh^2$$

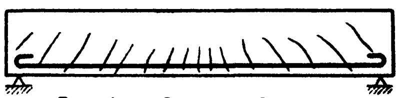
which is the familiar expression derived by substituting the values for a rectangular cross section in the general form of the relation, $M = fI/c$.

The moment of resistance (the couple formed by the internal fiber stresses) developed at any section of a beam equals the bending moment (the moment of the external forces acting on the beam to the right or to the left of the section) at that section, and so the expression

$$BM = MR = \tfrac{1}{6}fbh^2$$

gives a direct relation between the maximum fiber stress in a beam and the external loads, making it possible to proportion and investigate rectangular homogeneous beams as far as normal stress is concerned. The problem of shearing stress will be studied later.

7-4. Beams of Reinforced Concrete. *Working Loads.* A beam of plain concrete breaks under very small load on account of the weakness of the concrete in tension. If reinforced with steel rods as shown in Fig. 7-2 it will carry much more; the concrete will crack at the same load as though unreinforced but failure will be prevented by the steel.



Elevation Showing Cracks that Occur Under Load

FIG. 7-2

These cracks usually appear somewhat as shown in the illustration, inclined more and more toward the end of the beam. Each crack is approximately normal to the lines of maximum tension in its portion of

the beam, and it is this inclined tension that causes the crack. A large increase of load is possible if rods are placed crossing those sections where a failure might occur, as sketched in Fig. 7–18*a*. The problem of designing such web reinforcement is considered in a later section: At present the discussion is limited to the main tension steel.

An expression for the moment of resistance of a rectangular reinforced concrete beam can be derived by following the general argument of the preceding article. It is customary to make these preliminary assumptions: that initial stresses set up by the shrinkage of the concrete are negligible; that there is no slipping between the concrete and the steel; that the stress-strain curve for concrete is a straight line for working loads (pages 36 and 73); and that the concrete carries no tension. The latter is not strictly true, as the concrete plainly will resist tension for a small distance below the neutral plane, where the elongation is not so large as to cause cracking, but it is safer and simpler to make the assumption.

Consider the portion of a rectangular reinforced concrete beam to the left of any section *mn* such as that shown for a homogeneous beam in Fig. 7–1*c*. The resisting moment is considered to be supplied by the compression in the concrete and the tension in the steel since the tension

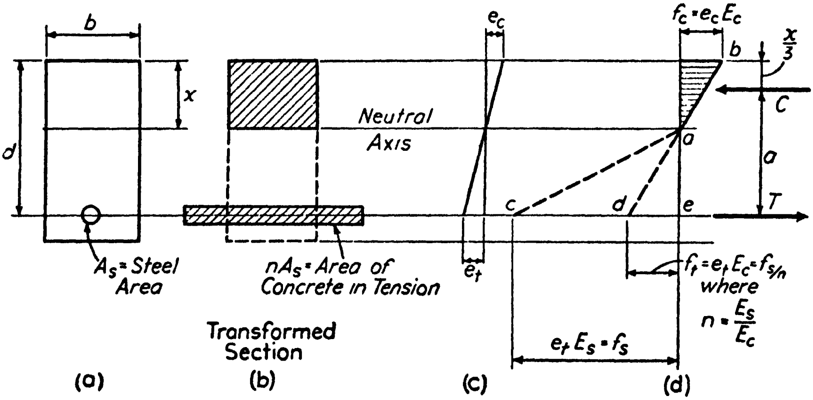


FIG. 7–3

in the concrete is neglected. As a plane section before bending remains plane after bending, the strain diagram is a straight line as before (Fig. 7–3*c*), the neutral fiber being an unknown distance x from the compressive face of the beam. To obtain the stress diagram (Fig. 7–3*d*) each abscissa of the strain curve above the neutral axis is multiplied by E_c, the modulus of elasticity for the concrete (assumed to be constant), giving the line *ab;* the strain at the level of the steel, e_t, is multiplied

by E_s, the modulus for steel, giving the abscissa $ec = f_s$, the thickness of the layer of steel being neglected. As with the homogeneous beam, the total compression may be indicated as $C = \frac{1}{2} \cdot f_c bx$, acting at a distance $x/3$ from the compression face. The total tension is $f_s A_s$ where f_s is the unit steel stress and A_s the steel area, and the arm of the couple is $a = d - x/3$. To evaluate x, the unknown distance of the neutral axis from the compressive face of the beam, the steel-concrete section is transformed into one entirely of concrete, as shown in Fig. 7–3*b* by replacing the steel with concrete, or rather with a hypothetical concrete having the same modulus of elasticity as the concrete in the compression portion but differing from it in its assumed ability to carry tension. The area, nA_s, of this tension concrete is determined by the following considerations. This transformed beam is the elastic equivalent of the steel-concrete beam and therefore the strain curve is unchanged by the transformation. The stress curve is changed only below the neutral plane, since the modulus of elasticity of the assumed tension-carrying concrete differs from that of steel and equals that of the compression part. Therefore the absicissa ec is replaced by $ed = E_c e_t$. The points a, b, d all lie on one straight line. The total tension equals $f_t n A_s$, where nA_s is the unknown amount of concrete used in substitution for the steel. The value of n is determined by making use of the fact that the total tension is the same for the steel-concrete beam and for the transformed homogeneous beam; that is, $f_s A_s = f_t n A_s$ or, putting stress in terms of strain, $E_s e_t A_s = E_c e_t n A_s$; whence $n = E_s/E_c$. Or, more briefly, since the deformation in the steel equals that in the tension concrete, the unit stresses in the concrete and the steel bear to each other the ratio of their moduli; hence the tension areas must be to each other inversely as their moduli, since the total tension is the same in both cases.

Since the neutral axis of a homogeneous beam passes through the center of gravity of the cross section, that of any given reinforced concrete beam, the steel area of which is known, may be located by determining the centroid of the transformed section (the shaded portions of Fig. 7–3*b*). Since the concrete of the original beam carries no tension its outline is shown by dotted lines below the neutral axis, where it serves simply to transmit shearing stresses. If the value of x is known the rest of the problem follows directly.

Although no formulas have been written, all the means are now at hand for the design and investigation of reinforced concrete beams of any shape provided the section is symmetrical about the plane of bending. It is much easier to solve problems by the simple methods that follow logically and simply from the outline just given than it is to use formulas, unless plots and tables are at hand to use with the formulas.

7–5. Rectangular Beams with Tension Reinforcement. Three problems arise regarding rectangular concrete beams reinforced in tension only: (1) the design of a beam to carry a stated bending moment at given stresses; (2) investigation of the maximum stresses at a given section subjected to a known bending moment; (3) investigation of the maximum permissible bending moment for a given beam with certain limiting stresses given. It is better to consider the cases of investigation first. The problem of design is by far the most common in practice.*

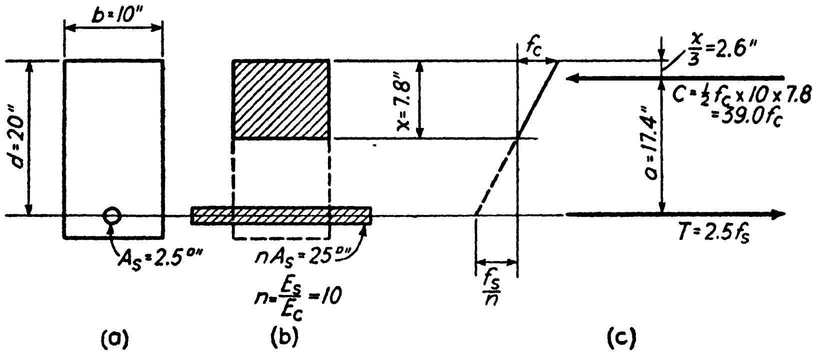


Fig. 7–4

Example 7–1. The beam here shown (Fig. 7–4*a*) carries a bending moment of 55,000 lb-ft; E_s = 30,000,000 psi; E_c = 3,000,000 psi. What are the maximum fiber stresses?

Solution. Following the argument of the preceding article the transformed section is that here given (Fig. 7–4*b*). The neutral axis is located by noting that the statical moment of the compression area about that axis (which passes through the center of gravity) equals the statical moment of the tension concrete area about the same axis, giving the expression

$$\frac{x}{2}(10x) = 25(20 - x)$$

$$x^2 + 5x = 100$$

Completing the square†

$$x^2 + 5x + 6.25 = 106.25$$

$$x = 10.31 - 2.50 = 7.8 \text{ in.}$$

Consequently the lever arm of the couple is 17.4 and as $MR = BM$

$$C = T = \frac{55{,}000 \times 12}{17.4} = 37{,}900 \text{ lb}$$

* Before checking the following problems read Art. 17–5, "Precision."

† This method of solving a quadratic equation is logical and easily remembered; solving by use of a memorized formula is a dangerous and foolish strain on that useful mental function, the memory.

The maximum fiber stresses then are obtained as follows:

$$C = \tfrac{1}{2}f_c \times 10 \times 7.8 = 37{,}900 \text{ lb} \qquad f_c = 970 \text{ psi}$$

$$T = f_s A_s = 2.5 f_s = 37{,}900 \text{ lb} \qquad f_s = 15{,}200 \text{ psi}$$

Example 7–2. What is the maximum moment that can be carried by the beam of Ex. 7–1, the limiting fiber stresses being $f_s = 20{,}000$ psi and $f_c = 1350$* psi? $n = E_s/E_c = 10$†.

Solution. As before, transform the section (Fig. 7–4b), locate the neutral axis, and determine the arm of the resisting couple, $a = 17.4$ in. If f_c has its maximum value of 1350 psi, C equals $\frac{1}{2} \times 1350 \times 10 \times 7.8 = 52{,}600$ lb; if f_s has its maximum value, T equals $20{,}000 \times 2.5 = 50{,}000$ lb. If the former value is attained T also equals 52,600 lb and f_s exceeds the limit of 20,000 psi. It is necessary to limit C and T to 50,000 lb each, thus using the concrete at a lower stress than the permissible. The limiting moment accordingly is

$$50{,}000 \times 17.4 \times \tfrac{1}{12} = 72{,}500 \text{ lb-ft}$$

Another, but slightly longer, method of carrying through this problem is to complete the stress diagram by assuming either unit stress as realized, and determining by proportion the simultaneous value of the other. For example, assuming f_s at 20,000 psi gives $20{,}000/10 = 2000$ as the tension stress in the transformed section, the neutral axis of which has been located. Then‡

$$f_c = 2000 \times \frac{7.8}{12.2} = 1280 \text{ psi}$$

Since this is less than the allowable 1350 the beam is limited by the strength of the steel. It is not necessary to determine the stress in the concrete since this is less than the allowable.

Example 7–3. Design a beam to carry a total moment of 55,000 lb-ft with stresses of $f_s = 20{,}000$ psi, $f_c = 1350$ psi, $n = E_s/E_c = 10$.

Solution. " Balanced reinforcement " results when both concrete and steel are stressed to their safe working capacity. The problem is to determine the breadth b, the depth to the steel d, and the steel area A_s, such that the given stresses will be realized simultaneously when the moment equals 55,000 lb-ft. The section and stress diagrams then appear as shown in Figs. 7–5, a and b.

* Art. 878, Table 7, of the J.C. Report (1940) recommends $f_c = 0.45f'_c$. For $f_c = 1350$ the concrete must test 3000 psi in standard 6 by 12 in. cylinders at 28 days. The designer should be responsible for seeing that at least this ultimate strength is obtained. The student is advised to obtain copies of the 1940 J.C. and 1941 A.C.I. reports and any subsequent revisions, and to have them available for use throughout the reading of this text.

† The standard notation uses the letter n to designate the ratio of the moduli and it will be employed for that purpose henceforth.

‡ Many students have difficulty in writing simple proportions dealing with similar triangles. They will fare better if they observe the rule of taking the unknown as the first term and the corresponding quantity in the *other* triangle as the second. This permits the equation above to be set down at once, whereas any other order is usually more difficult.

In order that these stresses be realized it is necessary that

$$x = d\left(\frac{1350}{2000 + 1350}\right) = 0.403d^*$$

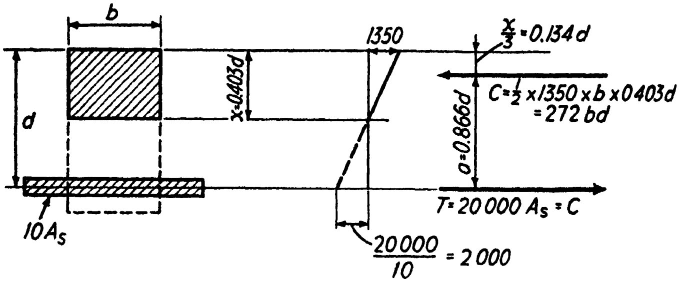


FIG. 7-5

There results $a = 0.866d$ and the moment of resistance in terms of the concrete stress is

$$MR = \tfrac{1}{2} \times 1350 \times b \times 0.403d \times 0.866d = 236bd^2$$

which must equal the bending moment. Therefore

$$bd^2 = \frac{55{,}000 \times 12}{236} = 2800$$

which condition is satisfied by $b = 10$ in., and $d = 16.8$ in.†

To obtain the steel area

$$C = \tfrac{1}{2} \times 1350 \times 10 \times 0.403 \times 16.8 = 45{,}700 \text{ lb} = T = 20{,}000A_s$$

$$A_s = 2.29 \text{ sq in.}$$

In practice the depth would commonly be made an integral number of inches.

The next two problems are given, therefore, to illustrate the principles involved in this change from the theoretical dimensions.

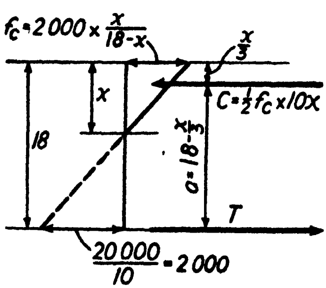


FIG. 7-6

Example 7-3a. What is the steel area required for the beam of Ex. 7-3 if d is made 18 in. and $b = 10$ in.?

Solution. Since the beam is deeper than required, the compression area furnished is in excess of that needed. The maximum concrete stress can be made lower than the limit and that of the steel equal to the limiting value by the use of the proper steel area as determined below. The stress diagram then is shown in Fig. 7-6. To determine x proceed as before; write an

* The best usage requires that the decimal point be preceded by a cipher. See Proceedings, American Concrete Institute 1923, p. 289, Art. 7d. There is less chance of error thus.

† In small rectangular beams b is commonly made ½ to ¾ of d; ¼ to ⅛ in large beams.

expression for MR thus:

$$C \cdot a = \left[\tfrac{1}{2}(2000)\left(\frac{x}{18 - x}\right)(10x)\right]\left(18 - \frac{x}{3}\right) = 55{,}000 \times 12$$

$$x = 6.84 \text{ in.}$$

whence

$$a = 15.7 \text{ in.}$$

and

$$f_c = 1230 \text{ psi}$$

Then

$$C = T = \tfrac{1}{2} \times 1230 \times 10 \times 6.84 = 42{,}000 \text{ lb}$$

and the steel area required

$$A_s = 42{,}000 \div 20{,}000 = 2.10 \text{ sq in.}$$

Approximate Solution. The above exact method is laborious and, in general, unnecessary. Assuming that the lever arms of the couples vary as the depths d, the steel area required for $d = 18$ in. is

$$A_s = 2.29 \times \frac{16.8}{18} = 2.14 \text{ sq in.}$$

Example 7-3b. What is the steel area required for the beam of Ex. 7-3 if d is made 16 in. and $b = 10$ in.?

Solution. Since the beam is smaller than the theoretical beam, if enough steel is used to make the ratio of fiber stresses the same as before, the given stresses, 20,000 − 1350, will be exceeded when the required moment of

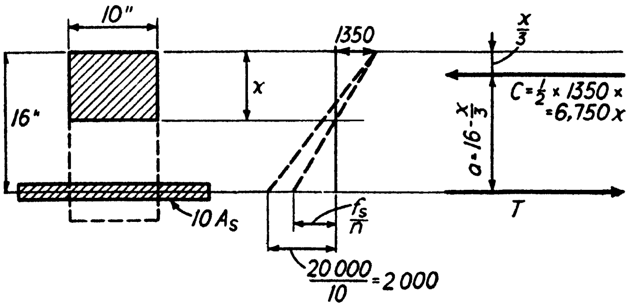


FIG. 7-7

resistance is realized. The lever arm of the resisting couple is smaller than previously and, in order to secure the necessary increase in the compressive force of the resisting couple, the compression area must be made larger, the extreme fiber stress being limited. The lowering of the neutral axis to give this increased area is accomplished by making the steel area larger, for this lowers the center of gravity of the section. The steel stress will be less than the limiting value, as is indicated by the stress curve of Fig. 7-7. The value of

x is arrived at as before:

$$C = \tfrac{1}{2} \times 1350 \times 10x = 6750x$$

$$Ca = (6750x)\left(16 - \frac{x}{3}\right) = 55{,}000 \times 12$$

$$x = 7.20 \text{ in. and } a = 13.6 \text{ in.}$$

$$C = T = 55{,}000 \times 12 \div 13.6 = 48{,}500 \text{ lb}$$

$$f_s = 10\left(1350 \times \frac{8.8}{7.2}\right) = 16{,}500 \text{ psi}$$

$$A_s = 48{,}500 \div 16{,}500 = 2.94 \text{ sq in.}$$

Less labor is needed to determine the steel area required by means of the known position of the neutral axis. Since this is the center of gravity of the cross section the moment of the compression *area* about this unknown axis equals that of the tension *area.*

$$10 \times 7.20 \times 3.60 = 8.8 \times 10A_s$$

$$A_s = 2.94 \text{ sq in.}$$

When the depth is smaller than the theoretical depth, there is too much error in the assumption of proportionality between depth d and lever arm a to permit the approximate solution previously used. For example, this assumption would give

$$A_s = 2.29 \times \frac{16.8}{16.0} = 2.40 \text{ sq in.}$$

where 2.94 sq in. are actually required.

The several examples of this article have demonstrated these important facts:

1. *In any given beam* (b, d, A_s, n = E_s/E_c *known*) *the neutral axis has a fixed position* (*through the centroid of the transformed section*) *and therefore the ratio of the maximum unit stresses in steel and concrete is constant.* It is evident this ratio is independent of any stresses that may be set as limits in future investigations of the beam.

2. *An increase of tension steel area lowers the neutral* (*gravity*) *axis.*

3. *A lower value of* n = E_s/E_c, *such as results from the increase of the concrete modulus with age, raises the neutral axis,* since the tension area of the transformed section is thereby decreased.

7-6. The Ratio E_s/E_c in Beams. It has been stated (Art. 5-7) that the stress-strain curve for concrete in compression is approximately a parabola, as shown in Fig. 7-8, with plastic flow increasing the strain readings even during the relatively rapid loading of the testing laboratory. It is customary to neglect the curvature of the stress-strain curve within the working range and to take the modulus of elasticity of the

concrete as the slope of the secant Ob (Fig. 7–8), instead of the slope of the initial tangent, Oa. If the piece is repeatedly loaded within the range Ob, there will be permanent set Oc, and the stress-strain curve will tend to straighten out as indicated by the dotted line cb. However, the major use of the modulus is to measure the relative stresses of adjacent concrete and steel which are equally deformed; this necessitates taking account of the total deformation, and so the secant modulus is the proper one to choose.

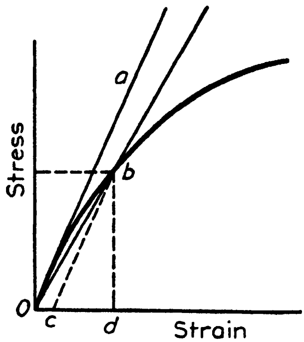
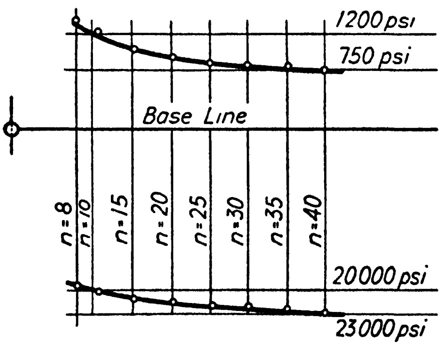


FIG. 7–8

In addition to the uncertainty introduced by plastic flow another element affecting the value of the modulus, and so the modular ratio n, is the increase of the stiffness of concrete with age. Evidently the value of the ratio n used in computation is none too certain a figure and the approximation adopted by the A.C.I., $30{,}000/f_c'$, is justified. The effect of changes in n on the computed magnitudes of stress in a given beam with known loading is shown in Fig. 7–9.



Variations in theoretical stresses due solely to changing assumed values of $n = E_s/E_c$ *in rectangular beam of known size, reinforcement, and loading.*

FIG. 7–9

It should be kept in mind that, since the modulus of elasticity for the concrete has been assumed constant, it is not possible to compute by the methods outlined here the breaking strength or the effect of loads that cause high concrete stresses. For such problems it becomes necessary to consider the actual shape of the stress-strain curve for concrete, as considered in Art. 7–7.

7–7. Flexure at Ultimate Load. The straight line variation of stress in a reinforced concrete beam is merely a useful design assumption. Although this is reasonably accurate at working loads, the fact that the stress-strain relationship of concrete is curved rather than straight (page 36) makes this assumption untenable at ultimate capacities. Fig. 7–10 shows various stages in the loading of a simple rectangular beam built of 3000 psi concrete. In the first stage the concrete is stressed

1350 psi (the normal working value), and the stress variation is very nearly triangular, with the vertex of the stress-strain curve far above the extreme fiber. In the second stage the load has been increased and the stress is in an intermediate position. In the third stage the load has been carried up to the ultimate capacity and the vertex of the stress-strain curve now lies in the extreme fiber of the beam. The assumption of a uniformly varying stress-strain relation is clearly too far from the facts to be of any value.

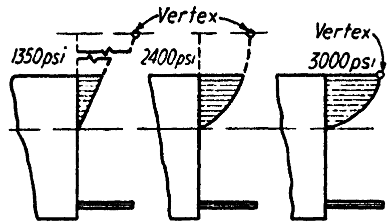


FIG. 7-10

The determination of concrete and steel stresses at ultimate capacity are important for two reasons: (1) for the interpretation of tests to failure; (2) for the designing of structures to have a given ultimate capacity with a definitely known factor of safety.

Several theories have been proposed for the determination of flexural stresses at failure, at least four of which are of sufficient importance to deserve comment. These will be identified by the shape of the stress-strain curve in compression.

1. *Simple parabola* with vertex in extreme fiber.

2. *Cubic parabola*,* also with vertex in the extreme fiber. The study of tests indicated to its author that the ultimate compressive fiber stress should be taken as 85 per cent of the cylinder strength, and that the neutral axis can never lie lower than $0.80d$, regardless of the amount of tension steel; this leads to the conclusion that k has the value 0.80 at failure.

3. *Fifth power parabola*† with vertex at the extreme fiber and with $k = 0.50$ at failure. At the moment of failure the resisting moment developed is closely

$$M = \tfrac{1}{3}f'_c bd^2$$

and the unit stress in the steel is

$$f_s = \frac{M_{\max}}{\frac{11}{14}dA_s}$$

From these relationships we may determine the steel ratio which must be supplied if the yield point in the steel and the ultimate compression of the concrete are reached simultaneously; $p = 0.42\dfrac{f'_c}{f_y}$, about 3 per cent for ordinary, good quality materials.

* Proposed by L. J. Mensch, *Journal*, A.C.I., Vol. 33, 1937, pp. 498-9 *ff*.

† Proposed by Inge Lyse, *Der Beiwert* n *in Eisenbetonbau*, Beton und Eisen, Heft. 7, April 5, 1937.

4. *Rectangle.** Those advancing this theory fix the unit steel stress at failure at the yield point and the unit compressive stress either at the ultimate (f'_c; Seunson) or at 85 per cent of the cylinder strength (Whitney). This simplifies the problem by stating that any stress-strain curve can be approximated by a rectangle, the two requirements being that the size is such that the mean value of the compressive stress will be that required, and the location of the neutral axis will be such as to produce a resisting couple of the correct magnitude. From a study of all available test data Whitney fixes the value of k as 0.537.

Probably the simple parabolic variation has had the most consideration. The parabola fits the stress-strain curve reasonably well; tests have shown that the assumption that a section plane before bending remains plane after bending is close to the facts; the other relations of this method are directly derived from these. The other three methods are partly empirical, as it was necessary in each case to make assumptions from the results of tests to determine the location of the neutral axis.

The table on page 76, summarizes all four of these suggestions for comparison. As this subject is of more importance to the testing laboratory than to the designing engineer, no further explanation of these theories will be presented here.

Example 7–4. By each of the four methods above, determine the ultimate resisting moment of a 12 by 20 in. rectangular concrete beam reinforced with four 1 in. square bars of hard steel having a yield point in excess of 85,000 psi and with ends anchored by standard hooks. Take d = 18 in. and f'_c = 3000 psi, n = 10. Compare with the value that would have been obtained with a straight line stress variation.

Solution.
$$p = \frac{4 \times 1.00}{12 \times 18} = 0.0185$$

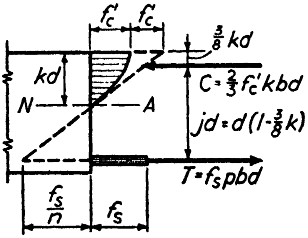
f_s		k	j	*Computation of M*	M (*lb-in.*)
56,000	parabolic	0.517	0.806	$\frac{2}{3} \times 3000 \times 0.806 \times 0.517 \times 12 \times 18^2$	3,240,000
82,500	cubic	0.80	0.68	$0.346 \times 3000 \times 12 \times 18^2$	4,040,000
67,500	quintic	0.50	0.786	$0.327 \times 3000 \times 12 \times 18^2$	3,810,000
74,000	rectangular	0.537	0.731	$0.333 \times 3000 \times 12 \times 18^2$	3,880,000
36,500	straight line	0.452	0.849	$\frac{1}{2} \times 3000 \times 0.452 \times 12 \times 18^2$	2,640,000

Lines 2, 3, and 4 approximate the most probable value. Lines 1 and 5 are too low.

* This was first proposed by E. Seunson, *Ingeniøren*, 1912, p. 508. A variation of this with study of test results is suggested by C. S. Whitney, *Journal*, A.C.I., Vol. 33, 1937, pp. 483 *ff*., and elaborated in detail in "Proc., A.S.C.E.," Dec., 1940.

METHODS OF DETERMINING ULTIMATE CAPACITY OF RECTANGULAR CONCRETE BEAMS

Simple Parabolic



$$\frac{f_s}{2nf'_c} = \frac{1-k}{k} \quad \textit{(Navier's Hypothesis)}$$

$$T = C;\ f_s pbd = \frac{2}{3} f'_c bkd$$

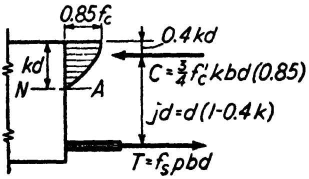
$$k = \sqrt{3pn + \left(\frac{3}{2}pn\right)^2} - \frac{3}{2}pn$$

$$M_c = Cjd = \frac{2}{3} f'_c jkbd^2$$

$$M_s = Tjd = f_s pjbd^2$$

$$p = \frac{\frac{2}{3}}{\frac{f_s}{f'_c}\left(\frac{f_s}{2nf'_c}+1\right)}$$

Cubic



$$T = C;\ f_s pbd = \frac{3}{4}\times 0.85 f'_c bkd$$

$$\therefore k = 1.57\frac{pf_s}{f'_c}$$

$$M_s = f_s pbdjd = pf_s bd^2(1-0.4k)$$

$$= pf_s bd^2\left(1-0.6275\frac{pf_s}{f'_c}\right)$$

Taking $k = 0.80$:

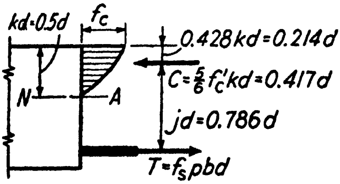
$$0.80 = 1.57\frac{pf_s}{f'_c}$$

$$p = 0.51\frac{f'_c}{f_s}$$

$$M = bd^2(0.51 f'_c)(1-0.6275\times 0.51)$$

$$= 0.346\, bd^2 f'_c$$

Quintic

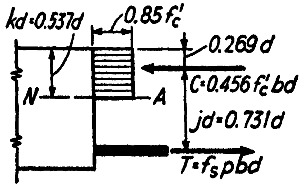


Assuming N. A. at mid-depth:

$$M = 0.417 f'_c bd\,(0.786\,d)$$

$$= 0.327 f'_c bd^2$$

Rectangular



Assuming $k = 0.537$

$$M = 0.333 f'_c bd^2$$

7-8. Tee-Beams. In practice tee-beams usually form part of a floor system and act integrally with the slab on either side, which forms a flange giving added strength in the compressive part. When beams are widely spaced the compressive stresses are not uniform in intensity across the whole width of slab (page 275). In order to investigate or design the usual tee-beam it is necessary to make some assumption regarding the width of slab that may reasonably be considered to act with the stem and be uniformly stressed over the whole width. If this width is assumed too large, the total compression will not only be given an exaggerated value provided enough steel is used to develop it, but also excessive shearing stresses will be induced at the junctions of the flange and stem. Formerly there was more or less disagreement among design specifications regarding the proper limit for flange width but the Joint Committee rule of 16 times the flange thickness plus the stem width is now commonly accepted.

There are two approaches to the problem of designing a tee-beam with the flange provided by a floor slab. One method considers the full width of flange available to be in action. Usually the compressive stress in the concrete is found by this assumption to be low in value. The other method assumes that the limiting working stresses are realized and that the width of slab called in to play is only that necessary. Usually this width is less than the limit set by the codes. The neutral axis and the lever arm of the resisting couple have different values by these two methods. Of course, the actual values are uncertain and neither assumption is more than a convenience which gives satisfactory results.

The design of tee-shaped beams consists in proportioning the stem or web (the portion below the slab) and determining the tension steel area. Since proportioning the stem requires consideration of the shearing stresses, that part of the problem is deferred until later. The following examples illustrate the determination of the required steel area in a given tee-beam and the investigation of a given beam. These examples are carried through with greater precision than is necessary in order to illustrate clearly the method of analysis.

The design of an independent tee-beam not connected with a floor slab is considered in Art. 7-10 and the design of tee-beam stems in Chapter XIII.

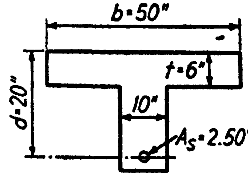


FIG. 7-11

Example 7-5. Locate the neutral axis of the tee-beam (Fig. 7-11) $E_s/E_c = n = 10$.

Solution. The thickness of the flange leads one to suspect that the neutral axis falls within it, and accordingly the investigation proceeds as for a rectangular beam. Taking moments of the areas

of the *transformed* section about the neutral axis:

$$\tfrac{1}{2}x(50x) = (10 \times 2.5)(20 - x)$$

whence

$$x = 4.0 \text{ in.}$$

It is plain that this is simply a rectangular beam with a portion of the concrete below the neutral axis removed, a change that does not affect the moment of resistance. The methods of Art. 7–5 suffice for this beam. Had the axis fallen in the stem the procedure would have been that of the succeeding examples.

Example 7–6. With a total moment of 55,000 lb-ft, what are the maximum fiber stresses in the tee-beam of Fig. 7–11, the flange thickness being made 3 in. instead of 6 in.? $n = 10$.

Solution. Exact Method. (Considering the compression in the stem.) Sketch the transformed section (Fig. 7–12) and locate the center of gravity in

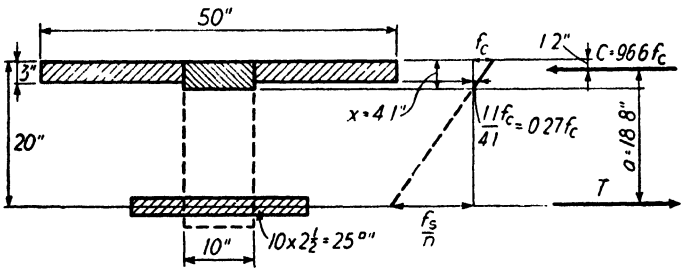


FIG. 7–12

order to find the neutral axis; divide the compression area into the two parts indicated and take moments about the unknown neutral axis:

$$(10x)(\tfrac{1}{2}x) + (50 - 10)(3)(x - 1\tfrac{1}{2}) = 25(20 - x)$$

$$x = 4.1 \text{ in.}$$

The line of action of the resultant compression may be located by taking moments about the top fiber, the total stress being equal to the compression that would exist if the slab had a thickness of 4.1 in., less the stress on the area below the actual slab.

Compression	*C*	*Arm*	*Moment*
$(\tfrac{1}{2}f_c)(50 \times 4.1) =$	$102.5f_c$	$\frac{4.1}{3}$ in.	$140.1f_c$
$(\tfrac{1}{2} \times 0.27f_c)(40 \times 1.1) =$	$5.9f_c$	3.4 in.	$20.0f_c$
C = total compression =	$96.6f_c)$		$120.1f_c$
Distance of C from top fiber:			1.2 in.

The lever arm a equals $20 - 1.2 = 18.8$ in.

$$C = T = 55{,}000 \times 12 \div 18.8 = 35{,}100 \text{ lb}$$

The maximum unit stresses are

$$f_c = 35{,}100 \div 96.6 = 360 \text{ psi}^*$$

$$f_s = \frac{35{,}100}{2.5} = 14{,}000 \text{ psi}$$

Solution. Approximate Method. (Neglecting compression in the stem.) (Fig. 7-13.) By far the greater number of beams met in practice are of such proportions that the amount of compression in the stem below the flange is

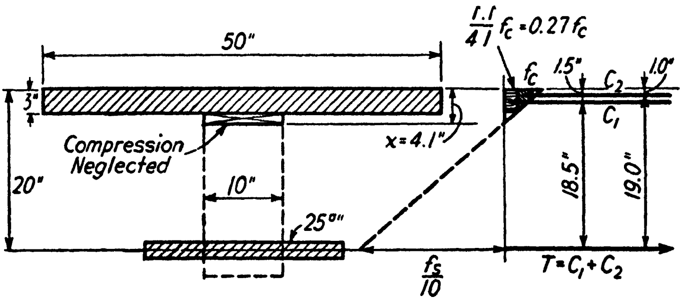


FIG. 7-13

small compared with that in the flange and can be neglected with small error. On this basis the following expression gives the location of the neutral axis, taking moments about that unknown axis.

$$(50 \times 3)(x - 1.5) = 25(20 - x)$$
$$x = 4.1 \text{ in.}\dagger$$

When the stress diagram is drawn, the intensity of the compression on the under edge of the slab is seen to be ${}^{1.1}\!/_{4.1} \cdot f_c = 0.27f_c$. The total compression is the sum of two forces: C_1, acting with a uniform intensity of $0.27f_c$ over the flange, and C_2, acting with an intensity varying from 0 to $0.73f_c$. Force C_1 acts at the center of the flange, 18.5 in. from the tension steel, and force C_2 acts at a distance of $(20 - {}^{3.0}\!/_{3}) = 19$ in. from the steel. Instead of computing the lever arm of the resultant as in the previous example, some prefer to consider the moment of resistance as consisting of two couples, $18.5C_1$ and $19.0C_2$. The total moment of resistance in terms of the unknown f_c then is:

Compression	*C*	*Arm*	*Moment*
$C_1 = (0.27f_c)(50 \times 3)$	$= 40.5f_c$	18.5	$749f_c$
$C_2 = (\frac{1}{2} \times 0.73f_c)(50 \times 3)$	$= 54.8f_c$	19.0	$1041f_c$
$C_1 + C_2 = C$	$= 95.3f_c$	$MR =$	$1790f_c$

Then $1790f_c = 55{,}000 \times 12$ and $f_c = 370$ (actually 369) psi.

$$C = 95.3 \times 370 = 35{,}300 \text{ lb} \qquad f_s = 14{,}100 \text{ psi}$$

* More precisely 363 psi. The third figure has no precision value.

† More quickly, $x = 1.5 + {}^{25}\!/_{175} \times 18.5 = 4.15$ in., since x = distance from extreme fiber to center of gravity of concrete plus moment of transformed steel area about this center of gravity divided by the *total* area of section.

It is equally simple to determine the lever arm of the resultant compression, $1790f_c \div 95.3f_c = 18.8$* in., and proceed as illustrated above in the exact solution.

Example 7–7. What is the maximum moment that can be carried by the beam of Ex. 7–6? (Fig. 7–11 with $t = 3$ in.) Limiting fiber stresses are $f_s = 20{,}000$ psi, $f_c = 1350$ psi. $n = 10$.

Solution. The exact method is so seldom used further illustration of it will not be made. As in the preceding problem, determine the location of the neutral axis and the lever arm of the resisting couple. The limiting value of C is $95.3 \times 1350 = 129{,}000$ lb; that of T is $20{,}000 \times 2.5 = 50{,}000$ lb, which limits. Accordingly the maximum moment of resistance is $50{,}000 \times 18.8 \div 12 = 78{,}300$ lb-ft.

Example 7–8. A beam of the same dimensions as that of Ex. 7–6 (Fig. 7–11 with $t = 3$ in.) carries a moment of 130,000 lb-ft. The limiting fiber stresses are $f_s = 20{,}000$ psi, $f_c = 1350$ psi. $n = 10$. What is the steel area required?

Solution. Any other than a cut-and-try solution would be exceedingly cumbersome. It is simple to assume the value of the lever arm of the couple and determine the area of steel required for a fiber stress of 20,000 psi. The results may be checked by either the exact or the approximate method as desired and the steel area corrected if necessary.

In a beam where the compression in the stem is so small that it may be neglected, the lower the neutral axis lies, the nearer the line of action of the resultant compression approaches the center of the flange. If the compression is assumed to act at this point the corresponding lever arm will be smaller than can ever be realized actually, and the values of C and T larger than the actual. Making this conservative and very common assumption (*i.e.*, that the lever arm equals the depth to the steel less half the flange thickness),

$$T = C = 130{,}000 \times 12 \div 18.5 = 84{,}300 \text{ lb}$$

$$\text{Steel area } A_s = 84{,}300 \div 20{,}000 = 4.22 \text{ sq in.}$$

The approximate average stress in the concrete is $84{,}300 \div (50 \times 3) = 560$ psi. A brief mental calculation shows that the neutral axis is below the flange† and so the maximum concrete stress is less than twice this approximate

* Once the neutral axis is located the lever arm of the total compression may be quickly determined from the stress diagram as follows. Let a 45° line represent the stress variation: then the f_c distance may be taken here as 4.15 units and the stress at the bottom of the slab as 1.15 units. (Query: May this not be assumed for any slope of the stress line?) The centroid of the trapezoidal prism representing the total compression is located a distance below the top of the beam (see page 153)·

$$d - a = \frac{3}{3}\left(\frac{4.15 + 2.30}{4.15 + 1.15}\right) = 1.22 \text{ in.}$$

Then compute $a = 20 - 1.2 = 18.8$ in. and $C = M/a = (55{,}000 \times 12)/18.8 = 35{,}100$ lb. The mean intensity of compressive stress at mid-depth of slab is (C/A_c), 234 psi, which, in terms of the maximum stress, equals $f_c\,(4.15 + 1.15)/(2 \times 4.15) = 0.638f_c$. Accordingly $f_c = 367$ psi.

† Taking moments about the bottom of the flange for a speedy estimate, assuming the steel area as 4 sq in.

$$50 \times 3 \times 1\tfrac{1}{2} < 10 \times 4 \times 17$$

showing the centroid to be more than 3 in. from the top.

average, and less than 1350 psi, the exact value being a matter of indifference.

The results thus obtained are safe but not economical. A check by the approximate method as in Ex. 7-6 will indicate what change to make.

All the examples so far have dealt with beams of fixed dimensions and correspond to the situation when the width of slab acting as flange is definitely assumed. The following examples illustrate the second method with working stress limits assumed to be realized.

Example 7-7a. (Same as Ex. 7-7 except that width of flange is not given.) What is the maximum moment that can be carried by the beam shown in Fig. 7-12 with limiting fiber stresses of $f_s = 20{,}000$ psi and $f_c = 1350$ psi? $n = 10$.

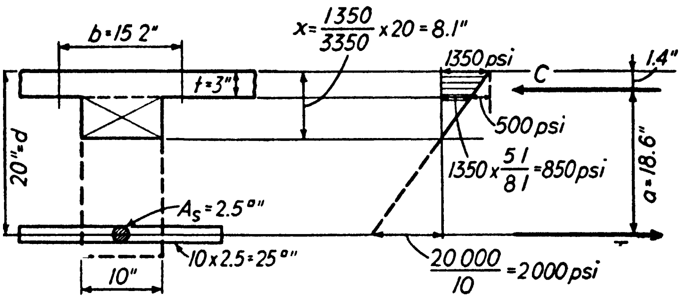


FIG. 7-14

Solution. As shown in Fig. 7-14 the stress diagram is drawn with extreme stresses taken as equal to the given limits, which locates the neutral axis. Following the procedure of Ex. 7-6, considering the compression acting on 1 in. width of flange:

Compression (C)	*Arm*	*Moment about Top*
$3 \times 850 = 2550$	$3 \times \frac{1}{2} = 1.5$ in.	3825
$\frac{1}{2} \times 3 \times 500 = 750$	$3 \times \frac{1}{3} = 1.00$	750
3300)		4575

Distance of C from top of beam 1.4 in.
Lever arm of resisting couple $= a = 20 - 1.4 =$ 18.6
Maximum moment of resistance $= T \cdot a = 2.5 \times 20{,}000 \times {}^{18.6}\!/_{12}$
$= 77{,}500$ lb-ft

Check of compressive stress: width of flange limit $= 16t + b' = 58$ in.
Width of flange required:

$$C = T = 50{,}000 \text{ lb} = \tfrac{1}{2}(1350 + 850) \times 3b$$

$$b = 15.2 \text{ in.}$$

which is less than the limit. The same result follows by using the fact that the neutral axis passes through the center of gravity of the cross section.

Example 7-8a. (Same as Ex. 7-8 except that the width of flange is not given.) A beam of the same dimensions as that shown in Fig. 7-14 carries a moment of 130,000 lb-ft. The limiting fiber stresses are $f_s = 20{,}000$ psi, and $f_c = 1350$ psi. $n = 10$. What steel area is required?

Solution. As in the previous example the stress diagram is constructed and the neutral axis and the lever arm of the resisting couple are calculated. Then

$$T = 130{,}000 \times 12 \div 18.6 = 83{,}900 \text{ lb}$$

$$\text{Steel area} = 83{,}900 \div 20{,}000 = 4.19 \text{ sq in.}$$

Check on width of flange:

$$C = \tfrac{1}{2}(1350 + 850) \times 3b = 83{,}900 \text{ lb}$$

$$b = 25+ \text{ in.}$$

which is less than the limit allowed.

7-9. Beams Reinforced for Both Tension and Compression. A concrete compression member is both stiffened and strengthened by longitudinal steel reinforcement, provided this reinforcement is properly restrained from buckling. Initially the two materials will deform equally, the stress being transmitted to the steel by the bond between the steel and the concrete. Accordingly the unit stress in the compression steel equals that in the concrete at that point multiplied by the value of n. The steel in the compression part of a beam acts the same way. However, such reinforcement is not economical because the ratio of the cost per unit volume of the steel as compared with the concrete is always greater than the value of n, the ratio of the stresses in the two materials at the same point.

Example 7-9. What are the maximum fiber stresses in this double-reinforced beam?* (Fig. 7-15.) The bending moment equals 80,000 lb-ft. $n = 10$.

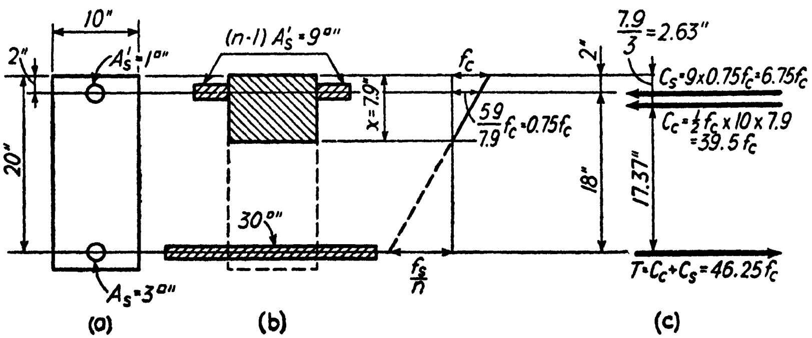


FIG. 7-15

Solution. To transform the steel-concrete section into its equivalent in concrete each area of steel is replaced by 10 times as much concrete at the same

* See p. 83 for the effects of plastic flow and the 1940 J.C. recommendations.

level in the beam as the steel it replaces. On removing the compression steel a hole of 1 sq in. is left in the concrete; this requires an equal amount of the added concrete to fill, leaving 9 sq in. in the compression wings. The neutral axis is found as usual.

$$\left(\frac{x}{2}\right)(10x) + 9(x - 2) = 30(20 - x)$$

$$x = 7.9 \text{ in.}$$

The total compression is considered to be the sum of two elements, C_c the compression on the rectangular 79 sq in., and C_s that on the 9 sq in. of the wings, where the stress intensity is $0.75f_c$ (Fig. 7-15c). The lever arm of the couple is the distance of this resultant compression from the tension steel (see Fig. 7-15 for values used):

$$\begin{aligned} C_s \times 18 &= 6.75f_c \times 18 = 121.5f_c \\ C_c \times 17.37 &= \underline{39.5f_c \times 17.37 = 686.1f_c} \\ & 46.25f_c\ \big)\ \underline{\qquad\qquad 807.6f_c} \\ a &= \qquad\qquad\qquad 17.46 \text{ in.} \end{aligned}$$

$$T = C = 80{,}000 \times 12 \div 17.46 = 55{,}000 \text{ lb}$$

$$f_c = 55{,}000 \div 46.25 = 1190 \text{ psi}$$

$$f_s = 55{,}000 \div 3 = 18{,}300 \text{ psi}$$

In replacing the compression steel with concrete the usual practice is to neglect the hole left by removal of the steel and assume the area added in the wings as n times the steel area. This is not unreasonable, considering the uncertainty that exists regarding the true value of the modulus of elasticity of the concrete.

Example 7-10. If the limiting fiber stresses are $f_s = 20{,}000$ psi, $f_c = 1350$ psi, what is the maximum moment of resistance of the beam of Ex. 7-9 (Fig. 7-15)? $n = 10$.

Solution. The transformed section is sketched and the neutral axis located as in Ex. 7-9. The procedure is the same as in Ex. 7-2.

Answer. 87,300 lb-ft.

The design of a double-reinforced beam to meet given conditions is considered in the next article.

The effect of plastic flow in a reinforced concrete compression member is to throw an increasing portion of the load on the steel, resulting in steel stresses approaching the elastic limit not infrequently. (See page 110.) On account of this phenomenon some designers assume the unit stress in compression steel in beams at a fixed figure regardless of the ratio n, for example, 16,000 psi. Where the usual methods are used in computation, as in this article, another way to allow for flow is to take the compression steel stress at some multiple of its calculated value; for instance, twice, as by the Joint Committee (Art. 804c) with a maximum of 16,000 psi.

To illustrate the application of these suggestions regarding allowance for plastic flow the beam of Fig. 7-15 is computed below in several ways.

Example 7–10a. (See Fig. 7–15.) Compute the resisting moment of this section (*a*) by the method of doubling the value of the compressive reinforcement, (*b*) by using the elastic theory with $n = 10$ for the concrete and the tension reinforcement, but using a straight 16,000 psi on the compressive steel, and (*c*) by using a value of $n = 20$. In all cases $f_c = 1350$ psi and $f_s = 20{,}000$ psi.

Solution. From Ex. 7–9:

	M_c (lb-in.)	M_t
Moment limited by compression: $46.25 \times 1350 \times 17.46$	1,090,000	
Moment limited by tension: $3 \times 20{,}000 \times 17.46$		1,048,000

(*a*) From Fig. 7–15:

$$\begin{aligned} C_c &= & 39.5f_c \\ C_s &= 2 \times 6.75f_c = & \underline{13.5f_c} \\ & & 53.0 \times 1350 = 71{,}500 \text{ lb} \\ & & a = 17.5 \text{ in.} \end{aligned}$$

Moment limited by compression: $71{,}500 \times 17.5$	1,253,000	
Moment limited by tension: $3 \times 20{,}000 \times 17.5$		1,050,000

(*b*) From Fig. 7–15:

$$\begin{aligned} C_c &= 39.5 \times 1350 = 53{,}300 \text{ lb} \\ C_s &= 1 \times 16{,}000 \quad = \underline{16{,}000} \\ & \qquad\qquad\quad 69{,}300 \text{ lb} \\ & \qquad a = 17.5 \text{ in.} \end{aligned}$$

Moment limited by compression: $69{,}300 \times 17.5$	1,213,000	
Moment limited by tension: $3 \times 20{,}000 \times 17.5$		1,050,000

(*c*) Locating the neutral axis:

$$\frac{x}{2}(10x) + 19(x - 2) = 60(20 - x)$$

$$x = 9.7 \text{ in.}$$

Compare Fig. 7–15:

$$\begin{aligned} C_c &= & 48.5f_c \\ C_s &= \frac{7.7 \times 19}{9.7} f_c = & \underline{15.1f_c} \\ & & 63.6 \times 1350 = 85{,}900 \text{ lb} \\ & & a = 17.1 \text{ in.} \end{aligned}$$

Moment limited by compression: $85{,}900 \times 17.1$	1,469,000	
Moment limited by tension: $3 \times 20{,}000 \times 17.1$		1,026,000

Discussion. In Ex. 7–10 the resisting moment determined by the compression slightly exceeds the limit set in tension: in this example the added value of the compressive reinforcement makes the tension steel the determining factor without question. The compressive side of the beam has had its total value increased by an amount varying from 15 per cent for cases (*a*) and (*b*) to 35 per cent for case (*c*). Probab'y cases (*a*) and (*b*) represent the preferable treatment of this problem. It might be noted that the fiber stress in the compressive reinforcement of (*a*) is 18,200 psi and in (*b*) it is 16,000 psi, so that (*b*) corresponds to J.C. 804*c*.

7-10. Beams of Limited Size. A beam is said to be of balanced design when both the tension steel and the concrete are stressed to their working limits simultaneously. It often happens that a beam must not exceed certain dimensions which are inadequate to provide the desired strength with a balanced design, when tension reinforcement only is used. Three different methods of strengthening such a beam follow.

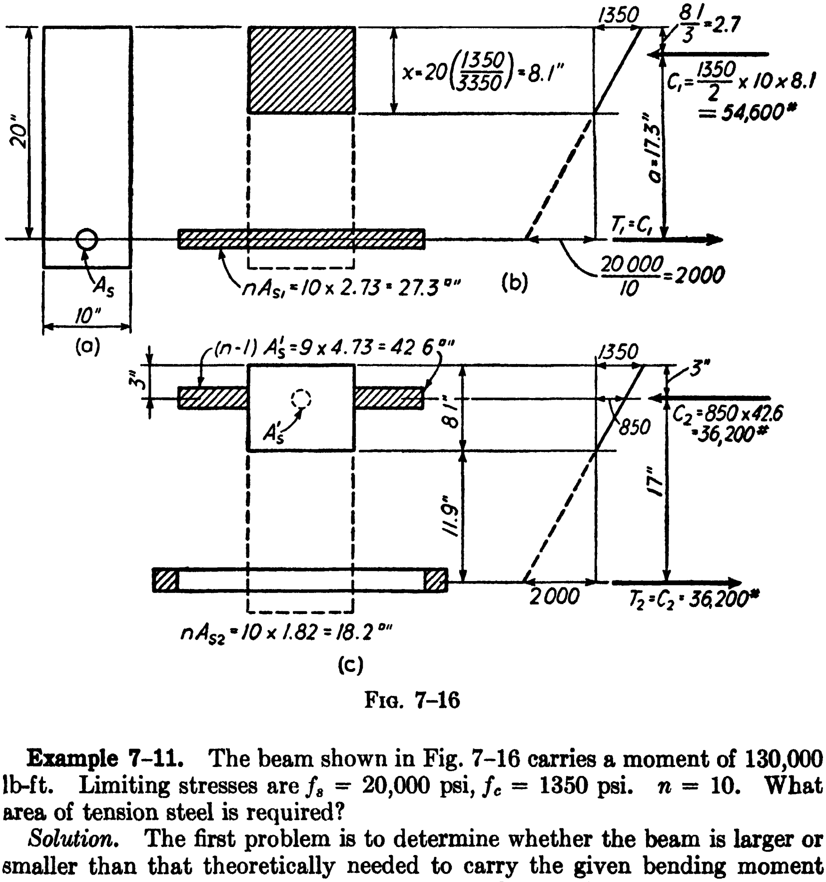


FIG. 7-16

Example 7-11. The beam shown in Fig. 7-16 carries a moment of 130,000 lb-ft. Limiting stresses are $f_s = 20{,}000$ psi, $f_c = 1350$ psi. $n = 10$. What area of tension steel is required?

Solution. The first problem is to determine whether the beam is larger or smaller than that theoretically needed to carry the given bending moment with the given fiber stresses, which is easily done as in Ex. 7-3 (page 69), where for the same limiting stresses the moment of resistance was shown to be $236bd^2$. Applying this information here this problem gives:

$$\text{Maximum } MR = 236 \times 10 \times 20^2 \div 12 = 78{,}700 \text{ lb-ft}$$

The beam being known to be smaller than the theoretical and the procedure being the same as in Ex. 7-3*b* (page 71), the result is 27.7 sq in. of steel required, the steel stress coming out 3800 psi. Whether or not it is possible to provide this amount of steel in the beam is not a matter of concern here.

Example 7-12. Data of Ex. 7-11. How much steel is required by the elastic theory, neglecting plastic flow and using both tension and compression reinforcement, the latter being placed 3 in. from the compression face?*

Solution. With balanced reinforcement the stress diagram will be as shown in Fig. 7-16*b*, with neutral axis 8.1 in. from the top. As in the beam discussed in Art. 7-9 the moment of resistance is considered to consist of two couples, one in terms of the compression on the rectangular part of the transformed section (see Fig. 7-15) and the other in terms of the compression on the projecting wings. The first step is to determine the maximum moment the beam can carry without compression reinforcement. From Ex. 7-11 we have MR = 78,700 lb-ft; or from the data of Figs. 7-16, *a* and *b* it is possible to write at once:

$$MR = C_1 \cdot a = 54{,}600 \times 17.3 \div 12 = 78{,}700 \text{ lb-ft}$$

The tension steel area required for this moment is

$$A_{s1} = 54{,}600 \div 20{,}000 = 2.73 \text{ sq in}$$

The difference between 130,000 lb-ft and 78,700 lb-ft, 51,300 lb-ft, must be provided for by adding A_s' sq in. of compression steel, whose effect on the transformed section is to add reinforcing wings of area $(n - 1)A_s'$ sq in. (Compare Fig. 7-15 and the first paragraph of Ex. 7-9.)

As shown in Fig. 7-16*c*, these concrete wings are 17 in. from the tension steel and at a level where the compressive stress is 850 psi. The total compression acting on these wings is 51,300 × 12 ÷ 17 = 36,200 lb. The compression steel area required is

$$9A_s' = 36{,}200 \div 850 = 42.6 \text{ sq in.}$$

$$A_s' = 4.73 \text{ sq in.}$$

The effect of adding this steel alone would be to shift the neutral axis, which is fixed by the conditions as noted. Since the neutral axis passes through the center of gravity of the transformed section, the addition of 42.6 sq in. at 5.1 in. from that axis must be balanced by adding 42.6 × 5.1 ÷ 11.9 = 18.2 sq in. = nA_{s2} sq in. on the other side, at the level of the tension steel. Whence A_{s2} = 1.82 sq in. of steel, making the total tension steel area

$$A_{s1} + A_{s2} = A_s = (2.73 + 1.82) = 4.55 \text{ sq in.}$$

The tension steel area can be found more easily thus:

$$A_s = \frac{T_1 + T_2}{f_s} = (54{,}600 + 36{,}200) \div 20{,}000 = 4.54 \text{ sq in.}$$

The total steel area required by this method of reinforcement is 9.27 sq in. as against 27.7 sq in. needed if only tension steel is used. Evidently only a relatively small excess moment can be cared for economically with tensile reinforcement only.

Example 7-13. Data of Ex. 7-11, except that there is no limit to width. Design a tee-beam to satisfy the requirements.

* The application of J.C. 840c to this problem to allow for plastic flow is illustrated in Ex. 9-10.

Solution. (Fig. 7–17.) Assume that the stem has been designed and a width of 10 in. determined upon. As in the previous example, the rectangular portion of the beam, 10 in. wide, can carry a moment of 78,700 lb-ft with the given limiting stresses, steel area = 2.73 sq in., leaving 51,300 lb-ft to be

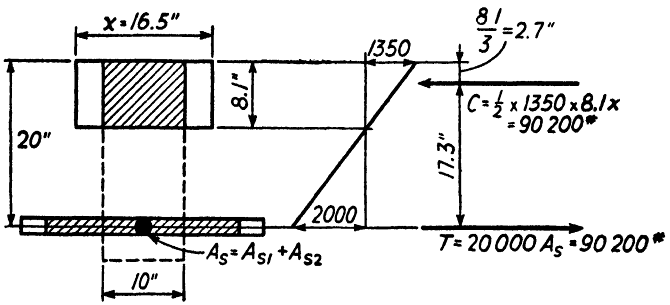


FIG. 7–17

carried by the compression in the flanges and the tension in the extra tension steel. Assume the flange 8.1 in. deep, thus keeping the beam as narrow as possible. Let its width equal x in. Since the arms of the tee are here brought down to the neutral axis the total width of the flange x is best found by one operation:

$$C = T = 130{,}000 \times 12 \div 17.3 = 90{,}200 = \tfrac{1}{2} \times 1350 \times 8.1x$$
$$x = 16.5 \text{ in.}$$

Had these arms been made thinner the procedure of the previous example would have offered a quick solution. The total tension steel equals 90,200 ÷ 20,000 = 4.51 sq in.

In practice the limiting depth of a beam given is the overall concrete dimension* and it is necessary for the designer to choose the depth to the steel that will conform to the proper placing of the reinforcing rods used. This is one of the inevitable cut-and-try problems that is solved easily only as one becomes experienced.

7–11. Bending of Longitudinal Tension Steel. Beams of reinforced concrete are usually of uniform section over their whole length and accordingly the area of longitudinal steel required at the section of maximum bending moment is greater than that required elsewhere. When this maximum area is supplied by two or more bars it is possible to dispense with some of them when not needed for main tension reinforcement. This is done by bending the surplus bars up into the web to act as reinforcement there. The ends of the cut bars should be bent

* One other possibility when a beam is limited in depth is to use a richer mix of concrete. Practical considerations require this rich mix for the entire floor or pour. For illustration, determine the minimum width of concrete for Ex. 7–13, using 4000 psi concrete with $f_c = 1800$, $f_s = 20{,}000$ psi, $n = 8$. Here $M/bd^2 = 325$ and b is found to be 12.0 in. as compared with 16.5 in. above.

up above the neutral axis and hooked to give anchorage. Many designers require that this be done in such manner as to keep the beam symmetrical about the vertical axis at all sections.

The point where a bar becomes unnecessary may be located by computing the moment of resistance of the section with that bar omitted and finding where the bending moment equals that moment of resistance. This method is cumbersome, however, and is not used. As has already been pointed out, the moment of resistance equals the total tension times the lever arm of the resisting couple; or

$$BM = MR = T \cdot a = f_s \cdot A_s \cdot a$$

using the notation already familiar. The arm a varies so little for varying tension steel areas that it may be assumed constant without serious error. With this assumption it becomes possible to lay down the principle that *the area of steel required varies directly with the bending moment and the same curve by proper choice of scale may serve as both moment curve and area-required curve.* Usually, practical considerations as to commercial size of bars result in the maximum area furnished being larger than that required. It is common to neglect this difference and compute bar lengths as though the maximum area furnished equaled that required.

Example 7-14. This reinforced concrete beam (Fig. 7-18) carries a uniform load. What are the minimum possible values of the dimensions a and b?

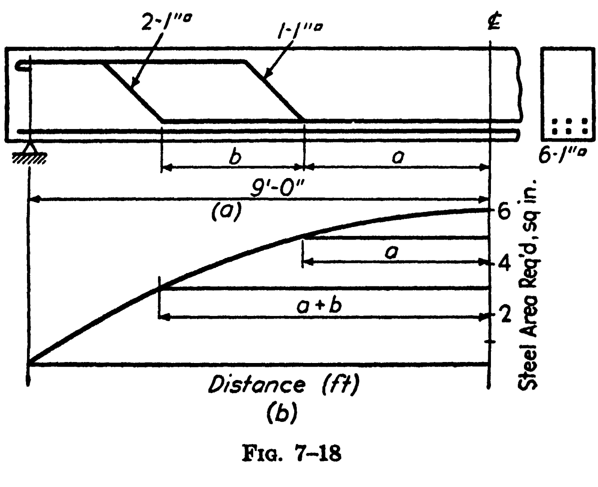


FIG. 7-18

Solution. The bending moment curve for this loading is a parabola with maximum ordinate at the center, and the area-required curve accordingly is the same. The parabola of Fig. 7-18*b* is drawn with the center ordinate

representing 6 sq in. One bar may be bent up when 5 sq in. only are required: so*

$$a = 9\sqrt{\tfrac{1}{6}} = 3.7 \text{ ft}$$

Similarly

$$a + b = 9\sqrt{\tfrac{1}{2}} = 6.4 \text{ ft}$$

This assumes the same average depths to the center of both layers of steel. Although the bent-up bars have a slightly smaller moment arm, it is not customary to consider this refinement.

7–12. Shearing Stresses in Homogeneous Beams. A brief review of the shearing stresses in homogeneous beams is desirable in order that a clear picture may be obtained of the web stresses in beams of all kinds. For rigorous demonstration of these matters the reader should consult the standard treatises on the strength of materials.

In a previous article the law governing the variation of normal stress intensity on any section was discussed and the present problem is the variation of the intensity of the tangential or shearing stress, the total value of which at any section equals the external shear V. It is easy to prove, by study of the shearing stress on an elementary prism, that the intensity of the vertical shear (see Fig. 7–19) and that of the horizontal shear at any point are equal. A knowledge of the variation of horizontal

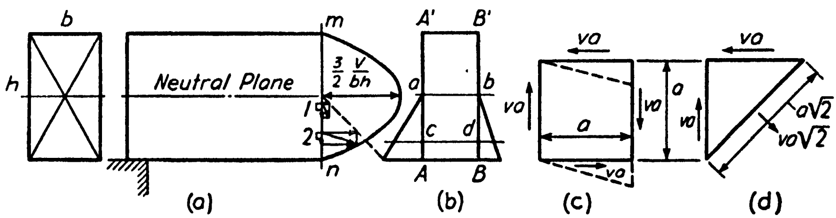


Fig. 7–19

shear intensity therefore gives also that of vertical shear. In Fig. 7–19*b* is shown a portion of a rectangular beam lying between any two sections, AA' and BB'. The variation of normal tension intensity at each section is indicated by the partial stress diagrams, the abscissas on AA' being shown larger than those on BB' on the assumption that the moment at AA' is the larger. Considering the stability of the small piece of beam $cdBA$, the pull on the cA face is larger than that on the dB face, and the only force available to balance the difference is the horizontal shear on the plane cd. A brief consideration of the problem shows that

* This follows from the property of a parabola that offsets vary as the squares of the distances; hence

$$\frac{a^2}{9^2} = \frac{1}{6}$$

the nearer the *cd* plane is to the neutral plane *ab*, the larger is the difference between the two tensions and the larger the horizontal shear. Therefore, the horizontal shear, and accordingly also the vertical shear, increase in intensity at a decreasing rate from zero at the extreme fiber to a maximum at the neutral plane. For a rectangular section the law of this variation is a parabola (Fig. 7–19*a*) with a maximum intensity of $1.5V/bh$.

The resultant intensity of stress at any point away from the extreme fibers, as, for example, on the vertical faces of the elementary prisms 1 and 2, Fig. 7–19*a*, must be inclined in direction and must act somewhat as shown. Referring again to the elementary prism shown in Figs. 7–19, *c* and *d*, the shearing forces there shown may be resolved into components along the diagonals, and *these components may be combined to give inclined tensile and compressive forces acting at 45° (Fig. 7–19d) with an intensity* v *equal to that of the shear.* This illustrates the case when the prism lies at the neutral plane where there is no direct stress. When it lies in the face of the beam there are neither horizontal nor vertical shearing stresses and the resultant tension is horizontal; it is that given by the usual formula for fiber stress.

A more detailed study of the state of stress at any point in this cross section would show that passing through it are two inclined planes, 90° apart, on which there is no shear, the resultant stress being compression on one and tension on the other, of an intensity greater than on any other plane through the point. These stresses are called the principal stresses at the point. Midway between these planes are those of maximum shear intensity.

In the web of a plate girder the action of the inclined tension is easily resisted by the steel but the diagonal compression tends to cause buckling and it is necessary to limit the minimum thickness of the web or to provide suitably spaced stiffeners, or both. In a concrete beam, on the other hand, the material easily resists the diagonal compression but is weak in tension. The wooden beam resists both.

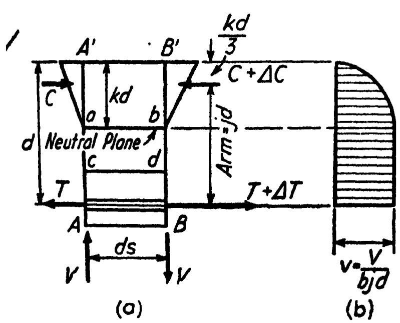


FIG. 7–20

7–13. Shearing Stresses in Reinforced Concrete Beams. The variation of shear intensity in a rectangular reinforced concrete beam may be ascertained by considering a small portion of such a beam between any two sections a small distance *ds* apart, as shown in Fig. 7–20*a*, the breadth of the beam being taken

as b inches. The forces acting on this bit of beam consist of the normal stresses C and T and the shear V; it is assumed that the moment at BB' is larger than that at AA', and that the sections are so close together that the two shears may be considered equal. These forces are in equilibrium, and if the condition $\Sigma M = 0$ is applied, there results:

$$\Delta T \times jd = V \times ds$$

The tendency of the small portion of the beam $cdBA$ to be pulled to the right is resisted by the horizontal shear on the cd plane which may be expressed as the intensity of shear on that plane v, assumed to be uniform, multiplied by the area bds. Then

$$v \cdot b \cdot ds = \Delta T$$

Combining these two equations gives:

$$v \cdot b \cdot ds = \frac{Vds}{jd}$$

and

$$v = \frac{V}{bjd} \qquad [7\text{–}1]$$

This relation may be even more simply deduced by consideration of the free body $ABdc$ in Fig. 7–20a only, from which we have as above $v \cdot b \cdot ds = \Delta T$. But ΔT may be obtained as $\Delta M/jd$, and ΔM may be written as $V \cdot ds$, from the well-known theorem that the difference in moment at two points on a beam equals the area of the shear curve between the two points. The equation follows at once.

As the concrete is assumed to take no tension, the shear intensity is accordingly assumed to be constant between the neutral plane and the steel; above that plane it varies as in a homogeneous rectangular beam (Fig. 7–20b). Accordingly equation 7–1 gives the maximum intensity of horizontal and likewise of vertical shear (as explained in Art. 7–12) at any section of a rectangular reinforced concrete beam.

This demonstration applies equally well in essential details to a rectangular concrete beam reinforced for both tension and compression and to a reinforced concrete tee-beam. Tests confirm the conclusion that in the matter of shear a tee-beam may be considered as equivalent to a rectangular beam of the same depth, with a width equal to that of the stem of the tee-beam. The standard notation for this tee-beam stem width is b', and so for tee-beams the formula is written

$$v = \frac{V}{b'jd} \qquad [7\text{–}1a]$$

The value of j does not vary greatly for a wide range of conditions and an average value of $7/8$ is usually taken for all shear computations. Since all computations in which the value of the shear is used are highly approximate greater precision than that obtained by the average value is unnecessary.

Consideration of Fig. 7–20 shows that the total bond, the grip of the concrete on the tension reinforcement which prevents slippage, equals the difference in total steel stress at the two sections, A and B : $\Delta T = v \cdot b \cdot ds$. Assuming uniform intensity u of bond stress we have also $\Delta T = u \cdot \Sigma o \cdot ds$ where Σo equals the sum of the bar perimeters. This results in

$$u = \frac{vb}{\Sigma o} \quad \text{or} \quad u = \frac{V}{jd\Sigma o} \qquad [7\text{–}2]$$

These equations for shear and for the bond in active tension steel (equations 7–1 and 7–2) take on physical significance when interpreted with the aid of the relation learned in the study of strength of materials. This states that the difference in bending moments at two sections of a transversely loaded beam equals the area under the shear curve between the two sections, that the shear at any section equals the rate of change of the bending moment at the section, and also equals the difference in bending moments at sections a unit distance apart where the shear is constant.

Study of test results shows that the theoretical relation for bond does not cover all the facts. "The usual methods of computing the bond stress in a reinforced concrete beam does not take into account all the phenomena of bond action" which "may be expected to greatly modify the distribution of bond stress over the length of the bar and otherwise to affect resistance to bond stress. However the nominal values for bond resistance, computed by the usual formula, form a useful basis for comparison in beams in which the dimensions and general make-up are similar."*

It is not often that the bond stress in compression steel needs investigation. Usually such steel is nearer the neutral axis than the tension bars and is made up of rods of about the same diameter. The rate of stress transfer to the compression steel will be to that of the tension steel as their respective unit stresses, which bear the ratio of the respective distances from the neutral axis. If the rods are of the same size the unit rate of stress transfer (or the horizontal shear on the bar perimeters) for compression steel is therefore generally less than that of the tension steel.

* Bull. 71, Univ. of Illinois.

Example 7-15. The beam shown in Fig. 7-21 is made of concrete with a 28-day strength in compression of 3000 psi, for which the following limiting stresses hold: shear when no web reinforcement is used, 60 psi; shear when web reinforcement is used, 180 psi; bond stress, 120 psi for plain bars, 150 psi for deformed bars; $n = 10$. Compute the shear and bond intensities at a point 24.5 in. from the end of the beam. How will these values compare with the maxima? Are the shear and bond limits exceeded in this beam?

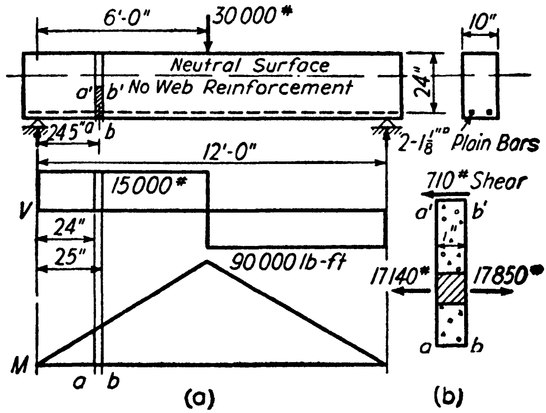


FIG. 7-21

Solution. The free body taken for study consists of a section of the beam *ab*, 1 in. long by the full width of the beam and extending upward from the bottom of the beam to any point short of the neutral surface. The pull in the steel at *a* is found to be 17,140 lb and that at *b* 17,850 lb. For equilibrium to exist there must be a shear of 710 lb on the top horizontal surface of the body as shown. This is the total horizontal shear on the horizontal top surface of the free body, an area 1 in. by 10 in. If uniform distribution of this shear is assumed, the resulting intensity is 71 psi.

This horizontal shear of 710 lb is also the total bond on the surface of the two reinforcing rods and, since the section is 1 in. long, this equals the rate of increase of total tension in the reinforcement at this point. The area of rods in contact with the concrete equals $2 \times 4 \times 1\frac{1}{8} = 9$ sq in. If uniform stress distribution is assumed again, the bond intensity equals 79 psi.

The student should check these answers by use of the formulas above given. Which is the more accurate, the solution just given, which utilizes sections slightly to the right and left of the section concerning which information is asked, or the formula solution, which uses the shear at the given section? The answering of the other two questions required by the problem is also left to the student, together with the following queries.

What is the total bond stress on the rods between the end and the center of the beam? How does the rate of change of steel stress vary in the half-length? What measures this rate of change?

How are these several results affected by disregarding the weight of the beam (about 280 lb per ft)?

Example 7-16. The beam of the preceding example is reinforced with two 1-in. squares and two ½-in. squares instead of the two 1⅛-in. squares shown in Fig. 7-21. Compute the bond stress at a point 24.5 in. from the end of the beam for this combination of steel.

Solution. The unit stress in the steel is found to be ($x = 8.7$ in.) 6830 psi at a point 24 in. from the support and 7110 psi 25 in. from the support. The difference in total tension at the two sections, accordingly, is $280 \times 2.5 =$ 700 lb, which equals the total shear on a horizontal plane above the steel, a unit shear of 70 psi. If this value is used in equation 7-2 a bond stress of 58 psi is obtained, a result largely in error as shown by examination of the two sizes of rods separately. The bond stress on the 1-in. squares equals $\dfrac{(7110 - 6830) \times 1}{4 \times 1} = 70$ psi; that on the ½-in. squares similarly equals 35 psi. Since the total tension is distributed uniformly over the bars according to their areas and the bond stress is prorated over the bars according to their perimeters, evidently equation 7-2 will not give reasonably accurate results unless the tension bars are all closely of the same size.

Problem 7-1. The beam of Ex. 7-15 is loaded with a total uniformly distributed load of 60,000 lb instead of the load shown. Compute the unit shear and bond at the reaction and at the quarter point.

Ans. $v = 143$ psi at end; 72 psi at quarter-point. $u = 159$ psi at end; 80 psi at quarter-point.

Query. What proportional difference in results will come with taking $a = \frac{7}{8} \cdot d$ instead of the value given by the transformed section?

7-14. Diagonal Tension in Reinforced Concrete Beams.

The concrete in a reinforced beam is no stronger in itself than when unreinforced and it cracks in any loaded beam when the tensile limit is exceeded, the line of cracking being indicated in a general way in Fig. 7-2, sloping less steeply toward the ends of the beam and tending to lie at right angles to the inclined web stress. The function of the reinforcement is not to prevent cracking, that being impossible, but to keep any one crack from opening up widely; thus it compels the formation of many minute cracks instead of a single large one which would cause failure.

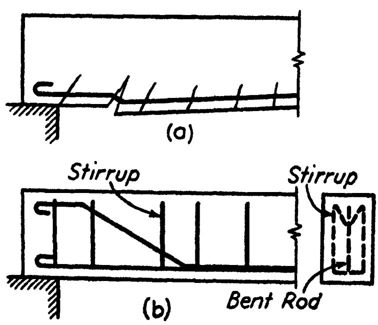


FIG. 7-22

It is plain that as long as the cracks are vertical the horizontal bars are effective reinforcement but, where they are inclined, horizontal bars are very ineffective, there being nothing but concrete to carry the vertical component of the inclined tension. When a beam is reinforced for normal stress only, a failure occurs under small load somewhat as pictured in Fig. 7-22*a*, the part of the beam toward the center drop-

ping below the end portion. To be accurate the sketch should show only gradual curves in the steel. There is insufficient strength in the concrete below the rods to the left of the rupture to resist the pressure brought upon it, and it spalls off in such a failure.

A beam is made secure against diagonal tension failure by supplying it with a sufficient amount of reinforcement, so placed as to cross a sufficient number of the inclined lines of potential failure. The more nearly perpendicular to the cracks the more effective are the rods. In practice use is made of stirrups (Fig. 7–22*b*), generally vertical, looped about the main steel, and of main longitudinal rods, bent up at an angle across the region of diagonal tension stress in those portions of the beam where they are no longer needed to resist the normal tension. In order to proportion such reinforcement, knowledge must be had of the amount of the diagonal tension. Unfortunately this cannot be computed accurately in a reinforced concrete beam since the concrete cracks irregularly and it is impossible to say just how much tension is taken by the steel. If there were no normal tension on any section below the neutral axis the maximum diagonal tension would act at 45° and have an intensity equal to that of the shear at the section. This is always the assumption made in design.*

In all discussions of diagonal tension these words from the 1916 Report of the Joint Committee should be kept in mind:

> In designing, recourse is had to the use of calculated vertical shearing stresses as a means of comparing or measuring the diagonal tension stresses developed, it being understood that the vertical shearing stress is not the numerical equivalent of the diagonal tensile stress, and that there is not even a constant ratio between them. . . . It does not seem feasible to make a complete analysis of the action of web reinforcement and more or less empirical methods of calculation are therefore employed.

Study of tests indicates that the concrete is effective in resisting small amounts of diagonal tension and may be counted on with safety to perform this duty unaided when the shearing stress and also the diagonal tension is less than 2 per cent of the ultimate compressive strength of the concrete, about 60 psi for ordinary 1–2–4 (3000 psi) mixes. If the tensile strength of the concrete is taken as one-tenth of the compressive strength this indicates a factor of safety of approximately 5 as regards diagonal tension failure.

When the shearing stress exceeds the $0.02f'_c$ limit reinforcing steel must be used in the web of the beam but the concrete may still be relied upon to carry its share of the web tension as long as the shear intensity

* Compare the statement of the 1940 J.C., Art. 816.

does not exceed 6 per cent of the compressive strength of the concrete. Tests have shown that still higher diagonal tension stresses may be carried by properly reinforced beams, up to a shearing stress limit of 12 per cent of the concrete compressive strength.

The use of the shear as a measure of the diagonal tension accounts for the fact that diagonal tension failure and diagonal tension reinforcement are very commonly, and erroneously, spoken of as shear failure and shear reinforcement. It is hardly worth while to quarrel with this usage as long as one keeps in mind exactly what the terms refer to.

7-15. Stresses in Diagonal Tension Reinforcement. No entirely satisfactory and consistent theory of the action of web reinforcement in concrete beams has yet been devised nor, very likely, ever will be. The usual methods as here presented are frankly approximate — little more than empirical rules which, experience has shown, give safe and reasonably economical results.

In a simply supported beam the area on the beam elevation which must be reinforced for diagonal tension lies between the tension steel and the neutral plane or somewhat above (frequently assumed for this purpose as lying at mid-depth) and between the end of the beam (sometimes the center of support and usually the edge of support) and that section where the intensity of shear measures a diagonal tension which

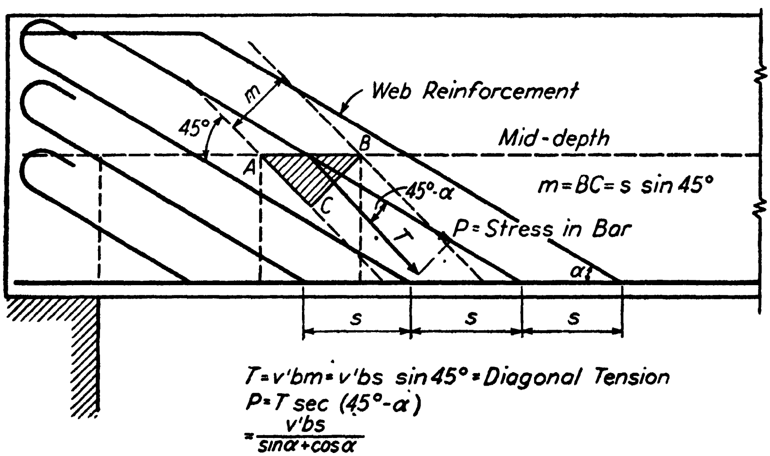


Fig. 7-23

can be carried by the concrete without assistance. If the web reinforcement consists of bars inclined at any angle α with the tension steel, with limiting values from 15° to 90°, spaced longitudinally at s inches, as shown in Fig. 7-23, the diagonal tension T to be resisted by any one

bar is that developed in the length s tributary to it, taken as the tension on a 45° plane, b (beam breadth) inches wide and distance BC long, a distance which may be thought of as lying along a potential diagonal tension crack crossing the bar. It is assumed that that part of this tension in excess of that carried by the concrete is carried entirely by the web bar: in other words, the bar stress component lying along a 45° slope equals the excess diagonal tension.

It is obvious that the horizontal component of diagonal tension actually is resisted by the tension steel and that only the vertical component need be considered in designing web reinforcement. This argument will be given consideration later. In the past there has been considerable difference in the computations for web steel by different designers, as the student may ascertain by examining the Joint Committee reports of 1916, 1921, 1924, and 1940. Apparently opinion has generally settled upon the formulas and essential arguments given in this chapter.

Example 7-17. A rectangular beam has a width (b) of 10 in. and a depth to the steel (d) of 24 in. and carries a total uniformly distributed load of 60,000 lb on a span of 12 ft (beam of Ex. 7-15 and Prob. 7-1). The web reinforcement consists of a series of ½-in. diameter rods inclined at an angle of 45° with the longitudinal steel to which they are welded at 8-in. intervals. Compute the stress in the bent rod 24 in. from the support, assuming the concrete to carry the diagonal tension up to the amount measured by $v = 60$ psi.

Solution. The shear at the given section is 20,000 lb and the intensity of horizontal shear is 95 psi, which is also the intensity of the diagonal tension. Since the concrete can carry the diagonal stresses without assistance up to a shear of 60 psi this leaves 35 psi as measuring the tension in the steel. Here the inclined rod lies along the line of action of the diagonal tension and its stress equals the excess tension developed in 8 in. — the excess of tension developed on an area 10 in. wide and 8 sin 45° = 5.7 in. long (see Fig. 7-23); $35 \times 10 \times 5.7 = 2000$ lb. This causes a unit stress in the steel of $2000 \div 0.20 = 10{,}000$ psi.

Example 7-18. Same as Ex. 7-17 except that the inclined ½ in. round bars make an angle of 65° with the longitudinal steel.

Solution. (See Fig. 7-23.) The student should redraw this figure making $\alpha = 65°$.

The angle between the diagonal tensile force of 2000 lb and the bar carrying it is 20° and accordingly the bar stress equals 2000 sec 20° = 2130 lb, which stresses the web bar to 10,700 psi.

Query. What would be the unit stress if the web bar were inclined at 25° with the longitudinal steel?

Problem 7-2. Same as Ex. 7-17 except that the web bars are made of ⅝ in. round steel spaced 6 in. along the tension reinforcement and inclined at 30° therewith.

Ans. 14,000 psi.

Example 7-19. Same as Ex. 7-17 except that the web reinforcement consists of vertical U-stirrups of $\frac{3}{8}$ in. round steel spaced 6 in. apart. Compute the unit stress in the stirrup lying 24 in. from the support.

Solution. The distance m (Fig. 7-23) equals 6 sin 45° = 4.24 in., and the excess diagonal tension acting at 45° with the tension steel in a distance of 6 in. equals $35 \times 10 \times 4.24 = 1485$ lb. When this is resolved into a vertical component and into one lying normal to itself, the vertical component equals 1485 sec 45° = 2100 lb, which is the load carried by the stirrup. When this is divided by the cross-sectional area of the stirrup, 2×0.11 sq in., a unit stress of about 9,500 psi is obtained. Note that the load on the stirrup equals the horizontal area tributary to the stirrup (6×10) multiplied by the excess shear intensity ($v'bs$).

Problem 7-3. Same as Prob. 7-2 except that the web steel consists of vertical U-stirrups of $\frac{3}{8}$ in. round steel spaced 10 in. apart.

Ans. 15,900 psi.

Formulas. A study of Fig. 7-23 reveals the excess diagonal tension to be carried by the steel represented by a 45° arrow and indicated as having a value of $T = v'bs \sin 45°$ where v' is the excess of shear intensity over that measuring a diagonal tension which the concrete can carry without aid. The other symbols need no redefining. Here the web steel is shown making an angle α with the main tension steel smaller than 45°; the same formula results when $\alpha > 45°$. The total stress P in this steel has its 45° sloping component equal to T. This gives the equation

$$P = A_v f_s = \frac{v'bs}{\sin \alpha + \cos \alpha} = \frac{V's}{jd(\sin \alpha + \cos \alpha)} \qquad [7\text{-}3]$$

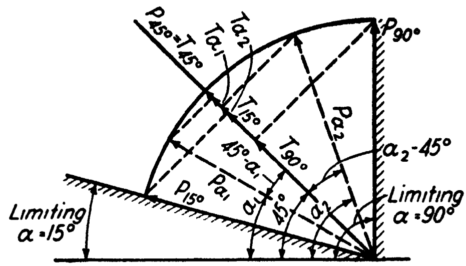
where V' is the external shear corresponding to v'. For $\alpha = 90°$ this equation takes this form

$$P = A_v f_s = \frac{V's}{jd} = v'bs \qquad [7\text{-}3a]$$

the stress in a vertical stirrup. Solving for s, these equations become those of the 1910 J.C. (Art. 819).

These equations suggest a rule-of-thumb way of remembering the stress relationships for web bars. The vertical component of diagonal tension developed in any distance along a beam (the other component being at right angles to the diagonal tension) equals the horizontal shear in that distance. The excess portion of this horizontal shear (vertical component) equals the stress in a vertical stirrup placed to care for the diagonal stress in this distance. This excess horizontal shear divided by the sum of the sine and cosine of the angle of inclination equals the stress in an inclined bar placed to care for the portion of stress not carried by the concrete.

The interpretation of web reinforcement action here given evidently is in accord with the Joint Committee recommendation which states as a basis for design that the effectiveness of this reinforcement is measured by its projection on the 45° line of diagonal pull (J.C. 816*d*2). This is illustrated in Fig. 7-24, which makes plain that a web bar, inclined at any angle other than 45°, working at the limit of its capacity P will develop or carry a lesser amount of diagonal tension T.



Effectiveness of diagonal tension rod at different inclinations

FIG. 7-24

Problem 7-4. Show that the effectiveness of web reinforcement is measured by cos (45° − α) and plot the curve of variation for the permissible range of α in value, 15° to 90° (J.C. 818).

Equation 7-3 is sometimes derived by a different argument which is summed up in Fig. 7-25, given here to enable the student to read other

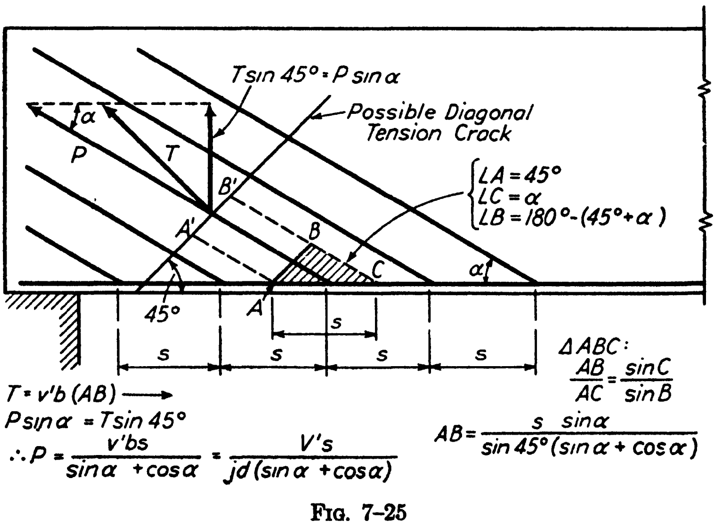


FIG. 7-25

textbooks with understanding. The argument is perfectly sound but perhaps not so convenient in implication as that first given. Here the amount of diagonal tension taken by an inclined bar is a portion of that

developed on the tributary area along a potential crack, an area which depends on the inclination as well as on the bar spacing. The horizontal component of this diagonal pull is assumed to be taken by the horizontal steel and the vertical component by the web steel; this results in this vertical component being also the vertical component of bar stress.

Examination of Figs. 7–23 and 7–25 makes it evident that the pull in an inclined bar below the neutral plane must be variable and a maximum at that plane, and that the shear intensity usually is maximum at that point since it is nearer the support than the lower end at the level of the main tension reinforcement. It is more logical to develop our theory by considering the pull downward from the neutral plane rather than upward from the tension steel as is common practice. This usual procedure, however, seems sufficiently accurate.

Further consideration of Fig. 7–25 will show that, in order that every potential 45° crack in the region of high shear intensity be crossed by reinforcement between the tension steel and mid-depth, a common requirement (J.C. Report, 819*c*), vertical stirrups must be spaced somewhat closer together than one-half the depth $d/2$, and 45° sloping bars (the usual inclination) must be spaced somewhat closer than the depth d.

Example 7–20. The beam shown in Fig. 7–26 carries a total load of 3100 lb per ft, the dead weight amounting to 600 lb per ft, including beam weight, and the balance being movable live load. The concrete used has a 28-day strength of 3000 psi which permits a shearing stress of $0.06f'_c$ with web reinforcement, there being no end anchorage of the longitudinal steel. (See J.C. Report, 878.) Determine whether the web reinforcement is sufficient, the stress being limited to 16,000 psi.

Note: The student should verify the values which are given without supporting computations.

Solution. With the live load extending over the whole span the end shear intensity is 105 psi, the allowable limit being 180 psi: with live load over one-half the beam the shear intensity at the center is 21 psi. For live load coverage between these limits the locus line of shear intensities is assumed to be straight as shown in Fig. 7–26*b*. *Query.* Is this on the safe side?

The concrete can carry diagonal tension up to the amount measured by a shear intensity of 60 psi and accordingly web reinforcement is needed in the end 55 in. of the beam.

Bent-Up Bars. Since we have only two pairs of inclined bars and not a series, it becomes necessary to delimit somewhat arbitrarily the effective range of action of each pair. The usual rule (see J.C. Report, 819*b*) limits the effectiveness to a distance of $3/8 \cdot d$ each side of the point where the bent rod crosses the mid-depth level of the beam. Accordingly the first pair of bars carries the diagonal tension in the end 9 in. of the beam, and the second pair that in the next 18 in. The reason for the horizontal spacing of bends at 18 in. is thus plain. The vertical component of diagonal tension in the end 9 in. (the other component being at 45° with the horizontal; Fig. 7–23)

equals the horizontal shear in the distance, 4480 lb; this, multiplied by sin 45°, is the diagonal tension and also the bar stress, the angle of inclination of both being 45°. The bars are stressed to $4480/\sqrt{2} = 3170$ lb or about 3600 psi. Similarly the second set is stressed to about 5300 psi which is much less than the limit set, 16,000 psi.

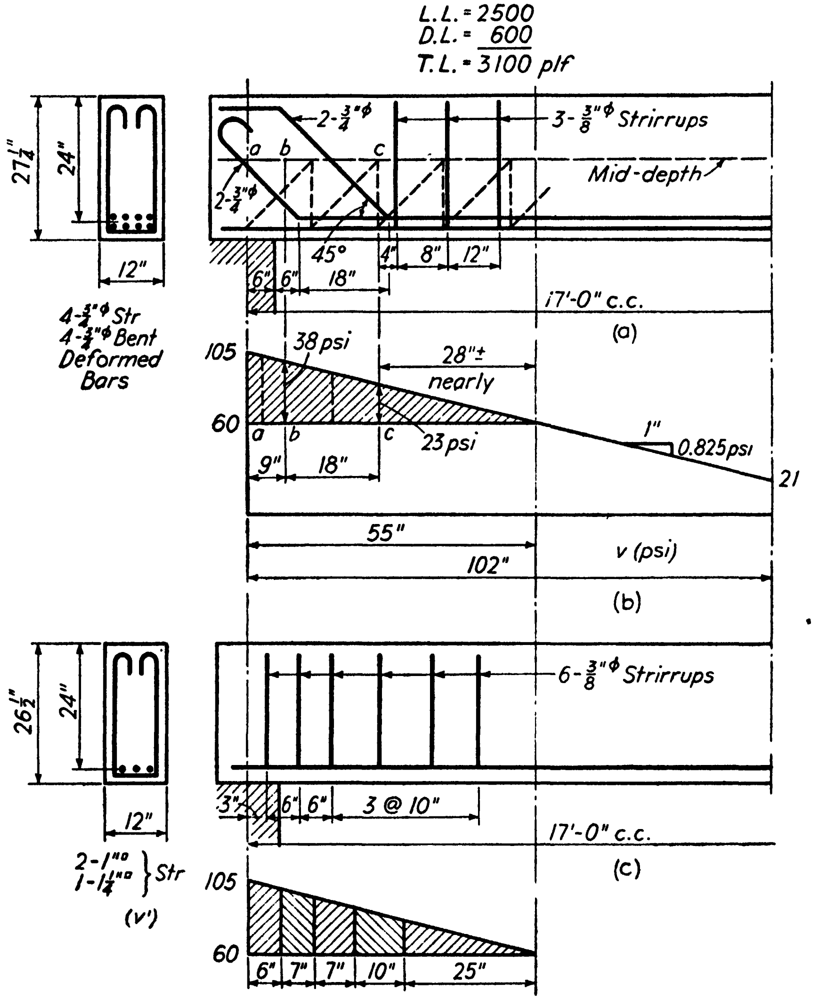


Fig. 7–26

Vertical Stirrups. One U-stirrup can carry $2 \times 0.11 \times 16 = 3.5$ kips. The total load to be carried is $\frac{1}{2} \times 23 \times 12 \times 28 = 3870$ lb. Two stirrups are required and three are used, since the limiting spacing of about $d/2$ would otherwise be exceeded.

Example 7–21. The beam of the previous example is redesigned with a reinforcement of 3 straight bars for main tension steel and vertical stirrups for

web reinforcement. Is the provision for diagonal tension satisfactory as shown in Fig. 7–26c?

Solution. The vertical component of diagonal tension to be carried in the end 55 in., which require web reinforcement, equals 14,850 lb, necessitating five $\frac{3}{8}$ in. round stirrups at 3500 lb each. Here 6 stirrups are used.

The theoretical spacing for 5 required stirrups may be obtained by dividing the triangular area representing the vertical component of diagonal tension in the end 55 in. into 5 equal parts, as shown in Fig. 7–26c, and placing a stirrup approximately midway of each division or at the third point of the last division, which is a triangle. Inspection of the figure shows that this is what has been done. Evidently refinement of placement is quite meaningless here; the important thing is to have enough stirrups spaced sufficiently close together so that no crack may open up between.

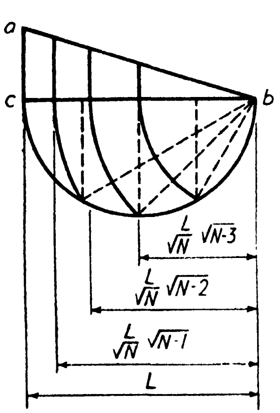


FIG. 7–27

A graphical method of dividing a triangular area into a specified number of equal parts is illustrated in Fig. 7–27 which should require no explanation. The student can easily verify the dimensions given, noting that N equals the number of specified divisions. These division points may be located easily by use of the slide rule with a single setting. Set the runner to L on the D scale and bring N on the B scale to the cross line, thus dividing L by $\sqrt{N}$. Then place the runner successively at N-1, N-2, etc., on the B scale, recording the readings on the D scale, which, subtracted from L, give the division lengths required. In this example, if 5 stirrups, are used, these readings are 55–49–42–35–25. The precision here used is greater than would be usually attempted in practice. A sufficiently accurate and quick variation based upon locating each stirrup at mid-length of its division consists in placing the runner successively at $N-\frac{1}{2}$, $N-1\frac{1}{2}$, $N-2\frac{1}{2}$, etc., on the B scale and without setting down any figures, mentally deducting each reading on the D scale from its predecessor. This gives directly the distance from face of support to first stirrup and from center to center of successive stirrups. A more usual method would be the computation of the required spacing at the end and other points and the placing of the stirrups roughly according to these requirements. Here the horizontal shear per inch at the end is $v'b = 45 \times 12 = 540$ lb per in., which equals the increment of vertical component of diagonal tension per inch. The end spacing equals the length which can be cared for by 1 stirrup, or $3500 \div 540 = 6.5$ in.; the spacing is twice this 27.5 in. from the support. These data suffice for placing the stirrups.

The examples considered in this article have all involved low shear stress, less than $0.06f'_c$. For larger diagonal tension and shear intensities tests have shown that it is necessary that both the web and main reinforcement be thoroughly anchored at the ends and that it is advisable that the web reinforcement be designed to carry all the diagonal tension

without reliance at all upon the concrete (J.C. Report, 817e and 878, Table 7). In this situation it is necessary also that each possible 45° line extending from tension steel to mid-depth of beam be crossed by at least two lines of web reinforcement (J.C. Report, 819c).

7-16. Anchorage and Bond Stress. Undesirable cracking and even failure of reinforced concrete structures result if there is slipping between the steel and the concrete. Two general considerations respecting bond strength must be kept in mind: the anchorage or length of embedment of rods and the rate at which stress passes from the concrete to the rod.

A simple case of anchorage is that of the reinforcement of the cantilever beam shown in Fig. 7-28 projecting from a supporting column. The length of embedment L must be such that the resistance to pulling out, developed with the allowable bond stress, equals or exceeds the total stress P in the rod at the face of the column. Then if u is the allowable unit stress in bond, if Σo is the total perimeter of the stressed bars, if D is the diameter of round or side of square bar, and if f_s is the unit tension in steel, there results as a general expression

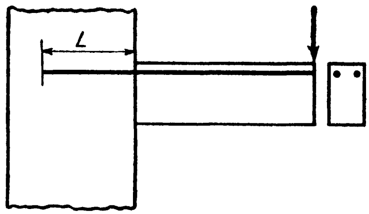


FIG. 7-28

$$u \cdot \Sigma o \cdot L = P \qquad [7\text{-}4]$$

and

$$u \times 4D \times L = f_s \times D^2$$

for a single square bar, and

$$u \times \pi D \times L = f_s \times \tfrac{1}{4}\pi D^2$$

for a round bar, giving for both rounds and squares

$$L = \frac{f_s}{4u} D \qquad [7\text{-}4a]$$

It is suggested that the student note that the relation is the same for both round and square bars and make no attempt to learn the formula, but work it out for a square bar of size D on each occasion as needed.

Another way to secure anchorage is to hook the end of the bar. The 1940 J.C. Report (Art. 828) recommends that the bar be bent in a full semi-circle with a radius of bend of not less than three diameters and with an extension at the free end of at least four bar diameters. These proportions are fixed on to insure that the additional length of bar provided by hook and extension be enough to develop the required anchor-

age by uniform bond stress over this length and also to insure that this anchorage be developed without bringing excessive compression on the concrete under the hook. In this country short, right-angle bends have been much used but they are ineffective because the concrete tends to be crushed and split by the excessive bearing and thus permits the bar to pull loose at low stress. The most effective hook is useless, however, if the mass of concrete in which it is embedded is too small to resist the stresses brought upon it. The question of proper length of embedment arises wherever there is stressed steel in concrete. Whether the stress be tension or compression a rod must extend beyond any stated point of stress a distance sufficient to develop in bond the total stress there existing. The bearing of the end of a rod on concrete is always considered to be negligible. Good practice limits the capacity of a hook to 10,000 psi stress in the anchored bar. Arts. 826 to 830 of the J.C. Report should be studied carefully in connection with these matters.

The complex interaction of concrete and reinforcement in beams is such that we turn to tests rather than to theoretical analysis for the final answer as to proper procedures and limitations. For example, it is difficult to justify the action of vertical stirrups by theory since observation fails to show elongations at all in the order of those theoretically demanded, but tests show conclusively that stirrups are very effective and that beams so reinforced can carry large increments of diagonal tension over that possible without stirrups. Similarly, tests show that beams with well-anchored reinforcement can carry safely largely increased bond stresses. Bond resistance increases with the compressive strength of the concrete as does also the shear and diagonal tension resistance. The effectiveness of diagonal tension reinforcement is particularly dependent upon thorough bond between steel and concrete and, accordingly, web steel is required to be securely anchored.

Example 7-22. Investigate the adequacy of the bond and anchorage of the reinforcement of the beam of Ex. 7-20 and Fig. 7-26. Assume deformed bars of structural grade and $f'_c = 3000$ psi.

Solution. (*a*) *Bond.* At the edge of the support the rate of increase of the horizontal shear is nearly $105 \times 12 = 1260$ lb per in. of length of beam. This is the total bond stress acting on four $\frac{3}{4}$-in. rounds which have a skin area of $4 \times \frac{3}{4} \times \pi = 9.4$ sq in. per in. The bond stress accordingly is $^{1260}/_{9.4} =$ 134 psi, which is less than the permissible limit for deformed bars, $0.05f'_c =$ 150 psi (J.C. 878). Plainly, further investigation of the bond stress for the main tension steel is not needed.

(*b*) *Anchorage. Main Tension Steel, Straight Lower Layer.* Since the sketch is closely to some unspecified scale it is plain that these four $\frac{3}{4}$ in. round bars have about 3 in. of anchorage beyond the center line of the support, sufficient to develop $(3 \times \frac{3}{4} \times \pi)150 = 1060$ lb of pull at the point where theoretically the rods just start to be stressed. Just how much anchorage is

here required is debatable. The Joint Committee requires that one-third of the positive steel be extended within the supports sufficiently to develop one-half the allowable stress in the bars (J.C. 829*d*). Here 4 of 8 bars extend about 9 in. beyond the support, sufficient to develop $4(9 \times \frac{3}{4} \times \pi)150 =$ 12,800 lb. The anchorage should be required to develop one-sixth of the capacity of all the rods, $\frac{1}{6} \cdot (8 \times 0.44)20{,}000 = 11{,}700$ lb. Special anchorage is not required.

(*c*) *Anchorage. Main Tension Steel. Bent Rods.* The diagonal tension stress in these bars is very low, 3600 psi in the end pair and 5300 psi in the second pair. The second set requires a length of embedment above the middle of the beam of

$$L = \frac{f_s}{4u} D = \frac{5300}{4 \times 150} \times 0.75 = 7.0 \text{ in.}$$

Plainly both sets extend sufficiently far above the neutral surface.

(*d*) *Anchorage. Stirrups.* Since 3 stirrups are used where less than 2 are required the unit stress in each is less than $\frac{2}{3} \times 16{,}000 = 10{,}700$ psi. The hook alone will develop 10,000 psi (J.C. 828) and the length of stirrup from mid-depth of beam to center of hook will add about $150 \times 10 \times \pi \times \frac{3}{8} =$ 1770 lb, or $\frac{1770}{0.11} = 16{,}000$ psi, a total well within requirements.

Experience has shown that if the active tension bars are anchored at the ends, as by suitable semi-circular hooks, bond failures are much less likely to occur. All codes permit considerably higher unit bond stresses for this special anchorage, as it is called. (J.C. 878, Table 7, and 826–830.) The reason can readily be seen by referring to Fig. 7–29*a*, which shows a simple rectangular reinforced concrete beam having the bars hooked at each end and with the concrete removed from the bars for most of the length. In the theory of bond stress just developed assumption was made of perfect adhesion of concrete to steel so that the entire increment of bending moment (the external shear) was transmitted from concrete to steel within the differential lengths *ds*. In Fig. 7–29*a* this is manifestly impossible, as the steel and concrete are in contact only near the ends of the beam. Yet such a beam will carry a load, the action being, perhaps, somewhat as indicated in Fig. 7–29*b*.

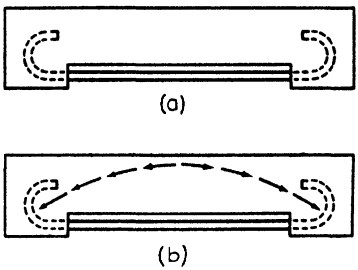


FIG. 7–29

The higher bond values for specially anchored bars are in recognition of a combination of the condition of ordinary bond as previously discussed, with the condition of hooked ends shown in Fig. 7–29.

It will be noticed in Fig. 7–29*b* that the tension in the bars is constant from the time they leave the embedment at one end until they enter it at the other.

When computing the point at which tension bars may be bent up (see Fig. 7–18) in beams where high bond stresses are permitted by reason of special anchorage, care should be taken to carry the bars farther along the tension side than this theoretical distance.

7-17. Summary. In this chapter simple cases have been discussed of flexure, shear, and bond in reinforced concrete beams of rectangular or tee-section with or without compressive reinforcement. This by no means completes the study of beams, and in Chapters XIII and XIV additional details are considered. Before proceeding with beams it is well to study the action of columns as presented in Chapter VIII.

Problems. A wealth of problem material is available in Chapter XVII which covers slabs and all types of beams. The abbreviated form of computation there used affords excellent practice in reestablishing the detailed figures, with frequent opportunity for checking. Chapter XIII explains thoroughly the methods used in Chapter XVII, should any questions arise.

CHAPTER VIII

COMPRESSION MEMBERS

8–1. The common type of reinforced concrete compression member has a circular or rectangular concrete section with a row of rods, parallel to the longitudinal axis of the piece, set 2 or $2\frac{1}{2}$ in. back from the surface all around the perimeter. These main reinforcing bars are held in place either by being wired to a series of encircling hoops or ties (made of $\frac{1}{4}$ or $\frac{3}{8}$ in. round material, spaced 8 to 12 in. apart), or to a closely spaced spiral (properly a helix) of steel wire.

The elastic theory, on the basis of which all reinforced concrete column design has proceeded until recently, assumes, as we have already seen, that concrete is a perfectly elastic material for the range of stresses met with in construction; that there is no slippage between the steel and the concrete, and accordingly that in a centrally loaded column without bending, the steel and concrete deform equally and their unit stresses are to each other as their moduli of elasticity. Later we shall consider the effect of shrinkage and plastic flow upon column action; these phenomena completely overshadow the effect of the variation of the concrete stress-strain curve from a straight line. The Building Regulations of the A.C.I., adopted in 1928, designed columns according to the elastic theory, using for spiral columns a fiber stress which varied with the amount of reinforcement — a method of taking account of flow and shrinkage, as will be seen later. The J.C. Report of 1940 and the 1941 A.C.I. Code propose design methods for columns which take account of plasticity and shrinkage.

According to the elastic theory the vertical reinforcement with either ties or spirals acts in exactly the same manner, deforming the same as the surrounding concrete as the column shortens under load. The action of the ties is to bind the rods to each other and into the mass of concrete in such a way that they will not buckle and cause the failure of the column. Owing to the shrinkage of the concrete in setting, the longitudinal reinforcement has a heavy initial stress before any load comes on the column and the function of the ties is therefore very important. This task is more efficiently performed by the spiral which also serves to restrain the lateral deformation of the enclosed concrete core that follows upon its shortening. In consequence the spiral column is a

much tougher and more dependable member than the tied column. However, the spiral does not come into active service until the load upon the column passes the elastic limit, so most authorities consider it improper to count directly upon the increased strength that its use affords. The spiral greatly increases the ultimate compressive strength and the resistance to shear.

Most reinforced concrete columns are so short that their ultimate strength is not limited by any tendency toward bending or buckling, their length being ordinarily less than 10 times their least lateral dimension. When they are more slender the working stresses must be reduced from those allowable on short columns of the same cross section (J.C. Art. 858).

8-2. Design of Tied Columns. The fundamental principles of column design are illustrated by the following examples.

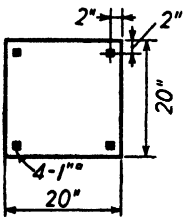


FIG. 8-1

Example 8-1. What are the fiber stresses in the column shown in Fig. 8-1 under an axial load of 290,000 lb? $n = 10$.

Solution. The steel in the column is equivalent to $10 \times 4 = 40$ sq in. of concrete, which makes a net increase of 36 sq in. (Compare Ex. 7-9, page 82). The stress in the concrete equals 290,000/436 = 664 psi, and that in the steel is $10 \times 664 = 6640$ psi.

Example 8-2. What is the allowable load on the column of Ex. 8-1 if $f_c = 540$ psi and $n = 10$?

Solution. Transformed area equals $400 + (10 - 1)(4) = 436$ sq in. Allowable load then is $436 \times 540 = 235{,}000$ lb.

Example 8-3. Design a column to carry a load of 88,000 lb. $f_c = 540$ psi. $n = 10$.

Solution. The total area of concrete required for the transformed section is 88,000/540 = 163 sq in. A 12 by 12 in. section furnishes 144 sq in., leaving 19 sq in. as the excess area furnished by transforming the steel area to equivalent concrete; that is, $(n - 1)$ or 9 times the steel area equals 19 sq in. The reinforcement required, accordingly, is 19/9 = 2.11 sq in. If the cross-section and steel first chosen are unsatisfactory for any reason further trials must be made. There is no direct road to a final design without the use of tables or diagrams.

In columns exposed to fire hazard the steel must be protected by at least 1½ or 2 in. of concrete and the greater portion of this protective cover is often not counted on in computing the strength of the column. The above examples have considered the gross area to be effective, and this is permitted by the Joint Committee. Computations of spiral reinforced columns would not differ from the preceding except that some specifications require that the effective area be taken as that of the concrete inside the spiral, which is itself covered by 1½ in. of concrete.

The value of n is taken the same for columns as for beams, that is, somewhat higher than the actual given by test specimens at low working load. This takes account of the fact that the value of n increases with increasing stress, thus throwing a larger and larger proportion of the load on the steel.

8–3. Shrinkage and Plastic Flow. *Shrinkage.* The effect of shrinkage upon the stresses in a reinforced concrete member may be learned by consideration of Fig. 8–2 which shows a member of unit length, of cross-sectional area A, symmetrically reinforced with steel in amount pA. If the piece were without reinforcement the shortening or shrinkage would be s but the action of the steel reduces the actual change of length to s_a. It is evident, therefore, that the interaction of the two materials results in a compressive stress in the steel and a tensile stress in the concrete, the two stresses being equal in magnitude for equilibrium. The figure identifies the strain effects related to these two stresses. We may write $f_cA_c = f_c(1 - p)A = f_spA$, which shows that the two unit stresses are related as $f_c = pf_s/(1 - p)$.

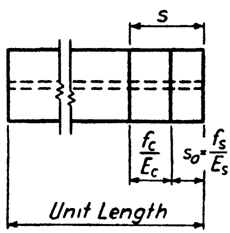


FIG. 8–2

In order to learn the relation of s_a to s we write

$$s = \frac{f_c}{E_c} + \frac{f_s}{E_s} = \frac{npf_s}{E_s(1 - p)} + \frac{f_s}{E_s}$$

$$= s_a \frac{1 + p(n - 1)}{1 - p} \qquad \text{(Note Fig. 8–2.)}$$

and

$$s_a = \frac{s(1 - p)}{1 + p(n - 1)},$$

The unit tension in the concrete equals

$$f_c = (s - s_a)E_c = sE_c \frac{pn}{1 + p(n - 1)} \qquad [8\text{–}1]$$

The unit compression in the steel equals

$$f_s = s_aE_s = \frac{sE_s(1 - p)}{1 + p(n - 1)} \qquad [8\text{–}2]$$

The use of these relations is attended with much uncertainty because of the impossibility of knowing at all exactly the value of the shrinkage coefficient s. Note that s is given in inches per inch;

a fair value is perhaps 0.0004 or more* for protected members in buildings.

Plastic flow acts to reduce the stress set up in the concrete by shrinkage and this may be allowed for by using an increased value of n in the computations, as demonstrated in the paragraphs below.

To cover the effect of flow in reducing shrinkage Glanville† states that n is generally about 15 and will usually lie between 10 and 20. Other investigators have used a much larger value.**

Plastic Flow.‡ If a piece of concrete is held at constant strain in a testing machine, for the reading of extensometers for example, the scale beam falls; the load required to maintain this strain becomes progressively less with time. If a constant load is maintained on a piece of concrete there is a slow increase in the initial deformation. Consideration of these two examples of plastic flow makes it plain that in a reinforced concrete column there must be a progressive transfer of stress from concrete to steel, a process which lasts for perhaps 6 years before equilibrium is reached. This phenomenon is in addition to the volume changes of concrete with changing temperature and changing moisture content.

The stress effect of plastic flow may be approximated analytically as outlined below. In this derivation the modulus of elasticity of concrete is assumed to be constant although it actually increases with the age of the concrete.

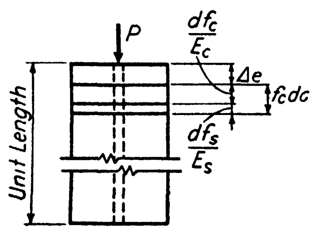


FIG. 8-3

Fig. 8-3 shows a block of concrete with a central reinforcing rod undergoing compression due to an applied load P. Upon application of the load the column undergoes an elastic shortening Δe. With the passage of time the column shortens still further owing to plastic action, which is progressive, increasing with age. Use the following notation:

c = the plastic flow of a unit length of plain concrete under unit stress from the time of loading to any given time under consideration.

* C. T. Morris, "Effect of Plastic Flow and Volume Changes in Design," *Journal, A.C.I.*, Vol. 33, 1937, p. 123.

† Quoted by C. S. Whitney in *Journal, A.C.I.*, Vol. 28, 1932, p. 500.

** For example, Turneaure and Maurer in Principles of Reinforced Concrete Construction, 4th ed., p. 351, where n is taken as 45 for this case.

‡ Professor J. R. Shank, in *Journal, A.C.I.*, Vol 31, Sept.-Oct., 1935, p. 72, gives a thorough treatment of shrinkage and plastic flow. This subject is not yet sufficiently clarified and understood that mathematical treatment is universally approved. See remarks by Prof. Probst on p. 191, *Journal, A.C.I.*, Vol 31, 1935.

dc = the plastic flow of a unit length of plain concrete for a given short time interval dt due to a unit stress intensity.

f_c = the intensity of stress in the concrete at the time interval dt due both to the applied load and the effects of plastic flow up to that time, and assumed constant through the small interval dt.

If the block in Fig. 8-3 were of plain concrete it would shorten plastically an amount $f_c dc$ in addition to the original elastic deformation. Assuming full adhesion between the concrete and steel, this flowing concrete pulls against the resisting steel and thus shortens less than the full plastic amount.

Let df_c (with proper algebraic sign included) be the *change* of stress in the concrete and df_s be the *change* of stress in the steel. The unit changes of length of concrete and steel are df_c/E_c and df_s/E_s respectively. Then,

$$f_c dc + \frac{df_c}{E_c} = \frac{df_s}{E_s} \qquad [a]$$

Also, the total change in load carried by the steel must equal the total change in load carried by the concrete

$$+A_s df_s = -A_c df_c \qquad [b]$$

Solving equations a and b by eliminating df_s gives

$$\frac{df_c}{f_c} = \frac{-dc}{\dfrac{A_c}{A_s E_s} + \dfrac{1}{E_c}} \qquad [c]$$

Integrating:

$$\log_e f_c = \frac{-c}{\dfrac{A_c}{A_s E_s} + \dfrac{1}{E_c}} + K \qquad [d]$$

When $c = 0$, $f_c = f_{co}$, the original concrete stress due to elastic action only, and $K = \log_e f_{co}$. Hence,

$$\log_e f_c = \frac{-c}{\dfrac{A_c}{A_s E_s} + \dfrac{1}{E_c}} + \log_e f_{co} \qquad [e]$$

or

$$\frac{f_c}{f_{co}} = e^{\frac{-c}{A_c/A_s E_s + 1/E_c}} \qquad [f]$$

and

$$f_c = \frac{f_{co}}{e^{\frac{c}{A_c/A_s E_s + 1/E_c}}} = \frac{f_{co}}{e^{\frac{E_s cp}{p(n-1)+1}}} \qquad [8\text{-}3]$$

since $E_s/E_c = n$; $A_s = pA$; $A_c = A(1 - p)$.

Let f_s equal actual unit steel stress combining elastic and plastic effects: f_{so} equal original steel stress from elastic action only; Δf_c equal total change in concrete stress due to plastic flow (+ being compression and − tension), and Δf_s equal total change in steel stress due to plastic flow (algebraic signs having same significance as before).

$$f_s = f_{so} + \Delta f_s \qquad [g]$$

$$-\Delta f_c = f_{co}\left(1 - \frac{1}{e^{\frac{c}{A_c/A_sE_s + 1/E_c}}}\right) \qquad [8\text{–}4]$$

$$\Delta f_s = -\Delta f_c \frac{A_c}{A_s}$$

and since $f_{co} = f_{so}/n$ we have

$$\Delta f_s = f_{so}\frac{A_c}{nA_s}\left(1 - \frac{1}{e^{\frac{c}{A_c/A_sE_s + 1/E_c}}}\right) \qquad [8\text{–}5]$$

The final stresses may be approximated by using an increased value of n in the ordinary formulas or in the application of the method of the transformed section, a fact indicated by examination of equation 8–3. If this is rewritten and the denominator is expanded into a series $\left(e^x = 1 + \frac{x}{\underline{|1}} + \frac{x^2}{\underline{|2}} + \cdots\right)$, and the first two terms taken, the result is:

$$f_c = \frac{f_{co}}{1 + \cfrac{c}{\cfrac{A_c}{A_sE_s} + \cfrac{1}{E_c}}} = \frac{f_{co}\left(A_c + \frac{E_s}{E_c}A_s\right)}{A_c + \left(\frac{E_s}{E_c} + cE_s\right)A_s} = \frac{P}{A_c + n_rA_s} \qquad [8\text{–}6]$$

The numerator represents the original elastic concrete stress multiplied by the transformed area with $n = E_s/E_c$; this equals the load on the column. The denominator represents the transformed area, using a modified n, called n_r. This combines elastic and plastic effects, with $n_r = E_s/E_c + cE_s$. See equation 8–6. If $c = 0.000,001$ (a common value), $E_s = 30,000,000$ psi and $E_c = 3,000,000$ psi, $n = 10$ and $n_r = 10 + 30 = 40$. The effect of plastic flow may be approximated by increasing the value of n from 10 to about 40* and using the same method as for purely elastic deformation.

* Glanville, Studies in Reinforced Concrete III, p. 24, indicates that the maximum error in this approximation will be about 23 per cent at age 1¾ to 2 years, and is usually much less.

Professor Davis applied the term "modulus of resistance" to a decreased value of E_c to combine the effects of elastic and plastic deformation. The modulus of elasticity of concrete is $E_c = 1/e$ and $n = eE_s$, e being the strain for a unit value of the stress. We may call the modulus of plastic flow the ratio $1/c = E_f$ and $n_f = cE_s$. The term modulus of resistance as used by Professor Davis is $R = 1/(e + c)$ and $n_r = (e + c)E_s = n + n_f$. If $c = 0.000,001$ and $e = 1/3,000,000 = 0.000,000,33$, $n_r = (0.000,001,33)\ 30,000,000 = 40$ as before.

The effect of shrinkage may be included by taking a still larger value of n,* or the procedure suggested above may be followed and the results added to those for elastic and plastic action under load.

Example 8–1a. (See Ex. 8–1.) Estimate the effects of shrinkage, plastic flow, and elastic deformation on a 20 by 20 in. tied column reinforced with four 1 in. square rods under a total load of 290,000 lb (145,000 lb dead and 145,000 lb live). Let $E_c = 3,000,000$ psi, $c = 0.000,001$, and $s = 0.000,4$.

Solution. Shrinkage. Analyze shrinkage first because in ordinary construction shrinkage occurs before much of the load is applied.

	f_c	f_s
(Here $1-p$ was taken as unity.)		
$p = 4/400 = 0.01 \quad s_a = 0.000,4/(1 + 0.01 \times 10) = 0.000,364$		
$f_c = 0.000,036 \times 3,000,000 =$	108 (T)	
$f_s = 0.000,364 \times 30,000,000 =$		10,920 (C)
Elastic Deformation under combined dead and live loads (see Ex. 8–1).	664 (C)	6,640 (C)
Plastic Flow occurs only for the dead load increased possibly by a small portion of the live load to represent permanent furniture or fixtures. In this case, since the dead load is one-half of the total, the concrete stress under dead load only will be 332 psi. From equation 8–4		
$\Delta f_c = 332\left[1-\dfrac{1}{e^{\frac{0.000,001}{(396/4 \times 30,000,000) + 1/3,000,000}}}\right] =$	82 (T)	
$\Delta f_s = \dfrac{82 \times 396}{4} =$		8,100 (C)
Totals	474 (C)	25,660 (C)

If it is desired to determine approximately what value of n_r will produce comparable results note the following.

* See Turneaure and Maurer, Principles of Reinforced Concrete Construction, John Wiley & Sons, Inc., 4th ed., 1935, p. 187. The approximate increase in n to cover shrinkage is given as sE_s/f_c, s being the shrinkage factor, about 0.000,4 on the average.

For the combination of elastic and plastic deformation without shrinkage the values are $f_c = 582$ psi and $f_s = 14{,}740$ psi. Trying $n_r = 30$:

$$\begin{aligned} A_c &= 20 \times 20 = 400 \\ (n-1)A_s &= 29 \times 4 \;\; = \underline{116} \\ \text{Total} & \qquad\qquad\quad\;\, 516 \end{aligned}$$

$f_c = 290{,}000/516 = 562$ psi $\quad f_s = 30 \times 562 = 16{,}860$ psi

To determine a value of n_r representing the combination of all three effects try $n_r = 50$:

$$\begin{aligned} A_c &= 20 \times 20 = 400 \\ (n-1)A_s &= 49 \times 4 \;\; = \underline{196} \\ \text{Total} & \qquad\qquad\quad\;\, 596 \end{aligned}$$

$f_c = 290{,}000/596 = 487$ psi $\qquad f_s = 50 \times 487 = 24{,}400$ psi

Factor of Safety. Tests made at the University of Illinois and at Lehigh University under the auspices of the American Concrete Institute* indicate that the ultimate strength of a reinforced concrete column may be taken as the compressive strength of the concrete, 85 per cent of the cylinder strength (f_c'), plus the yield point strength of the steel.

Example 8–2a. Same as Ex. 8–2, assuming that a factor of safety of 4 is desired. The steel here used is intermediate grade with a yield point of 40,000 psi: $f_c' = 3{,}000$ psi.

Solution. Ultimate load:

$$\begin{aligned} 396 \times 3{,}000 \times 0.85 &= 1{,}010{,}000 \text{ lb} \\ 4 \times 40{,}000 &= \underline{\;\;160{,}000\;\;} \\ & \;\;\; 1{,}170{,}000 \text{ lb} \end{aligned}$$

Dividing by the factor of safety, 4, gives the capacity of the column as 293,000 lb.

For design, J.C. 855 prescribes for axially loaded tied columns a working stress of $0.18f_c'$ for the concrete and 32 per cent of the yield-point strength for the steel, and recommends basing the concrete capacity on the gross area of the section, not deducting for the space occupied by the steel.

Example 8–2b. Same as Ex. 8–2*a* above except that the Joint Committee rules are to be followed.

Solution. Design load:

$$\begin{aligned} 400 \times 3{,}000 \times 0.18 &= 216{,}000 \text{ lb} \\ 4 \times 40{,}000 \times 0.32 &= \underline{\;\;51{,}000\;\;\;\;} \\ \text{Total} \;\; & \;\;\; 267{,}000 \text{ lb} \end{aligned}$$

* Proc. A.C.I., Vol. 29, 1933, p. 275. The entire report of this investigation should be studied by the student.

Comparison with Ex. 8–2*a* shows that this gives a factor of safety of about 4.4. Compare the statement of the Joint Committee in the first paragraph of Art. 851.

Problem 8–1. Same as Ex. 8–2*b* except that the reinforcement consists of 16 bars 1 in. square. Compute the factor of safety. Check answer with J. C. 851*a*, paragraph 1.

Ans. Load = 421,000 lb; F.S. = 3.9.

Problem 8–2. The design loads in Ex. 8–2*b* and Prob. 8–1 were computed with these working stresses: f_c = 540 psi; f_s = 12,800 psi. The ratio of the steel area to that of the whole cross section is 1 per cent in the first case; 4 per cent in the second, the Joint Committee limiting values. Compute the percentage variations in unit stresses in these two cases under the action of shrinkage and plastic flow. $n_r = 40$.

Ans.

	Ex. 8–2*b*	Prob. 8–1
f_c	−11%	−24%
f_s	+50%	+29%

Discussion. This result shows that shrinkage and plastic flow have a less severe effect upon steel stresses when the amount is relatively large. This is one reason for demanding the use of at least 1 per cent of steel and the reason why some engineers prefer, when possible, to use larger percentages. However, the lower percentages are the more economical. (The student should verify this and take particular note of the relative stresses in these examples and also of the fact that per unit volume steel costs perhaps 50 times as much as concrete.)

The acceptance of the 1940 J.C. Report will increasingly lead to the disuse of the elastic theory (Ex. 8–1, 2, 3) for axially loaded columns. This remains, however, the method for use with eccentric loads and consequent bending added to the direct column loading and also the simplest means for estimating the effect of flow and shrinkage, a matter usually of minor importance since the factor of safety is readily computed, as has been shown.

8–4. Spiral Columns. In 1903 Considère published tests of columns reinforced with longitudinal steel and spirals, on the basis of which he concluded that the steel in the spiral is 2.4 times as effective in increasing the strength of the column as that placed vertically. Several cities of the United States have adopted in their building codes spiral column formulas based upon this supposed effectiveness of the closely spaced hooping. Conservative practice is represented by the Joint Committee recommendations which take no direct account of the spirals.

Example 8–4. What is the allowable load on the column section shown in Ex. 8–7, according to Considère's theory? The spiral reinforcement consists of ⅜-in. rolled wire with a pitch of 2 in. Allowable f_c = 675 psi; $n = 10$.

Solution. In 1 ft of length of this column there are 12/2 = 6 turns of wire, making a total length of $6 \times \pi \times 16.63 \times 1/12 = 26.1$ ft. If this length were used as vertical reinforcement it would provide 26.1 bars, each with an area of 0.11 sq in., making a total of 2.87 sq in. This spiral area is assumed to

be 2.4 times as effective as an equal amount of vertical steel, so it is equivalent to $2.4 \times 2.87 = 6.9$ sq in., making the total area of reinforcement $6.9 + 6 = 12.9$ sq in. The transformed area is $\frac{1}{4} \times \pi \times 17^2 + (10 - 1)(12.9) = 343$ sq in. The allowable load is $675 \times 343 = 232{,}000$ lb.

Note that in this solution the cross-sectional area of the column was taken as that inside of the outside surface of the spiral, the core. As the load on the column approaches its ultimate value and the steel stress reaches the yield point, the shell of the column, the concrete outside the spiral, tends to spall off and throw all the load on the core and the reinforcement.

J.C. 854 recommends that a spiral column be designed for a working stress in the concrete of $0.225f'_c$, on the gross area of the column (the constant value 0.225 having been chosen to make due allowance for the presence of the longitudinal steel) and a steel working stress equal to 40 per cent of the yield-point strength. The Committee considered it permissible to take the full area of the cross section since it also specified that the spiral should be more than sufficient to equal the strength of the shell so, that should the shell spall, it could be considered replaced by the spiral.

Example 8-5. Column of Ex. 8-4 and Fig. 8-8. Compute the allowable load by the J.C. rules. Yield point of steel = 40,000 psi: $f'_c = 3{,}000$ psi.

Solution.

$$\begin{array}{lrl} \text{Concrete:} & 675 \times 314 = & 212{,}000 \text{ lb} \\ \text{Steel:} & 16{,}000 \times 6 = & \underline{96{,}000} \\ & & 308{,}000 \text{ lb} \end{array}$$

In order that the gross area be taken the Joint Committee lays down a requirement (Formula 10, Art. 854*d*), which is equivalent to saying that the strength supplied by the spiral shall be $12\frac{1}{2}$ per cent greater than that of the shell, assuming the spiral to be twice instead of 2.4 times as effective as an equal amount of longitudinal steel, and assuming the concrete of the shell to have an ultimate strength of $0.80f'_c$ the same as for tied columns.* The useful limit of stress in the spiral wire is set at 40,000 psi for the rolled rod used in this case. From Ex. 8-4 we know that this particular spiral placed longitudinally would supply 2.87 sq in. of reinforcement. Accordingly, we should have

$$2 \times 40{,}000 \times 2.87 \geq 112.5 \times 0.80 \times 3000(314 - 227)$$
$$230{,}000 < 235{,}000$$

The strength of the spiral is practically $112\frac{1}{2}$ per cent that of the shell so the design may be considered satisfactory in this point.

In practice it is, of course, easier to solve the equation given by the Joint Committee and determine the required spacing (pitch) of any given size of spiral wire directly.

The method of designing spiral columns endorsed by the 1928 Building Regulations for Reinforced Concrete* of the American Concrete Insti-

* Proc. A.C.I., 1933, Vol. 29, p. 281.

tute was that of the ordinary elastic theory as developed in Art. 8-2, which takes no account of the strength furnished by the spiral reinforcement, and employs a variable fiber stress,

$$f_c = 300 + (0.10 + 4p)f'_c \qquad [8\text{-}7]$$

where p is the steel ratio, or ratio of steel area to that of the core.

The reason for the variable f_c in the above equation merits examination. In the 1921 Proceedings, A.C.I., Mr. F. R. McMillan presented a paper, " A Study of Column Test Data," in which he analyzed the effects of shrinkage and plastic flow on column strengths. This marked the first serious consideration given the subject by American engineers and led directly to the elaborate series of column tests conducted by the Institute in 1929–1933.† From this study of test data Mr. McMillan concluded that the total deformation due to shrinkage and flow may be taken at about $m = 0.0005$, corresponding to an initial stress in the steel of 15,000 psi, to which must be added the usual nf_c in order to arrive at the final stress. The total load on a column may be expressed as

$$P = EmpA + f_cA[1 + (n - 1)p] \qquad [8\text{-}8]$$

which Mr. McMillan placed in this form for concrete with a 28-day compressive strength of $f'_c = 2000$ psi.

$$\frac{P}{A} = 333(1 + 29p) + 15{,}000p \qquad [8\text{-}9]$$

Fig. 8-4 shows the variation of f_s with changing steel ratios for a constant average unit stress in the column of 1000 psi, these curves being obtained from equation 8-8 with $f_s = Em + nf_c$. Even for the usual elastic value of n the steel stress for small amounts of steel becomes very high. Based on the total strain in the concrete the value of n is very large. Accordingly it would seem wise to use a low concrete stress for small percentages of steel and so assist in keeping the steel stress as low as possible, as is done in equation 8-9 and in the 1928 A.C.I. formula. Some engineers have been so much impressed by Mr. McMillan's evidence that they favor the consistent use of large steel ratios. The usual practice is to use the smallest amount permitted by the code for the sake of economy.

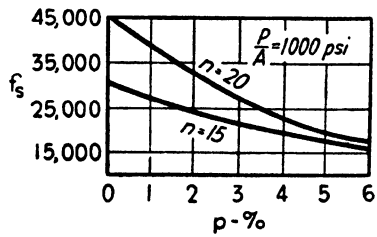


FIG. 8-4

† These tests are reported in Proc. A.C.I., Vols. 26, 27, 28, and 29. Together with the ensuing discussion these reports should be carefully studied.

Equation 8-9 is plotted in Fig. 8-5 in comparison with the 1924 and 1940 Joint Committee design recommendations.

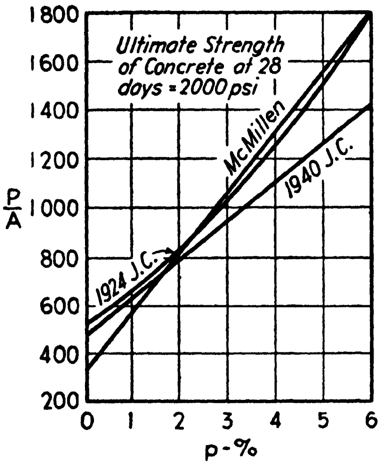


FIG. 8-5

Example 8-5a. Column of Ex. 8-5 and Fig. 8-8. Compute the allowable safe load by the 1928 A.C.I. Code.

Solution.

$p = 6.0$ sq in. of steel/227 sq in. of core $= 0.0265$

$f_c = [300 + (0.10 + 4 \times 0.0265)3000] = 918$ psi

$P = 227 \times 918(1 + 9 \times 0.0265) = 258{,}000$ lb

This code required the ratio for spiral steel to be at least $0.25p$, which in this case becomes 0.0066. The minimum spiral pitch using $\frac{1}{4}$ in. round wire is

$$\frac{12 \times \pi \times 17 \times 0.05}{227 \times 12 \times 0.0066} \text{ or } 1\tfrac{3}{4} \text{ in.}$$

8-5. Members Carrying Compression and Bending. When the load P on a homogeneous column is applied eccentrically on one of the axes of symmetry at a distance of e inches from the other axis, as shown

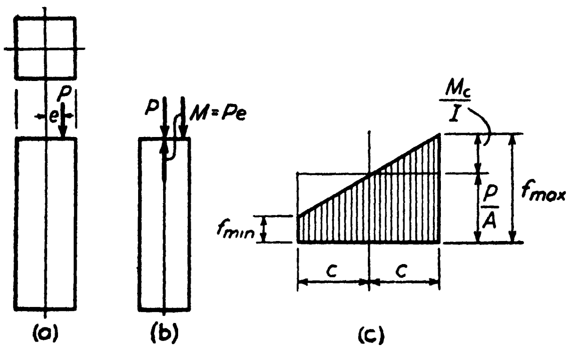


FIG. 8-6

in Fig. 8-6, it is equivalent to the same load applied axially, together with a moment of Pe lb-in. and the stress is found by use of the familiar relation

$$f = \frac{P}{A} \pm \frac{My}{I} = \frac{P}{A} \pm \frac{Pey}{I}$$

The application of this formula to the transformed section of a reinforced concrete column presents no difficulties if there is compression over the

whole section. Since concrete cannot carry high tensile stresses, where the eccentricity of the load is sufficient to cause a tension in excess of about 50 psi a modification of this method must be employed which is described in Ex. 8–8.

Example 8–6. The column shown in Fig. 8–7 carries a load, parallel to the column axis, of 210,000 lb which may be considered as applied at point indicated. What are the fiber stresses? $n = 10$.

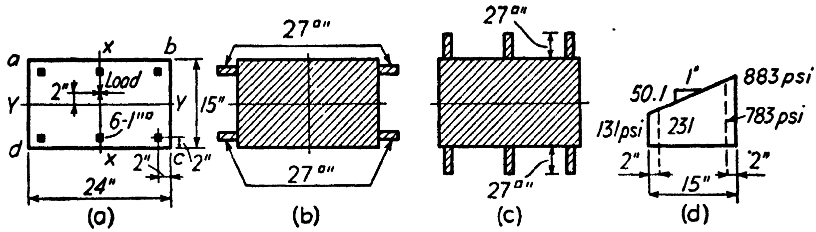


FIG. 8–7

Solution. The steel-concrete section may be replaced by its equivalent in concrete, as shown in Fig. 8–7*b*, the concrete in the wings acting at the same distance from the axis in the direction of the bending as the steel it replaces. If the arrangement of Fig. 8–7*c* were adopted, the transformed section would not be equivalent to the original, as far as resisting the given bending is concerned, since here the added wings have a greater leverage than the steel.

The transformed section and moment of inertia are calculated in the usual fashion, disregarding the small moment of inertia of the wings about their own axes.

$$\begin{array}{rclcrcl} 24 \times 15 &=& 360 & \times & 15^2/12 &=& 6750 \text{ in.}^4 \\ 9 \times 6 &=& \underline{54} & \times & (5\tfrac{1}{2})^2 &=& \underline{1630} \\ \text{Transformed area} &=& 414 & & \text{Transformed I} &=& 8380 \text{ in.}^4 \end{array}$$

Whence

$$f_c = 210{,}000/414 \pm 210{,}000 \times 2 \times 7.5/8380 = 507 \pm 376$$

$$= 883 \text{ psi for maximum stress}$$

$$= 131 \text{ psi for minimum stress}$$

giving the stress diagram shown. The maximum steel stress equals $783 \times 10 = 7830$ psi; the minimum, 2310 psi.

In Art. 861 of the J.C. Report is given a formula for determining the allowable unit concrete stress in a column subjected to bending which takes this form for a rectangular section:

$$f_c = f_a \frac{1 + 6\dfrac{e}{t}}{1 + C\left(6\dfrac{e}{t}\right)} \tag{8–10}$$

where f_a is the allowable average stress with no bending in the column
e is the eccentricity
t is the overall depth of the section
C equals the ratio of f_a to the permissible unit stress in flexure $= f_a/0.45f'_c$.

Computing f_a:

$$\begin{aligned}\text{Allowable load: } 675 \times 15 \times 24 &= 243{,}000 \text{ lb}\\ 16{,}000 \times 6 &= \underline{96{,}000}\\ &\ 339{,}000 \times 0.8 = 271{,}000 \text{ lb}\end{aligned}$$

$$\text{Transformed area} = 15 \times 24 + (10 - 1)6 = 414 \text{ sq in.}$$
$$f_a = 271{,}000/414 = 655 \text{ psi}$$

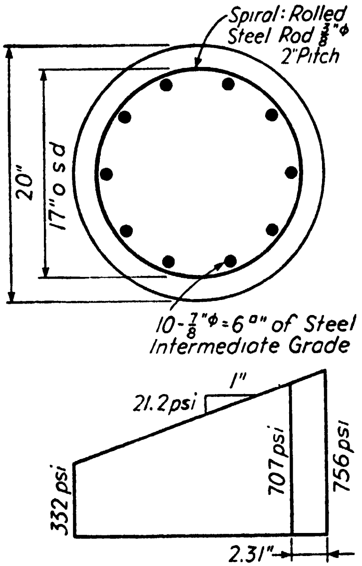


FIG. 8–8

Then

$$C = 655/(0.45 \times 3000) = 0.485$$

Accordingly,

$$f_c = 655 \frac{1 + 6 \times \dfrac{2}{15}}{1 + 0.485 \times 6 \times \dfrac{2}{15}}$$
$$= 655 \times \frac{1.8}{1.388} = 850 \text{ psi}$$

Example 8–7. What are the maximum stresses in the column shown in Fig. 8–8 under a direct load of 200,000 lb and a moment of 200,000 lb-in.? $n = 10$.

Solution. The area of the transformed section is 314* + 9 × 6 = 368 sq in. In computing the moment of inertia, consider the concrete that replaces the steel as a ring of 15⅜ in. mean diameter (17 − 2 × ⅜ − ⅞):

$$\text{20 in. circle}\quad I = \frac{\pi D^4}{64} = \frac{\pi D^2}{4} \times \frac{D^2}{16} = \text{Area} \times \frac{D^2}{16};\quad 314 \times \frac{20^2}{16} = 7850 \text{ in.}^4$$

$$\text{15⅜ in. ring}\quad I = \frac{\pi}{64}(D_1^4 - D_2^4) = \frac{\text{Area}}{16}(D_1^2 + D_2^2)$$
$$= \frac{\text{Area} \times D^2 \text{ mean}}{8} \text{ as limit;}$$
$$\text{i.e., } 54 \times \frac{(15\tfrac{3}{8})^2}{8} \qquad = \underline{1600}$$
$$I = \qquad = 9450 \text{ in.}^4$$

* The student will find it advantageous to familiarize himself with the easy slide-rule method of finding the area of circles. For the ordinary Mannheim rule, set the right-hand index of the B scale under the mark indicating $\pi/4 = 0.785$ on the A scale. Pace the runner cross-line on the diameter on the C scale and read the area on the A scale, thus solving Area $= \pi D^2/4$.

$$f_c = \frac{200{,}000}{368} \pm 200{,}000 \times \frac{10}{9450}$$

$$= 544 \pm 212 = 756 \text{ psi as maximum}$$

$$f_s = 10 \times 707 = 7070 \text{ psi}$$

The formula for maximum allowable unit stress is the same as for a rectangular section (Ex. 8-6 above), except that $6 \cdot e/t$ may be replaced approximately by $8 \cdot e/t$; this limit here is about 930 psi. The check is left to the student.

The solution of a circular column with tension over part of the section is shown in Ex. 8-10.

Example 8-8. Same column as in Ex. 8-6 with 110,000-lb load applied 5 in. from the centroid as shown in Fig. 8-9.

Solution. If the load on a homogeneous column acts on an axis of symmetry outside the middle third there is tension on the section. In this case the presence of the steel modifies the limiting boundaries of the area within which the load must act if compression only is to exist, but the load is so far without the middle third that preliminary investigation is hardly necessary.

To solve this problem consider a short section of the column *abcd*, shown in elevation in Fig. 8-9*b*, which is in equilibrium under the action of the forces shown acting upon it; the given 110,000 lb on the end *ab*, and the internal fiber stresses on the end *cd*. The diagram of stress variation over this end resembles those already met in beams, the neutral axis being at an unknown distance x from the compression face, and the extreme compressive fiber stress f_c also being unknown. These two unknowns may be found by solving the equations written by application of the two conditions of equilibrium of a coplanar system of parallel forces. The algebraic work is simplified by taking the center of moments on the line of action of the known force, as thus an equation results containing x only: $\Sigma M = 0$. Compare Figs. 8-9, *b* and *c*.

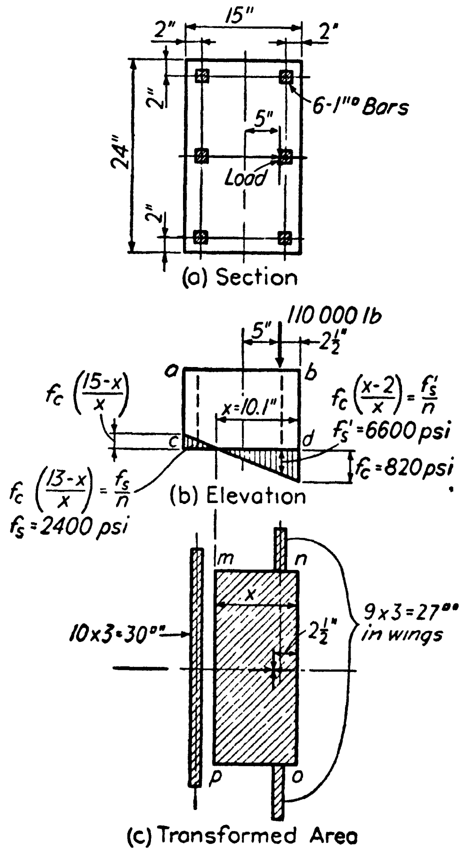


FIG. 8-9

$$-\begin{bmatrix}\text{Compression}\\ \text{on } mnop = \\ \frac{1}{2}f_c(24x)\end{bmatrix}\begin{bmatrix}\text{Arm}\\ \frac{x}{3} - 2.5\end{bmatrix} + \begin{bmatrix}\text{Compression}\\ \text{on wings} = \\ f_c\left(\frac{x-2}{x}\right)27\end{bmatrix}\begin{bmatrix}\text{Arm}\\ \frac{1}{2}\end{bmatrix}$$

$$+\begin{bmatrix}\text{Tension} = \\ f_c\left(\frac{13-x}{x}\right)30\end{bmatrix}\begin{bmatrix}\text{Arm}\\ 10\frac{1}{2}\end{bmatrix} = 0$$

giving

$$4x^3 - 30x^2 - 13\tfrac{1}{2}x + 27 - 4095 + 315x = 0$$

$$x^3 - 7.5x^2 + 75.4x = 1017$$

$$x = 10.1 \text{ closely}$$

Applying the condition $\Sigma V = 0$ gives:

$$\frac{1}{2}f_c(24x) + f_c\left(\frac{x-2}{x}\right)27 - f_c\left(\frac{13-x}{x}\right)30 - 110{,}000 = 0$$

or

$$121.2f_c + 21.6f_c - 8.61f_c = 110{,}000$$

$$f_c = \frac{110{,}000}{134.2} = 820 \text{ psi}$$

$$f_s = 10 \times 820 \times \frac{2.9}{10.1} = 2400 \text{ psi tension}$$

$$f_s' = 10 \times 820 \times \frac{8.1}{10.1} = 6600 \text{ psi compression}$$

The determination of the stress limit is similar to that in Ex. 8–6:

$$f_c = 655\,\frac{1 + \dfrac{6 \times 5}{15}}{1 + 0.485 \times 6 \times 5/15} = 1000 \text{ psi}$$

Since the bending is greater and the direct stress less than in Ex. 8–6 the allowable f_c is considerably higher.

It is very convenient to be familiar with the method of solving a case of direct compression and bending by the transformed section as plots given in most texts are of limited range and sections are often met, particularly in arch design, which require a careful solution.

Frequently the moment applied to a column is not in the plane of a principal axis. When the magnitude of such a moment is sufficient to cause any great amount of tension, the stresses may be computed from the transformed section. Graphic methods are given in various treatises.*

* Rich and Bigelow, "Stresses in Composite Structural Members," *Journal of the Boston Society of Civil Engineers*, Feb., 1926; M. S. Wolfe, "Graphical Analyses," in Hool and Johnson, Concrete Engineers' Handbook, page 406; Paul Anderson, "Square Sections of Reinforced Concrete under Thrust and Unsymmetrical Bending," Univ. of Minn., Bul. 41.

When there is compression over the whole section the familiar method of dividing the moment into components in the planes of the principal axes is used, as illustrated by the following example.

Example 8-9. Same column as that treated in Ex. 8-6. A load of 100,000 lb acts 1 in. from the Y axis and 3 in. from X in the quarter toward corner b. What is the fiber stress in each corner? $n = 10$.

Solution. This section is called on to carry a direct load of 100,000 lb, a moment of 100,000 lb-in. about axis YY and one of 300,000 lb in. about axis XX. The effect of each moment is determined independently as in Ex. 8-6. The transformed section for the 100,000 lb-in. moment is shown in Fig. 8-7b and that for the 300,000 lb-in. in Fig. 8-7c. Then

$$
\begin{aligned}
f_c &= \frac{P}{A} \pm \frac{M_1 y_1}{I_1} \pm \frac{M_2 y_2}{I_2} \\
&= \frac{100{,}000}{414} \pm 100{,}000 \times 1 \times \frac{7.5}{8380} \pm 100{,}000 \times 3 \times \frac{12}{20{,}880} \\
&= 242 \pm 90 \pm 172
\end{aligned}
$$

The maximum stress is 504 psi at b, and the minimum 20 psi tension at d. The amount of tension here found is so small that it may be assumed that the entire section is in action as assumed, without cracking of the concrete to destroy its ability to carry tension. Had this tensile stress been greater, say above 100 psi in this case (assuming tensile strength at one-tenth the compressive and using a safety factor of 3), it would have been necessary to assume a neutral axis, with only the steel acting on its tension side, and compute the resulting stress. Unless the correct axis has been guessed the results will not satisfy the necessary conditions but will indicate the change to make for a second try.* This is illustrated for a circular section in Ex. 8-10.

The solution of circular columns undergoing direct stress and bending for those cases where there is tension over part of the section are complicated, not because of anything different in theory but because of the difficulty of finding the properties of a segment of a circle. These can be obtained from formulas or by the use of curves. Solution by formula is shown below. The use of curves is considered in Chapter IX.

Example 8-10. Same column as in Fig. 8-8 with 40,000-lb load applied 12 in. from the center. Using the gross area of the section, determine maximum compressive stress on concrete. $n = 10$. (See Fig. 8-10.)

Solution. Collect the following formulas for properties of the segment of a circle (see sketch):

$$\text{Area} = \frac{r^2}{2}(2\alpha - \sin 2\alpha) = A$$

$$\bar{x} = \frac{2r^3 \sin^3 \alpha}{3A}$$

$$I_y = \frac{Ar^2}{4}\left(1 + \frac{2 \sin^3 \alpha \cos \alpha}{\alpha - \sin \alpha \cos \alpha}\right)$$

* This is illustrated in "A Simple Analysis for Eccentrically Loaded Concrete Sections," by L. G. Parker and J. H. Scanlon, in *Civil Engineering*, Oct., 1940, p. 656.

Trial and error is quicker than a direct algebraic solution. For first trial assume that the neutral axis coincides with the center line of the circle. ($\alpha = \pi/2$.) $A_c = \pi r^2/2 = \pi 10^2/2 = 157$ sq in. $x_c = 4r/3\pi = (4 \times 10) \div 3\pi = 4.24$ in. For the half-circle, $I_y = \pi r^4/8 = 0.393r^4$. The moment of inertia of the segment about an axis through its centroid parallel to Y–Y: $I_c = I_y - Ad^2 = 0.393r^4 - (8r^4/9\pi) = (0.393 - 0.283)r^4 = 0.110r^4$.

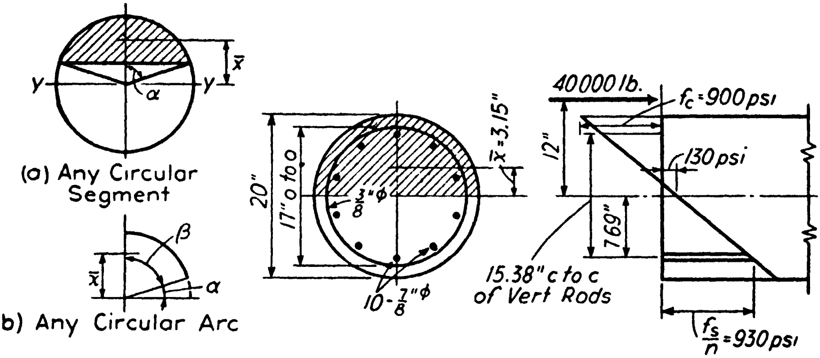


FIG. 8–10

The steel is transformed into a ring with its center at the center of the column. Properties of the combined section are computed:

Area	*Arm*	*Stat. Mom.*
157	× 4.24 =	666
54*	0	0
211		)666
	c of g =	3.15 in.

	Moment of Inertia	
Steel	$AD^2/8 = \frac{1}{8}(54 \times 15.38^2)$	$= 1600$ in.^{4}
	$Ad^2 = 54 \times 3.15^2$	$= 540$
Concrete	0.110×10^4	$= 1100$
	$157(4.24 - 3.15)^2$	$= 187$
		3427 in.^{4}

$$\frac{P}{A} = \frac{40{,}000}{211} = 190 \text{ psi}$$

$$\frac{My}{I} = \frac{40{,}000(12 - 3.15)(10 - 3.15)}{3427} = 710 \text{ psi compression}$$

$$\frac{My}{I} = \frac{40{,}000(12 - 3.15)(7.69 + 3.15)}{3427} = 1120 \text{ psi tension}$$

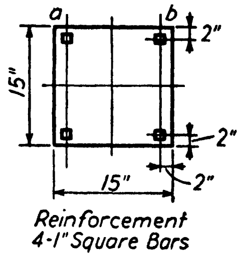
* This is approximate, replacing the steel by $(n - 1)$ times its area on tension as well as the compression side. To take account of the factor n in tension the student may add 3 sq in. to the area of the tension semi-circle, with a centroid $20/\pi$ distant from the diameter: the centroid of the whole stress-bearing area is shifted 3.02 in. instead of 3.15 in. and the moment of inertia is increased by about 70 in.^{4}; the resulting changes in unit stress are negligible, 10 to 20 psi. For checking this result the student will need these formulas relating to a circular arc (see Fig. 8–10*b*): Length $= r\beta$; $\bar{x} = r \cos \alpha/(\pi/2 - \alpha)$; $I_y = r^3(\pi/4 - \alpha/2 + \sin 2\alpha/4)$. In this case $\alpha = 0$. In applying these formulas to an annular ring the length of arc is replaced by the area of the given length of ring.

$$\frac{P}{A} \pm \frac{My}{I} = \begin{cases} 710 + 190 = 900 \text{ psi compression} \\ 1120 - 190 = 930 \text{ psi tension in the transformed concrete section at extreme fiber, which is at assumed ring of steel.} \end{cases}$$

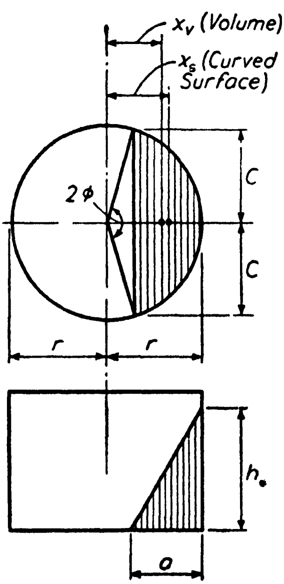
Combined stress on the center line (the assumed neutral axis) equals 130 psi tension, where it was assumed to be 0. Hence, the true neutral axis lies perhaps 1 in. from the center on the compression side. The above results are sufficient to indicate that the column is safe, the allowable compressive stress being about 1200 psi according to J.C. 861. If more exact results are desired use the method of Ex. 9-17 (page 142), or recompute, assuming the neutral axis located as suggested above. The resulting extreme unit stresses will be more closely consistent with the assumed neutral axis than before. When stresses and assumed neutral axis sufficiently agree a check may be made of the equilibrium of the section; the resultant total internal stress here should equal 40,000 lb and the moment of the total stress about the column center should equal 40,000 lb-ft. In this case the total internal compression considerably exceeds the actual axial load.*

Problem 8-3. At a certain section of this 15 by 15 in. column the compression in the concrete at all points on edge *ab* is 600 psi and the neutral axis is at the center of the column. $n = 10$. (*a*) What is the intensity of the load on the column? (*b*) Where is its line of action? (*c*) What is the bending moment at this section?

Ans. (*a*) 32,870 lb; (*b*) 7.94 in. from axis; (*c*) 21,700 lb-ft.



PROB. 8-3



* To check this the student will require the formulas giving the volume of an ungula (the wedge cut from a cylinder by a sloping plane), the curved external surface of the wedge, and the locations of the centroids of volume and surface. These are:

$$V = \frac{h}{3a}\left(c(3r^2 - c^2) - 3r^2(r - a)\phi\right)$$

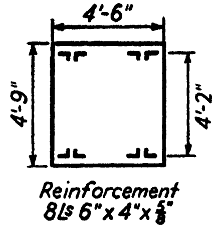
$$= \frac{hr^3}{a}\left(\sin\phi - \frac{\sin^3\phi}{3} - \phi\cos\phi\right)$$

$$x_v = \frac{1}{V}\left(\frac{hr^4}{12a}\right)(3\phi - 3\sin\phi\cos\phi - 2\sin^3\phi\cos\phi)$$

$$S = \frac{2rh}{a}[c - (r - a)\phi] = \frac{2hr^2}{a}(\sin\phi - \phi\cos\phi)$$

$$x_s = \frac{1}{S}\left(\frac{hr^3}{a}\right)(\phi - \sin\phi\cos\phi)$$

To obtain the tension in the ring representing the tension steel the surface area found above must be multiplied by the thickness of the ring.



PROB. 8-4

Problem 8-4. (*a*) The thrust in this reinforced concrete arch rib is 500,000 lb and it acts 31 in. above the axis. Determine the maximum stress in the concrete and in each layer of reinforcement. $n = 15$. (Note this arch was built about 1925 and this problem constitutes one of the design checks.)

(*b*) Is this design satisfactory by 1940 standards? Assume $f_c = 3000$ psi. $n = 10$.

Ans.

	(*a*)	(*b*)
Maximum compression in concrete	560 psi	659 psi
Maximum compression in steel	7200 psi	5560 psi
Maximum tension in steel	5640 psi	5460 psi
Allowable f_c		948 psi

Additional problems may be taken from the wealth of material in Chapter XVII. The abbreviated form of computation there used gives the student good practice in reestablishing computations with frequent opportunities for checking. The methods which are used if questions arise are explained in detail in Chapter XIII.

CHAPTER IX

FORMULAS, DIAGRAMS, AND TABLES

9–1. In routine office practice it is essential that all work be carried on with the greatest speed consistent with excellence and accuracy. The general methods that have been outlined are fundamental but their application is somewhat cumbersome. In order to save time all designers provide themselves with many data in convenient form, and especially with tables and diagrams for design which are based upon relations or formulas developed by means of the method of the transformed section. The principal use for the many formulas found in this and other textbooks on reinforced concrete is to compute tables and charts which give values for the various relations for different conditions. Since the formulas are easily written upon any diagram it is foolish to memorize any of these literal expressions whose relations cannot be easily visualized, especially as they are liable to subtle metamorphoses during periods of misuse and thus become a source of error. Their unthinking use tends to obscure the nature of the fundamental process being employed. In the absence of diagrams or tables recourse should be had to the method of the transformed section.

A large number of tables and diagrams have been published for use in reinforced concrete design. Many charts are good but some, of the more comprehensive type, are of very limited use on account of lack of precision and difficulty of reading. In general tables are quicker to use than diagrams and fewer mistakes are made in taking from them the desired data. Curves, however, offer many advantages as they show the changing values of the variables plotted and generally permit direct reading without interpolation. Limitations of space make it impossible to include a comprehensive range of designing data in this volume. Only a few typical curves and tables have been printed.

The formulas and notation employed here are those made standard by the Joint Committee.

9–2. Rectangular Beams with Tension Reinforcement. Notation:

f_s = tensile unit stress in steel
f_c = compressive unit stress in extreme fiber of the concrete
E_s = modulus of elasticity of steel
E_c = modulus of elasticity of concrete

$n = E_s/E_c$

M = moment of resistance or bending moment in general

b = breadth of beam

d = depth of beam to center of steel

A_s = cross-sectional area of tension steel reinforcement

k = ratio of depth of neutral axis to depth, d

j = ratio of lever arm of resisting couple to depth, d

z = depth from compression face to resultant of the compressive stresses

$jd = d - z$ = arm of resisting couple (hitherto called a)

p = steel ratio $= A_s/bd$ (often expressed as a percentage)

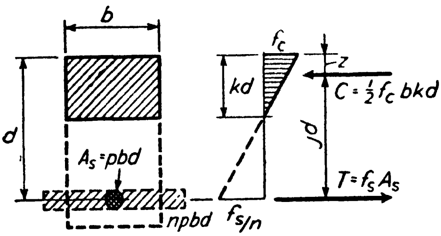


FIG. 9-1

By applying to the solution of the beam shown in Fig. 9-1 the method of the transformed section, a useful series of formulas may be derived. In terms of the steel the moment of resistance is

$$M = Tjd = (f_s pj)bd^2 \qquad [9\text{-}1]$$

and in terms of the concrete

$$M = Cjd = (\tfrac{1}{2} f_c kj)bd^2 \qquad [9\text{-}2]$$

The neutral axis may be located by finding the center of gravity of the transformed section:

$$bkd \times \frac{kd}{2} = npbd(d - kd)$$

whence

$$k = \sqrt{2pn + (pn)^2} - pn \qquad [9\text{-}3]$$

This formula is of service when investigating a beam where the steel area is known. When the fiber stresses are known, as in design, a simple proportion gives this result:

$$\frac{kd}{d} = \frac{f_c}{f_c + \dfrac{f_s}{n}}$$

$$k = \frac{1}{1 + \dfrac{f_s}{nf_c}} \qquad [9\text{-}4]$$

a formula in quite general use. For the purpose of this text it is best to return always to the original proportion instead of solving the formula and no table is given to aid in its use.

In design, where it is desired that both limiting stresses be realized simultaneously, the steel area must be such that the neutral axis lies at the level indicated by the expression just derived. It becomes necessary, therefore, to express the steel ratio p, which measures the steel area, in terms of the fiber stresses, which may be done by equating T and C (Fig. 9-1):

$$f_s pbd = \tfrac{1}{2} f_c bkd$$

$$p = \frac{f_c}{2f_s} k$$

The elimination of k by inserting its value from equation 9-4 gives:

$$p = \frac{1}{2} \frac{1}{\dfrac{f_s}{f_c}\left(\dfrac{f_s}{nf_c} + 1\right)} \qquad [9\text{-}5]$$

which is the value of the steel ratio for "balanced reinforcement."

For determining the lever arm of the resisting couple it is to be noted that

$$j = 1 - \frac{k}{3} \qquad [9\text{-}6]$$

On Fig. A-1 (in the Appendix) are curves for k and j (equations 9-3 and 9-6). Inspection of the curve for j shows that it varies little for wide variations of the steel ratio. Approximate values of j are therefore often used. A good approximation is

$$j = \tfrac{7}{8} \quad (k = \tfrac{3}{8})$$

which is exact for all values of

$$\frac{f_s}{nf_c} = 1.67$$

as occurs with

$$f_s = 20{,}000 \text{ psi}$$
$$f_c = 1200 \text{ psi}$$
$$n = 10$$

Another approximation is

$$j = 0.866$$

which is correct for the common combinations: 20,000–1350, $n = 10$; and 18,000–1125, $n = 12$.

The use of these approximate values greatly facilitates much design work.

The most used formula in reinforced concrete work is this:

$$M = Tjd = f_s A_s jd \qquad [9\text{-}7]$$

which is so simple and useful a relation that it should be remembered, not arbitrarily as a collection of letters, but in the form of a pictured relation; i.e., *the moment of resistance equals the tensile force of the resisting moment times the lever arm — the tensile force equaling the unit stress multiplied by the area stressed.*

The formulas for moment of resistance (9–1 and 9–2) are best combined for use as

$$M = Rbd^2 \qquad [9\text{-}8]$$

where $R = f_s pj$ or $\frac{1}{2} \cdot f_c kj$ according as the moment is expressed in terms of the steel or of the concrete. The quantity R, commonly called the coefficient of resistance, is seen to be a function of three variables, f_s or f_c, p and n (k being a function of p and n). Figs. A–2 and A–3 (in the Appendix) show the variation of R with the steel ratio p, n having the values of 8, 10, 12, and 15, a separate curve being drawn for each fiber stress desired. The steeper curves are for the values of f_s. Evidently the intersections of any two curves, as for $f_s = 20{,}000$ and $f_c = 1200$, should be at the value of p determined for these stresses by equation 9–5.

The following examples illustrate the use of formulas and Figs. A–1 and A–2 and are the same as those previously given in Chapter VII.

Example 9–1. (Same as Ex. 7–1, page 68.) What are the fiber stresses for this beam? $b = 10$ in.; $d = 20$ in.; $A_s = 2.5$ sq in.; $n = 10$; $M = 55{,}000$ lb-ft.

Solution. The first step in investigating a beam is to compute the value of the steel ratio.

$$p = 2.5 \div (10 \times 20) = 0.0125 \text{ or } 1.25 \text{ per cent}$$

Also

$$R = \frac{55{,}000 \times 12}{10 \times 20^2} = 165 \text{ psi}$$

The value of R is the second important criterion of the status of a beam. The designer quickly forms the habit of basing his judgments on these two factors. (See further discussion on page 237, paragraph *h.*) In this case the intersection on Fig. A–2 of the two ordinates just computed gives stresses of about 15,200 psi for f_s and 970 psi for f_c.

Example 9-2. (Same as Ex. 7-2, page 69.) What is the maximum moment that can be carried by the beam of Ex. 9-1 if the limiting fiber stresses are $f_s = 20{,}000$ psi and $f_c = 1350$ psi? $n = 10$.

Solution. As before, first determine the steel ratio, $p = 0.0125$. Enter Fig. A-2 with this value and follow vertically upward to the 20,000 line, where $R = 218$, with the 1350 line lying higher and thus giving a still larger value of R. Therefore, the allowable moment is

$$M = Rbd^2 = 218 \times 10 \times 20^2 \div 12 = 72{,}600 \text{ lb-ft}$$

Evidently the beam is limited by the steel, as a moment that stresses the concrete to 1350 psi causes a stress of about 21,000 psi in the steel, estimating from the plate the reading at the intersection of $p = 1.25$ per cent and $f_c = 1350$ psi. The intersection of the 20,000 and 1350 lines gives the value of p needed if these stresses are to be realized simultaneously, that is, 0.0136. If more steel is used the beam is overreinforced and the concrete limits.

Example 9-3. (Same as Ex. 7-3, page 69.) Design a beam to carry a moment of 55,000 lb-ft with stresses of $f_s = 20{,}000$ psi and $f_c = 1350$ psi. $n = 10$.

Solution. Fig. A-2 gives the information that for these stresses $p = 0.0136$ and $R = 236$. Accordingly, $bd^2 = (55{,}000 \times 12) \div 236 = 2800$; and for $b = 10$ this gives $d = 16.8$. For the steel, $A_s = pbd = 0.0136 \times 10 \times 16.8 = 2.29$ sq in.

Example 9-3a. (Same as Ex. 7-3*a*, page 70.) What is the steel area required for the beam of Ex. 9-3 if d is made 18 in. and $b = 10$ in.?

Solution. The actual value of R is $(55{,}000 \times 12) \div (10 \times 18^2) = 204$. On Fig. A-2, following horizontally from this figure, the 1350 line is reached first; but using the steel ratio there indicated would give a steel stress of about 26,000 psi. Farther to the right the intersection with the 20,000 line calls for $p = 0.0116$ and $A_s = 0.0116 \times 10 \times 18 = 2.09$ sq in.

Example 9-3b. (Same as Ex. 7-3*b*, page 71.) What is the steel area required for the beam of Ex. 9-3 if d is made 16 in. and $b = 10$ in.?

Solution. $R = (55{,}000 \times 12) \div (10 \times 16^2) = 258$. The intersection of this horizontal with the 20,000 line on Fig. A-2 indicates too high a concrete stress; its intersection with the 1350 line is at $p = 0.0180$, giving $A_s = 0.0180 \times 10 \times 16 = 2.88$ sq in.

In designing reinforced concrete beams keep in mind that *for any given value of the steel ratio* p *the neutral axis* k *is fixed and also the ratio of fiber stresses* f_s/f_c; *that in order to realize any given fiber stresses in a beam it is necessary to employ the proper value of the steel ratio.*

9-3. Tee-Beams. Notation as before except (see Fig. 9-2):

b = width of flange
b' = width of stem
t = thickness of flange

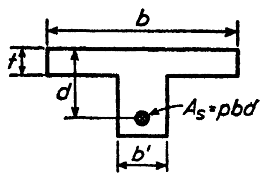


FIG. 9-2

When the neutral axis lies in the flange the formulas for rectangular beams are to be used. When the neutral axis lies in the stem the fol-

lowing are the standard approximate formulas which neglect the compression in the stem. Their derivation follows the general procedure of Art. 9-2. No exact formulas taking account of the compression in the stem will be given because no diagrams are available for their solution. The exact formulas are unnecessary as well as cumbersome. An exact analysis of a tee-beam may be made by the procedure explained in Ex. 7-8, page 80.

Approximate formulas, neglecting compression in the stem:

$$kd = \frac{2ndA_s + bt^2}{2nA_s + 2bt} \qquad [9\text{-}9]$$

$$z = \frac{3kd - 2t}{2kd - t} \times \frac{t}{3} \qquad [9\text{-}10]$$

$$jd = d - z \qquad [9\text{-}11]$$

$$f_c = \frac{Mkd}{bt\left(kd - \frac{t}{2}\right)jd} \qquad [9\text{-}12]$$

$$= \frac{f_s}{n} \times \frac{k}{1 - k} \qquad [9\text{-}13]$$

$$k = \frac{1}{\left(\frac{f_s}{nf_c} + 1\right)} \qquad [9\text{-}14]$$

The expression for kd (equation 9-9) can be written thus:

$$k = \frac{2pn + \frac{t^2}{d^2}}{2pn + 2\frac{t}{d}} \qquad [9\text{-}15]$$

Fig. A-4 (Appendix) gives curves for k for various values of pn. Note that the righthand termination of each curve marks where $k = t/d$, that is, where the neutral axis lies at the edge of the flange. For convenience the ratios of f_s/nf_c corresponding to the values of k are set down at the left.

By substituting in equation 9-11 the values of z and k from formulas 9-10 and 9-15 there results:

$$j = \frac{6 - 6\left(\frac{t}{d}\right) + 2\left(\frac{t}{d}\right)^2 + \frac{\left(\frac{t}{d}\right)^3}{2pn}}{6 - 3\left(\frac{t}{d}\right)} \qquad [9\text{-}16]$$

Curves for j from this equation are plotted on Fig. A-4.

For the moment of resistance in terms of f_s we may write:

$$M_s = f_s(pbd)jd = \left(\frac{f_s}{n}\right) pnjbd^2$$

$$= C_s \frac{f_s}{n} bd^2 \qquad [9\text{-}17]$$

where

$$C_s = pnj$$

Equation 9-12 is more easily understood when expressed as below — a convenient formula for moment of resistance in terms of f_c,

$$M_c = f_c\left(\frac{1 - \frac{t}{d}}{2k}\right)\frac{t}{d}jbd^2$$

$$= C_c f_c bd^2 \qquad [9\text{-}18]$$

where

$$C_c = \left(\frac{1 - \frac{t}{d}}{2k}\right)\frac{t}{d}j$$

From equations 9-17 and 9-18, Figs. A-5 and A-6 (in the Appendix) were prepared.

The following examples illustrate the use of these formulas and diagrams.

Example 9-4. (Same as Ex. 7-5, page 77.) Locate the neutral axis of this tee-beam. $b = 50$ in.; $b' = 10$ in.; $t = 6$ in.; $d = 20$ in.; $A_s = 2.5$ sq in.; $n = 10$.

Solution.

$$\frac{t}{d} = 6 \div 20 = 0.30$$

$$p = 2.5 \div (50 \times 20) = 0.0025 \qquad pn = 0.025$$

By entering Fig. A-4 with the above value of t/d and looking for the $pn =$ 0.025 line, a point to the right of and above the righthand ends of the k curves is located; this shows that t/d is greater than k and that this is essentially a rectangular beam. From Fig. A-1, k is found to be 0.2, making $kd = 4$ in.

Example 9-5. (Same as Ex. 7-6, page 78.) Same data as for Ex. 9-4 except that $t = 3$ in. The beam carries a total moment of 55,000 lb-ft. What are the maximum fiber stresses?

Solution.

$$p = 2.5 \div (50 \times 20) = 0.0025 \qquad pn = 0.025$$

$$\frac{t}{d} = 3.0 \div 20 = 0.150$$

From Fig. A-4,

$$j = 0.94 \text{ approximately, and } \frac{f_s}{nf_c} = 3.8$$

Then

$$f_s = \frac{55{,}000 \times 12}{2.5 \times 0.94 \times 20} = 14{,}000 \text{ psi}$$

$$f_c = 14{,}000 \div 38 = 370 \text{ psi}$$

Example 9-6. (Same as Ex. 7-7, page 80.) Beam of Ex. 9-4, except that $t = 3$ in. What is the maximum moment of resistance for limiting stresses of $f_s = 20{,}000$ psi and $f_c = 1350$ psi? $n = 10$.

Solution. As in all cases of investigation, compute pn and t/d, 0.025 and 0.150 respectively, as in the previous example. The ratio f_s/nf_c for 20,000 – 1350 is 1.48. For this beam, according to Fig. A-4, the values of pn and t/d fix the ratio of f_s/nf_c at 3.8, showing it to be limited by the steel, with a maximum f_c of $20{,}000/38 = 526$ psi.

Reading $j = 0.94$ from Fig. A-4 gives

$$M = 20{,}000 \times 2.5 \times 0.94 \times 20 \div 12 = 78{,}300 \text{ lb-ft}$$

or, from Fig. A-5 with $t/d = 0.150$ and $pn = 0.025$, $C_s = 0.024\pm$ and

$$M = 0.024 \times \frac{20{,}000}{10} \times 50 \times 20^2 \div 12 = 80{,}000 \text{ lb-ft}$$

Example 9-7. (Same as Ex. 7-8, page 80.) Beam of same dimensions as that of Ex. 9-4, except that $t = 3$ in., carrying 130,000 lb-ft. What is the steel area required? Stresses: $f_s = 20{,}000$ psi; $f_c = 1350$ psi; $n = 10$.

Solution. As in Ex. 7-8 it is possible to assume the lever arm and compute the steel area as 4.22 sq in. However, it is easy to make a slightly closer approximation than before by figuring $t/d = 0.150$ and inspecting the probable value of j as given by Fig. A-4, which may be estimated at about 0.94. Then

$$A_s = \frac{130{,}000 \times 12}{20{,}000 \times 0.94 \times 20} = 4.15 \text{ sq in.}$$

A revision by computing $pn = 0.0415$, reading the more exact value of $j = 0.935$, making $A_s = 4.17$ sq in., adds nothing materially to the precision of the solution. Since $f_s/nf_c = 2.6$ (Fig. A-4) for $pn = 0.0415$, $f_c = 20{,}000/(10 \times 2.6) = 770$ psi.

Frequently the problem is to design a tee-beam to carry a known moment at selected stresses. Table A-1 (in the Appendix), which applies only for balanced reinforcement, is useful for this purpose.

Example 9-7a. (Same as Ex. 7-8a, page 82.) Design a tee-beam for a moment of 130,000 lb-ft with $f_s = 20{,}000$ psi; $f_c = 1350$ psi; $n = 10$; $t = 3$ in., and $d = 20$ in.

Solution. From Table A-1 with $t/d = 0.15$, $R = 153$, then

$$b = \frac{130{,}000 \times 12}{153 \times 20 \times 20} = 25.5 \text{ in.}$$

Problem 9-1. Using the transformed section and standard notation, work out the complete derivations of all formulas listed in Art. 9-3.

9-4. Beams Reinforced for Both Tension and Compression. The formulas frequently used for this problem are approximate in that they make no allowance for the holes left in the concrete by the compression steel upon transforming the section, making the wings equal nA'_s instead of $(n - 1)A'_s$ (Fig. 7-15, page 82). The diagrams usually employed with them are advantageous chiefly for investigation of a given beam (Figs. A-7, A-8, A-9). For use in design Fig. A-10 is given, which is corrected for $(n - 1)A'_s$. The student will find it a simple matter to construct similar diagrams for other stresses.*

Example 9-8. (Same as Ex. 7-9, page 82.) What are the maximum fiber stresses in this beam? $b = 10$ in.; $d = 20$ in.; $d' = 2$ in.; $A_s = 3.0$ sq in.; $A'_s = 1.0$ sq in.; $n = 10$; $M = 80{,}000$ lb-ft.

Solution. Compute first the steel ratios, $p = 3.0/(10 \times 20) = 0.0150$, $p' = 1.0/(10 \times 20) = 0.0050$. Then $pn = 0.150$ and $p'n = 0.050$. Also $d'/d = 1/10$. Reference to Fig. A-8 shows that for these values $R = 0.205$. The stresses then are

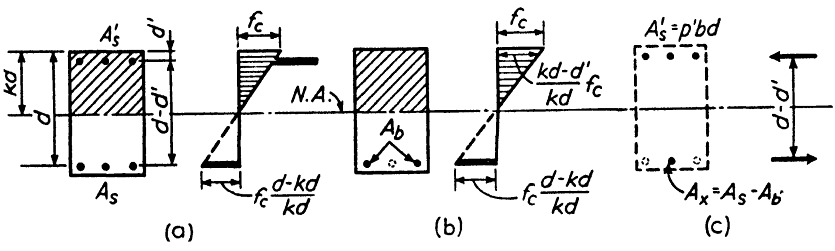
$$f_c = \frac{M}{Rbd^2} = \frac{80{,}000 \times 12}{0.205 \times 10 \times 20^2} = 1190 \text{ psi}$$

Also

$$j = 0.875$$

$$f_s = \frac{80{,}000 \times 12}{3.0 \times 0.875 \times 20} = 18{,}300 \text{ psi}$$

* Visualize the double reinforced beam a in the sketch as made up of a rectangular beam b with balanced reinforcement A_b plus a supplementary internal couple c of the compression in the steel A'_s and the tension in the additional steel A_x.



For balanced reinforcement as in b, $R_b = p_b f_s j = 1/2 \cdot f_c kj = 236$ for 20,000/1350/10. For the couple in c, the resisting moment M_x can be obtained as the product of the transformed compressive area $(n - 1)p'bd$, the stress intensity $\left(\frac{kd - d'}{kd}\right) f_c$ and arm $(d - d')$. From this $R_x = M_x/bd^2 = (n - 1)p'\left(1 - \frac{1}{k}\frac{d'}{d}\right) f_c \times \left(1 - \frac{d'}{d}\right)$. Then $R = R_b + R_x$, gives values for the chart.

The extra tension steel A_x being farther from the neutral axis and not displacing any flexurally stressed concrete, will be less in amount than A'_s, since $p_x = \left(\frac{kd - d'}{d - kd}\right)\left(\frac{n - 1}{n}\right)p'$. If p_b is the ratio for balanced reinforcement (0.0136 for 20,000/1350/10), $p = p_b + p_x$ gives values for plotting.

To check the solution of Ex. 7–9 use $p'n = \dfrac{n-1}{n}(p'n \cdot \text{actual})$.

Example 9–9. (Same as Ex. 7–10, page 83.) Same beam as in Ex. 9–8. If the limiting fiber stresses are $f_s = 20{,}000$ psi and $f_c = 1350$ psi, what is the maximum moment of resistance of this beam?

Solution. As in the previous problem, enter Fig. A–8 and find $R = 0.205$, $j = 0.875$.

$$M_c = f_cRbd^2 = 1350 \times 0.205 \times 10 \times 20^2/12 = 92{,}300 \text{ lb-ft}$$

$$M_s = A_sf_sjd = 3 \times 20{,}000 \times 0.875 \times 20 \times \tfrac{1}{12} = 87{,}500 \text{ lb-ft}$$

To check the solution of Ex. 7–10 use $p'n = [(n-1)/n] \times 0.050$.

Example 9–10. (Same as Ex. 7–12, page 86.) What areas of tension and compression steel are required for this beam? $b = 10$ in.; $d = 20$ in.; $d' = 3$ in.; $f_s = 20{,}000$ psi; $f_c = 1350$ psi; $n = 10$; $M = 130{,}000$ lb-ft.

Solution. In order to use Fig. A–10 there are required the values of $d'/d = 3/20 = 0.15$ and $R = (130{,}000 \times 12)/(10 \times 20^2) = 390$. The figure gives $p = 0.023$ and $p' = 0.022$, making $A_s = 4.60$ sq in. and $A_s' = 4.40$ sq in. This value of A_s' is that required by the elastic theory. J.C. 804*c* allows for plastic flow (page 83) by permitting a reduction of 50 per cent in A_s' providing f_s' does not then exceed 16,000 psi. The stress $f_s' = \left(\dfrac{kd - d'}{kd}\right)nf_c$, where k varies with p, p', n, and d'/d. In this case* if k is assumed equal to 0.40, $f_s' = [(0.4 \times 20 - 3)/(0.4 \times 20)] \times 10 \times 1350 = 8400$ psi and A_s' can be reduced to 8400/16000 of 4.40 = 2.31 sq in. Had A_s' been taken as 0.50×4.40 then f_s' would have been 16,800 psi or more than 16,000 psi allowed. It is desirable to have in mind certain beam depths above which taking one-half the compressive steel area would result in stresses higher than 16,000 psi. For $k = 0.40$, $d' = 3$ in., $f_c' = 3000$ psi, $n = 10$, we can write $\dfrac{2\,(0.40d - 3)}{0.40d} \times 10 \times 1350 = 16{,}000$; $d = 18\tfrac{1}{2}$ in.; if $d' = 2$ in. with the other factors unchanged $d = 12$ in. For smaller values of d the stress will be less than 16,000 psi; for larger values the stress would exceed 16,000 psi and it becomes necessary to take $\dfrac{(kd - d')nf_c/kd}{16{,}000}$ of A_s' instead of one-half, as illustrated in Chapter XVII.

9–5. Columns. Additions to notation:

P = total safe load on ordinary short column
A = total effective area of column cross section
p = ratio of area of longitudinal reinforcement to effective column area = A_s/A
f_s' = minimum yield-point strength of steel.

* Fig. 7–17 indicates that $kd = 8.1$ in. or $k = 0.405$, but for most practical purposes and particularly for checking negative moment at the supports it is frequently satisfactory to take $k = 0.4$. More precise values could be found from Figs. A–7, A–8, and A–9.

For a short column with vertical reinforcement and ties only and without spiral hooping the elastic theory outlined in Art. 8–2 and Ex. 8–1 gives:

$$P = Af_c[1 + (n - 1)p] \qquad [9\text{–}19]$$

This is the method recommended in the 1928 A.C.I. Code which limits p between ½ and 2 per cent.

If the safe stress is based on the yield-point strength as recommended by the 1940 J.C. Report and the A.C.I. Building Code, as shown in Art. 8–3 and Ex. 8–2*b*:

$$P = Af_c + A_sf_s \qquad [9\text{–}20]$$

These codes permit A to be taken as the gross area of the column and recommend that f_c be taken as 18 per cent of f'_c and f_s as 32 per cent of f_s, with the percentage of vertical steel limited between 1 and 4 per cent.

Formulas for columns with spirals based on Considère's work take the general form:

$$P = A(f_c[1 + (n - 1)p] + 2.4nf_cp_1)$$

where p_1 is the ratio of the volume of the spiral wire to the volume of the enclosed concrete.

The 1928 A.C.I. Code for spiral columns uses formula 9–19 as for tied columns but increases the allowable stress on the concrete because of the spiral hooping to

$$f_c = [300 + (0.10 + 4p)f'_c] \qquad [9\text{–}21]$$

This code limits A to the core area inside of the outside face of the spiral hooping and the vertical steel to between 1 and 6 per cent. Also p_1 is arbitrarily taken as ¼ of p.

The 1940 J.C. Report and the 1941 A.C.I. Code use formula 9–20 for spiral columns but increase the allowable stresses 26 per cent to allow for the toughening effect of the spirals. Thus f_c becomes 22½ per cent f'_c and f_s becomes 40 per cent of f'_s, with p limited between 1 and 8 per cent. The spiral percentage is established as

$$p_1 = 0.45(R - 1)\frac{f'_c}{f''_s} \qquad [9\text{–}22]$$

where R is the ratio of the gross area to the core area of the column and f''_s is the stress in the spiral.

Example 9–11. (Same as Ex. 8–2, page 108.) What is the allowable load on a tied column 20 by 20 in. with four 1 in. square vertical bars, if f_c = 675 psi and n = 10, using the elastic theory of the 1928 A.C.I. Code?

Solution. See Table A–3 in the Appendix, and read directly P = 294,300 lb.

Example 9–12. (Same as Ex. 8–3, page 108.) Design a column to carry a load of 110,000 lb, with f_c = 675 psi, n = 10, and using the elastic theory of the 1928 A.C.I. Code.

Solution. See Table A–3, and read the following choices:

12 by 12 with four ⅞ in. round rods carries 111,800 lb
12 by 12 with four ¾ in. round rods carries 107,900 lb
13 by 13 with four ⅝ in. round rods carries 121,600 lb

The first choice is acceptable. The second is too weak and the third is too strong.

Example 9–11a. (Same as Ex. 8–2*b*, page 114.) What is the allowable load on the column of Ex. 9–11, using the yield-point theory of the 1940 J.C. Report?

Solution. See Table A–4, and read directly P = 267,200 lb.

Example 9-12a. See Ex. 9–12 and design a column to carry 110,000 lb, using f'_c = 3000 psi, f'_s = 40,000 psi, and using the yield-point theory of the 1940 J.C. Report.

Solution. See Table A–4, and read the following choices:

12 by 12 with six ¾ in. round rods carries 111,600 lb
12 by 12 with four ⅞ in. round rods carries 108,500 lb
13 by 13 with four ¾ in. round rods carries 113,800 lb.

The first choice is satisfactory. The second is too weak and the third is slightly too strong.

Example 9–13. (Same as Ex. 8–5, page 116.) What is the allowable load on the spirally reinforced column of Fig. 8–8, page 120, using the 1940 J.C. yield-point methods with f'_c = 3000 psi and f'_s = 40,000 psi?

Solution. See Table A–6, and read directly P = 308,100 lb (10-⅞ $\phi \approx$ 6–1□). The proper spirals can also be read directly from this table as ⅜ in. diameter hot-rolled wire at 2-in. pitch.

Example 9–13a. (Same as Ex. 8–5*a*, page 118.) What is the allowable load on the spirally reinforced column of Fig. 8–8, page 120, using the method of the 1928 A.C.I. Code with f'_c = 3000 psi and n = 10?

Solution. See Table A–5, and read directly P = 258,100 lb. At the same time the proper spiral can also be read as ¼ in. round wire at 1¾-in. pitch.

9–6. Direct Stress and Bending. As can be seen by reference to Art. 8–5 (page 118), charts or tables for the cases of direct stress combined with bending become complicated for several reasons. The most important point is whether the normal thrust is inside or outside the kern of the section, determining whether the combined stress is all compression or compression on one side and tension on the other. Ex. 8–8, page 121, shows the cubic equation solution which is inherent in this problem. Among the variables that have to be taken into account are: n, d'/d, f_c, f_s, p, p', and e = the eccentricity of the thrust. Although a family of curves can be arranged for *three* independent arguments, so numerous a group of independent variables requires a whole set of charts. For that reason two types of diagrams are pre-

sented. The first group, consisting of Figs. A-12 to A-21 inclusive, from Turneaure and Maurer's Principles of Reinforced Concrete Construction, cover rectangular and circular sections for the two cases of compression only, and compression with tension. To limit the number of diagrams the assumption is made in every case that $p' = p$, i.e., that an equal amount of steel is used in each face of the member. This somewhat limits their range of usefulness.

The second type of diagram, Figs. A-22 to A-25, is suggestive only of a straight line chart which the designer can quickly prepare for any combination of variables. For that reason the derivation of the necessary formulas is given on page 141. These formulas themselves have proved quite useful in attacking this problem, as the method employed results in linear functions and saves the solution of awkward cubic equations.

Figs. A-12 to A-21 may be applied to the solution of problems as follows:

Example 9-14. (Same as Ex. 8-6, page 119.) Determine the fiber stresses in a member 24 by 15 in. with three 1 in. square bars 2 in. in from each of the longer sides when loaded with a thrust of 210,000 lb applied 2 in. along the short axis from the intersection of the two main center lines.

Solution. $p = p' = \dfrac{3}{(15 \times 24)} = 0.00833$; $d'/h = 2/15 = 0.133$; $e/h = 2/15 = 0.133$. This is quite apparently Case I with compression over the entire section, but if in doubt check from Fig. A-12. Entering with $pn = 0.0833$ and $d'/h = 0.10$, any value of e/h less than 0.188 will produce compression over the entire section.

It will be necessary to interpolate between Fig. A-14 ($d'/h = 0.10$) and Fig. A-15 ($d'/h = 0.15$) entering with $e/h = 0.133$ and $pn = 0.0833$ as follows:

$$\frac{d'}{h} = 0.10 \qquad C = 1.47 \qquad f'_c = 0.170 f_c$$

$$\frac{d'}{h} = 0.15 \qquad C = 1.49 \qquad f'_c = 0.150 f_c$$

$$\frac{d'}{h} = 0.133 \qquad C = 1.49 - \qquad f'_c = 0.157 f_c$$

Then

$$f_c = 1.49 \times \frac{210{,}000}{15 \times 24} = 870 \text{ psi}$$

$$f'_c = 0.157 \times 870 = 136 \text{ psi}$$

$$f_s = 10 \times 870\,[1 - 0.133(1 - 0.157)] = 7730 \text{ psi}$$

$$f'_s = 10 \times 870\,[0.157 + 0.133(1 - 0.157)] = 2350 \text{ psi}$$

These values check reasonably well with Ex. 8-6. Note that these four values are all obtained with one setting of the slide rule. Note also that it

is practically as easy to compute with the transformed section as to use the diagram as above.

Example 9-15. (Same as Ex. 8-8, page 121.) Determine the fiber stresses in the column of Ex. 9-14 if the load is 110,000 lb applied 5 in. from the centroid.

Solution. $p = p' = 0.0083$ (Ex. 9-14); $d'/h = 2/15 = 0.133$ (Ex. 9-14); $e/h = 5/15 = 0.333$. This example probably falls under Case II, but if in doubt check from Fig. A-12. Entering with $pn = 0.0833$ and $d'/h = 0.10$, any value of e/h greater than 0.188 produces tension over part of the section.

Interpolate between Fig. A-17 ($d'h = 0.10$) and Fig. A-18 ($d'/h = 0.15$), entering with $h/e = 3.00$ (the reciprocal of e/h) and $pn = 0.0833$:

$$d'/h = 0.10 \qquad C = 7.6 \qquad k = 0.70$$

$$d'/h = 0.15 \qquad C = 8.05 \qquad k = 0.66$$

$$d'/h = 0.133 \qquad C = 7.9 \qquad k = 0.67$$

Then

$$f_c = CM/bh^2 = \frac{7.9 \times 5 \times 110{,}000}{24 \times 15 \times 15} = 805 \text{ psi}$$

Steel stresses are best obtained from Fig. A-19. Entering with $k = 0.67$ and $d'/h = 0.133$, read $A = 0.25$ and $B = 0.80$. Then

$$f_s = nf_cA = 10 \times 805 \times 0.25 = 2010 \text{ psi tension}$$

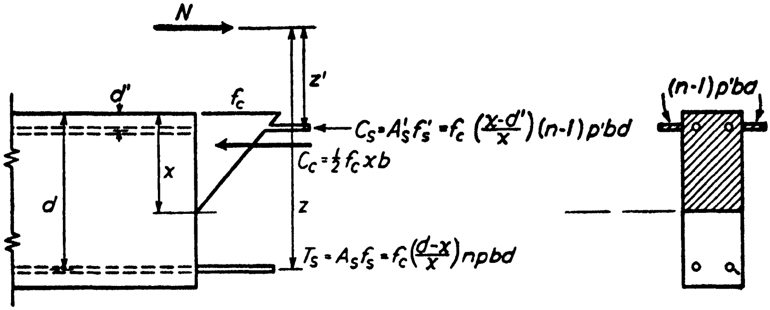
$$f_s' = nf_cB = 10 \times 805 \times 0.80 = 6440 \text{ psi compression}$$

These values agree with those computed in Ex. 8-8 as closely as values can be read from the diagram.

It will be noted that the use of the diagrams for Case II requires less time than computation by the use of the transformed areas and the solution of the resulting cubic equation.

The diagrams are limited, however, in their scope to those cases where $p = p'$. Frequently in design problems (and especially in the design of rigid frames and similar continuous structures) it is economical to work both the compressive and tensile reinforcement at approximately full capacity. In that case these diagrams are too limited to be of value. For that reason the following set of additional diagrams is included.

The diagrams on Figs. A-22 to A-25 for direct stress and bending are new. They cover cases where the tension and compression steel areas are not equal and apply to all conditions where the normal thrust is outside of the kern of the section — a condition that is readily established by reference to Fig. A-12. A word of caution in using the diagrams is necessary. As will be seen from the following derivation, moments are taken about the tension and the compression steel respectively and not about the centroid of the column. Hence, in determining values of R and R' for entering the diagrams, be sure that moments are taken about the tension and compression steel.



$$\textit{Mom. abt. Tens. Steel:}\quad Nz = M_s = A'_s f'_s (d - d') + \frac{1}{2} f_c x b\left(d - \frac{x}{3}\right) \tag{a}$$

$$\textit{Mom. abt. Comp. Steel:}\quad Nz' = M'_s = A_s f_s (d - d') - \frac{1}{2} f_c x b\left(\frac{x}{3} - d'\right) \tag{b}$$

$$\textit{Solving (b) for p:}\quad p = \frac{A_s}{bd} = \frac{M'_s + \frac{1}{2} f_c x b\left(\frac{x}{3} - d'\right)}{bdf_s(d - d')} \tag{c}$$

$$= \frac{\frac{M'_s}{bd^2} + \frac{1}{2} f_c x\left(\frac{x}{3} - d'\right)\frac{1}{d^2}}{f_s\left(\frac{d - d'}{d}\right)} = \frac{R' + \frac{1}{2} f_c x\left(\frac{x}{3} - d'\right)\frac{1}{d^2}}{f_s\left(1 - \frac{d'}{d}\right)} \tag{d}$$

$$\textit{Solving (a) for p':}\quad p' = \frac{A'_s}{bd} = \frac{M_s - \frac{1}{2} f_c x b\left(d - \frac{x}{3}\right)}{bdf'_s(d - d')} \tag{e}$$

$$= \frac{\frac{M_s}{bd^2} - \frac{1}{2} f_c x \frac{\left(d - \frac{x}{3}\right)}{d^2}}{f'_s\left(\frac{d - d'}{d}\right)} = \frac{R - \frac{1}{2} f_c x \frac{\left(d - \frac{x}{3}\right)}{d^2}}{f'_s\left(1 - \frac{d'}{d}\right)} \tag{f}$$

$$\textit{from (d):}\quad R' = pf_s\left(\frac{d - d'}{d}\right) - \frac{\frac{1}{2} f_c x\left(\frac{x}{3} - d'\right)}{d^2} \tag{g}$$

$$\textit{from (f):}\quad R = p'f'_s\left(\frac{d - d'}{d}\right) + \frac{1}{2} f_c x\left(\frac{d - \frac{x}{3}}{d^2}\right) \tag{h}$$

$$f'_s = \frac{x - d'}{x}(n - 1)f_c \quad \textit{approximately}\quad f_s = \frac{d - x}{x} nf_c \tag{k}$$

Substituting from (*k*) *and using* $x = kd$·

$$R' = p\frac{1 - k}{k} nf_c \frac{d - d'}{d} - \frac{1}{2}\frac{f_c kd\left(\frac{kd}{3} - d'\right)}{d^2} \tag{m}$$

$$= pnf_c\left(\frac{1 - k}{k}\right)\left(1 - \frac{d'}{d}\right) - \frac{1}{2} f_c k\left(\frac{k}{3} - \frac{d'}{d}\right) \tag*{[9–23]}$$

$$R = p'(n - 1)f_c\left(1 - \frac{d'}{d}\cdot\frac{1}{k}\right)\left(1 - \frac{d'}{d}\right) + \frac{1}{2} f_c k\left(1 - \frac{k}{3}\right) \tag*{[9–24]}$$

where $R = \frac{M_s}{bd^2}$: M_s ***being the moment about the Tens. Steel***

where $R' = \frac{M'_s}{bd^2}$: M'_s ***being the moment about the Comp. Steel***

FIG. 9–3

The derivation of the formulas is clearly shown on Fig. 9–3. Frequently they can be used to advantage for those design problems for which diagrams are not immediately available. Note that the refinement of taking the equivalent of the compression reinforcement at $n - 1$ instead of n times its value has been followed, allowing for the concrete displaced by the steel.

Charts are readily drawn from equations 9–23 and 9–24 for various stresses, d'/d ratios, and elastic properties.

Example 9–16. Required to determine the compressive and tensile steel areas in a 24 by 15 in. concrete member undergoing a thrust P of 110,000 lb and a moment M of 850,000 lb-in. about the longer axis through the centroid of the member. Use $f_c = 1350$ psi, $f_s = 20{,}000$ psi, and $n = 10$.

Solution. For tension steel use Fig. A–22. Since M is about the centroid, compute $M_s' = 850{,}000 - 110{,}000 \times 5\frac{1}{2}$ in. $= 245{,}000$. From this $R' = 245{,}000/24 \times 13 \times 13 = 60.4$. $d'/d = 2/13 = 0.154$. Entering Fig. A–22 with $R' = 60.4$ and interpolating for $d'/d = 0.154$ gives $p = 0.0029$, $A_s = 0.0029 \times 24 \times 13 = 0.91$ sq in.

For compression steel Fig. A–25 is close enough. Since M is about the centroid, compute $M_s = 850{,}000 + 110{,}000 \times 5\frac{1}{2} = 1{,}455{,}000$. From this $R = 1{,}455{,}000/24 \times 13 \times 13 = 358$. Fig. A–25 gives $p' = 0.0183$, from which $A_s' = 0.0183 \times 24 \times 13.0 = 5.71$ sq in.

The first set of plates is most useful for checking columns that are symmetrically reinforced and undergoing some bending due to eccentric loads. The second set of plates is most useful in designing members such as rigid frames or arches that are primarily undergoing bending but with a certain amount of direct stress.

Problem 9–2. Solve Ex. 9–16 using the method of transformed areas along the lines suggested on page 121.

For the design of circular sections undergoing combined stress and bending whole sets of diagrams are available (see " Handbook of Reinforced Concrete Building Design " of the American Concrete Institute, 1928, pages 56 to 65 inclusive) which have a range of usefulness. Fig. A–26 reproduced with the permission of Professor J. R. Shank, combines all the necessary information on one plate and, though the solution is by cut-and-try methods, results are obtained quite rapidly.

Example 9–17. (Same as Ex. 8–10, page 123.) The column of Ex. 8–7, page 120, carries a load of 40,000 lb applied 12 in. from the center. Determine the concrete and steel stresses. $n = 10$.

Solution. The first trial computation in the tabulation is that of Ex. 8–10, using Fig. A–26 instead of the formulas. Plotting the stress curve as in Fig. 9–4 suggests that the neutral axis lies about 9.0 in. from the tension extreme and the second trial assumes a shift of the neutral point 1.3 in. toward the compression side. The second solution suggests a further shift of about ⅛ in., too small a change to consider, having in mind the many uncertain elements which affect the possible degree of precision attainable in this problem. The maximum fiber stresses are approximately

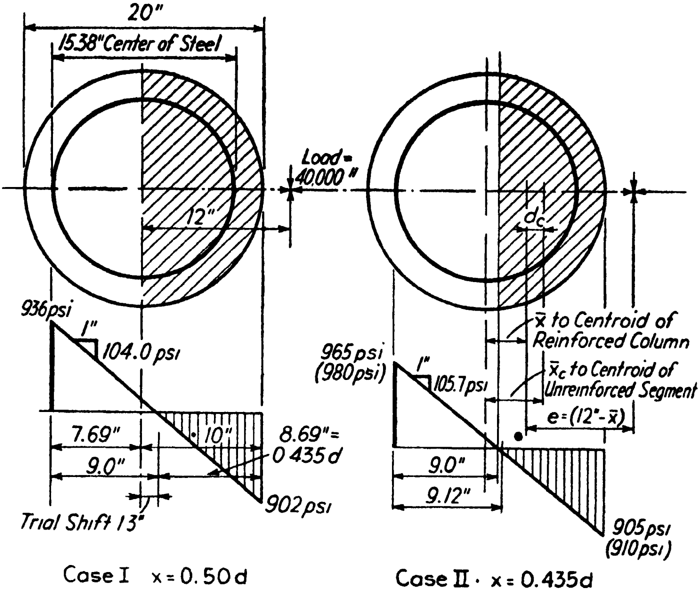


FIG. 9–4

the values given in parentheses on Fig. 9–4, 910 psi tension on the extreme concrete of the transformed section, i.e., 9100 psi for the steel and 980 psi compression for the concrete. It is suggested that the student carry through a third solution assuming that the neutral axis lies about 1.5 in. beyond the center of the column.

Neutral Axis from Edge	A	x_c	B	A_c	C	I_c	A_s	I_s	$A = A_c + A_s$
0.50d	0.420	4.2	0.395	158	0.110	1100	54	1600	212
0.435d	0.490	4.9	0.330	132	0.070	700	54	1600	186

Neutral Axis from Edge	$\bar{x} = \frac{A_c x_c}{A}$	$A_c d_c^2$	$A_s \bar{x}^2$	$I = I_c + A_c d_c^2 + I_s + A_s \bar{x}^2$	$\frac{P}{A}$	$\frac{Pey}{I}$	f_c at Assumed Neutral Axis
0.50d	3.13	181	530	3411	188	1124t / 714c	138t
0.435d	3.48	266	655	3221	215	1180t / 690c	13t

$d_c = (x_c - \bar{x})$ $\quad e = (12 - \bar{x})$ $\quad y_t = (7.69 + \bar{x})$ $\quad y_c = (10 - \bar{x})$

9-7. Principal Design Formulas. *Location of Neutral Axis in Rectangular Beams:*

$$k = \sqrt{2pn + (pn)^2} - pn \qquad [9\text{-}3]$$

$$k = \frac{1}{1 + \dfrac{f_s}{nf_c}} \qquad [9\text{-}4]$$

Arm of Internal Couple:

$$j = 1 - \frac{k}{3} \qquad [9\text{-}6]$$

Moment of Resistance:

Concrete $M = \frac{1}{2}f_c kjbd^2 = R_c bd^2$ [9-2; 9-8]

Steel $M = f_s pjbd^2 = A_s f_s jd = R_s bd^2$ [9-1; 9-7]

Balanced Reinforcement:

$$p = \frac{1}{2}\,\frac{1}{\dfrac{f_s}{f_c}\left(\dfrac{f_s}{nf_c} + 1\right)} \qquad [9\text{-}5]$$

Web Shear:

$$v = \frac{V}{bjd} \qquad [7\text{-}1]$$

Point Where No Web Reinforcement Is Required:

$$a = \left(\frac{V - V_c}{V}\right) \times \text{distance to zero shear}$$

Stress in Diagonal Tension Reinforcement:

$$p = \frac{V'S}{jd(\sin\alpha + \cos\alpha)} \qquad [7\text{-}3]$$

Stress in Vertical Stirrups:

$$P = \frac{V'S}{jd} \qquad [7\text{-}3a]$$

Bond:

$$U = \frac{V}{jd\Sigma o} = \frac{vb}{\Sigma o} \qquad [7\text{-}2]$$

Embedment to Develop Bar:

$$L = \frac{f_s}{4u}D \qquad [7\text{-}4a]$$

Column Formulas:

Elastic Theory: $P = Af_c[1 + (n - 1)p]$ [9–19]

Yield-Point Theory: $P = Af_c + A_sf_s$ [9–20]

Problems. There is a wealth of problem material in Chapters XV, XVI and XVII. The abbreviated form of computation there used affords the student an opportunity to reestablish the figures independently, with frequent opportunities for checking. If any questions should arise as to methods used, detailed explanations will be found in Chapter XIII.

CHAPTER X

RETAINING WALLS

10–1. Retaining walls are built to restrain a mass of earth or similar material and are of two types: the gravity wall of plain concrete or other masonry, which depends for its stability principally upon its own weight, and the reinforced concrete wall which depends, in addition, upon the weight of a portion of the earth back of it. These two types are illustrated in Fig. 10–1. The gravity wall there shown resists solely by its own weight the thrust of the earth behind it which tends to slide the wall along its foundation or tip it over; the reinforced concrete wall can neither slide nor tip except as the earth resting on its heel slides or tips with it. Had the gravity wall been built with a sloping back, the weight of the prism of earth above the sloping portion would assist the wall in retaining its position.

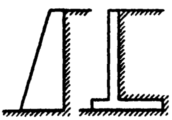


FIG. 10–1

The mathematical part of the design of walls consists in ascertaining the amount of earth thrust on the back and proportioning the wall so that it shall be structurally sufficient in every part and stable against sliding and overturning, without exerting too large a unit pressure upon the foundation.

The briefest of consideration given to the infinite variety of earthy materials makes plain the complexity and difficulty of the problem. Earth ranges all the way from sand without cohesion, the side of a bank of which naturally takes a rather flat slope, to clay which is strongly cohesive when dry and will maintain a vertical cut for a considerable time, and which when wet will, in some instances, flow like a thick, viscous liquid. We have available a method of analysis which gives us easily and quickly the pressure that dry, granular, cohesionless material exerts upon the back of any wall built to restrain it, namely, Coulomb's, published in 1773. If the earth back of a wall is cohesive we may distinguish two cases. If the material is sandy we may neglect the cohesion and apply Coulomb's theory, with an error on the safe side; if the material is clay there is no earth pressure theory available and reliance must be had upon experience as a guide to proportioning the wall. Modern investigators in the field of soil mechanics have not yet

reached general agreement as to phenomena and forces, although it is evident that great advances toward a solution have been made in very recent times.*

10–2. Coulomb's Theory of Earth Pressure. Coulomb's theory is empirical, and is based on the observation of a practical engineer that the failure of a laterally supported sand bank shows the sliding down of a wedge of the material along a sloping plane, such as bc in Fig. 10–2. In order to apply the principles of mechanics to the phenomenon without excessive mathematics four assumptions are necessary as to the conditions realized: (a) the intensity of shearing resistance S (psi) along any section through the sand mass is $S = p \tan \phi$, where p (psi) is the intensity of normal pressure on the section and ϕ is the angle of internal friction of the material; (b) the surface of sliding is a plane; (c) the earth mass is just on the point of motion so that the shearing resistance along the plane of sliding is fully developed; (d) each and every particle of the wedge, as well as the whole mass, is on point of movement with reference to its neighbors. The first three assumptions make it possible to compute the lateral pressure, and the fourth, which is valid only under certain conditions, makes it possible to find its location since, if realized, this state of homogeneous failure in the sliding wedge involves a hydrostatic distribution of the lateral pressure over the back of the support.

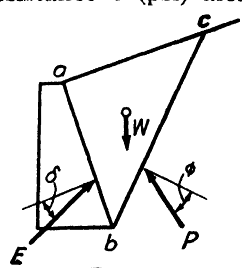


Fig. 10–2

For the realization of the third assumption, that of the full mobilization of the shearing resistance along the plane of sliding, there must be yielding of the support and actual movement of the mass, since stress

* The modern science of soil mechanics began with the publication of "Erdbaumechanik" by Karl von Terzaghi, an Austrian engineer, in 1925. Since that day the theories and the methods of research suggested by Dr. Terzaghi have been more and more accepted since they have been found to solve successfully an increasing number of the difficult problems met in engineering construction. It is but fair to say that, as might be expected with a science in its infancy, a great deal of controversy centers about some of the newer concepts.

For further study of the subject the reader is referred to these articles by Dr. Terzaghi: "A Fundamental Fallacy in Earth Pressure Computations," *Journal of the Boston Society of Civil Engineers*, 1936, reprinted in the Proceedings of the First International Congress of Soil Mechanics, held at Harvard University, 1936; "Soil Mechanics — A New Chapter in Engineering Science," the James Forrest Lecture of the Institution of Civil Engineers, London, 1939; "General Wedge Theory of Earth Pressure," Proc., A.S.C.E., Oct., 1939; and the text, Soil Mechanics, by Professor D. P. Krynine of Yale University, McGraw-Hill Book Co., 1941, which contains extensive bibliographies. The first volume of Dr. Terzaghi's text, Soil Mechanics, John Wiley & Sons, Inc., was in press the same time as this book and so was not available for the rewriting of this edition.

can exist only with corresponding strain. With sand this movement is very small, perhaps 0.001 of the depth, and probably every retaining wall yields sufficiently, almost imperceptibly, so that the assumption is valid. Simultaneous breakdown of the entire wedge will occur only if there is tilting of the surface of support about its lower point. This actually occurs with walls to a sufficient degree to justify the assumption, but very evidently the yield of the walls in a timbered cut in sand is very different in character. Consequently the variation of unit lateral pressure in these situations is not hydrostatic, but more nearly parabolic with maximum intensity at mid-depth or even higher.

The nature of the Coulomb theory and its application can be learned by a consideration of Fig. 10–2, which shows a retaining wall with sloping back supporting a mass of sand with sloping upper surface. Should the wall fail a wedge of sand, such as *abc,* would slide after it and accordingly the pressure on the back of the wall may be taken as equal and opposite to that needed (E) to hold the wedge in equilibrium, the other forces acting being the weight (W) of the wedge and the pressure (P) along the plane of sliding (bc). Since hydrostatic pressure variation is assumed both E and P act at the lower third points of their respective planes of application; since shearing resistance is assumed to be mobilized neither E nor P acts normal to the contact plane, but below the normal at an angle equal to the friction angle for the surfaces in contact, that of sand on concrete commonly for E, with $\delta = 25°$, on the average, that of sand on sand for the sliding plane, commonly taken as the angle of repose (ϕ) of the material, frequently assumed as 34°. Actually the friction angle ranges from 34° to 40°, increasing with the compactness of the sand.

Study of the force system of Fig. 10–2 shows an internal inconsistency in the theory; these three forces do not meet in a point as the laws of equilibrium require. However, the maximum error due to this cause is stated to be 3 per cent and so it may be ignored.

Nothing has so far been stated which would enable one to determine the size of the wedge in Fig. 10–2 which would exert the greatest possible pressure on the wall. Plane bc has, evidently, infinite possibilities of position and for some one position the magnitude of E will be a maximum. The equation expressing E is derived with recognition of this possibility and is necessarily quite complicated except for simple cases. Accordingly its use is not recommended since graphical solutions of Coulomb's theory are available and preferable. The best known of these graphical methods are those of Poncelet, Culmann, and Engesser. The method used here is that of Culmann, the simplest and most easily remembered of the three.

A deviation from the Coulomb assumption of a plane sliding surface is of interest. Actually the surface is slightly curved in the case of the wall being pushed out by the wedge of sand, a deviation from assumption which never causes the theory to be more than 3 per cent in error, a negligible amount.

Active and Passive Pressure. The pressure tending to tip a wall is known as the active earth pressure. Should some force be present which tends to push the wall into the earth mass a much larger resistance is developed to movement than this active thrust. This resistance is known as the *passive earth pressure* or *passive resistance.* The application of the Coulomb theory to the computation of the passive pressure involves very large error unless a proper curve is assumed for the lower surface of the wedge. Consideration of passive earth pressure is beyond the scope of this book.

10-3. The Culmann Construction. The magnitude of the forces E and P required to hold in equilibrium any wedge of sand of (computed) weight W may be determined graphically by the simple construction of

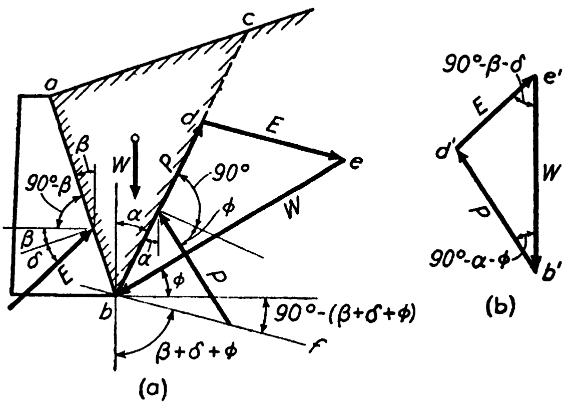


FIG. 10-3

Fig. 10-3*b*, where W is laid down vertically and the two lines of action of P and E are laid off as shown, their inclination with the vertical being known. Inspection of Fig. 10-3*a* will verify the values shown for these angles.

Instead of laying off W vertically it is quicker and easier to lay it off as shown in Fig. 10-3*a*, along a line making the angle ϕ with the horizontal, the vertex of the diagram being at the heel of the wall. This will bring the line of action of P coincident with the line representing the sliding surface *bc*. To complete the diagram, E, the line *de* is drawn parallel to a prepared reference line *bf*, making an angle $\beta + \delta + \phi$ with the vertical, where β is the angle between the back of the wall

and the vertical, and δ and ϕ are as already defined. This results in the angle *bed* having the value found necessary in Fig. 10–3*b* as triangle *bed* is similar to the force triangle $b'e'd'$.

The first step in the Culmann construction is the drawing of force triangle *bde* as shown in Fig. 10–3*a*, *bc* being any convenient plane through *b*. The next step consists in taking a slightly larger wedge and drawing a second force triangle overlying the first in part, as for wedge *ab2* in the figure for Ex. 10–1. This is repeated for a third still larger wedge and for others as may be desired. The curve (envelope) drawn through the points *d* (Fig. 10–3 and Ex. 10–1) defines the varying magnitudes of the wall pressure *E* for any size of wedge. The maximum value is found by drawing a tangent to the curve parallel to the line *be* representing the wedge weight. It will be noted that for varying positions of the line of slip, *P* will vary in magnitude and direction and *E* will vary only in magnitude, point of application being one-third of the distance up the wall, as required by the condition of hydrostatic pressure variation. It will also be noted that the plane of rupture lies above the plane of repose.

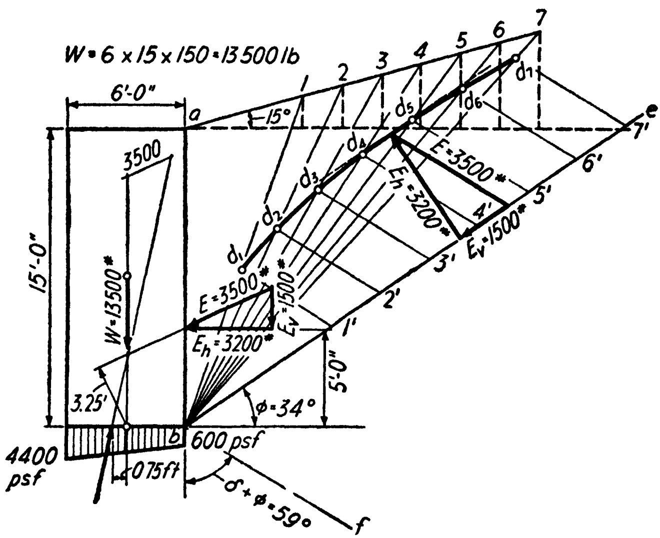


FIG. 10–4

Example 10–1. Is the wall shown in Fig. 10–4 stable against overturning and sliding? Data: weight of earth (sand) backfill, 100 pcf, weight of wall, 150 pcf, angle of repose of sand, 34°; angle of friction, sand and concrete, 25°; allowable pressure on foundation, 6000 psf.

Solution. The pressure on the back of the wall, 3500 lb acting downward at an angle of 25° above the normal, was determined graphically as above

outlined; first the line be was drawn upward from the heel of the wall, making an angle $\phi = 34°$ with the horizontal; then the horizontal dash line through a on which was laid off a series of points conveniently spaced at intervals of 2 ft on the scale of the drawing; projecting upward from these points the terminals of the several wedges on the sloping top surface, points 1, 2, 3, etc., were located. The cross-sectional area of the first wedge is 45 sq ft and the weight accordingly 4500 lb. The second wedge is larger by an increment of 15 sq ft or 1500 lb; each wedge differs from the preceding by the same increment. Thus it became possible to draw the several wedge sliding lines, $b1$, $b2$, $b3$, etc., and to lay off the weights on line be, $b1' = 4500$ lb by scale, $b2' = 6000$ lb, etc. Next the reference line for E was laid down through b, making an angle of $\delta + \phi$ with the vertical. Through the points 1′, 2′, 3′, etc., on line be, was drawn a series of parallels, each terminating at the sliding plane line of the related wedge at points d_1, d_2, etc. Each parallel $d_1 1'$, etc., represents the active pressure on the back of the wall exerted by the corresponding wedge. A line parallel to be drawn tangent to the curve through the d points is separated from be by an E distance representing 3500 lb pressure by scale, the maximum which is to be expected actively in this case. Accordingly the construction has given us the pressure acting on the back of the wall at a point 5 ft from the base at an angle of 25° above the normal. The horizontal and vertical components were recorded as shown.

The forces acting on a 1-ft length of wall, in addition to its own weight, are the earth thrust just described and the foundation pressure. Since this system of forces is in equilibrium, full information concerning the resultant of this unknown base pressure (but not concerning its distribution) may be obtained by applying the three equations of equilibrium. Accordingly, the horizontal component equals 3200 lb and the vertical 13,500 + 1500 = 15,000 lb. In order to locate the line of action of the resultant, moments may be taken about any convenient point in the plane of the force system, the center of the base being chosen in this case:

$$3500 \times 3.25 - 15{,}000 \times x = 0$$

$$x = 0.75 \text{ ft}$$

This location of the intersection of the resultant thrust and the base can be checked graphically, as shown on the diagram, by drawing through the intersection W and E their resultant, found by completing the triangle of forces involved. This cuts the base at 0.75 ft from the center.

Assuming the pressure to vary uniformly on the base, the extreme intensities are given by the familiar

$$f = \frac{P}{A} \pm \frac{Mc}{I} = \frac{P}{A}\left(1 \pm \frac{6e}{b}\right) = \frac{15{,}000}{6}\left(1 \pm \frac{6 \times 0.75}{6}\right)$$

$f = 4400$ psf at the toe and 600 psf at the heel, the resultant acting as shown. This maximum pressure is less than the allowable and the wall is safe against overturning by reason of failure of the earth under the toe pressure.

The fact that the resultant cuts the base inside of the middle third indicates that the wall will not overturn by rotating bodily about the toe. To provide a factor of safety against such overturning it is often stipulated that the

moment of resistance about the toe shall be a certain multiple of the overturning moment. This multiple varies from as low as $1\frac{1}{2}$ or 2 to 3 or more. Another method of obtaining a safety factor is to specify that the resultant shall cut the base inside of its middle third. (*Query.* In this example, if W were decreased until the resultant cut the base exactly one-third of the way from the toe, what would be the ratio of the resisting moment to the overturning moment? *Ans.* 2.41. *Hint:* The resisting moment is the moment, about the toe, of the wall weight plus back wall friction; the overturning moment is that of the horizontal component of the earth pressure.) (*Query.* What is the factor of safety, using the data obtained on the diagram for Ex. 10–1? *Ans.* 3.1.)

Another manner of expressing a factor of safety is the "factor of limitation," i.e., the ratio of the earth pressure which would cause the resultant to act through the outer middle-third point to the actual earth pressure.

Since the angle of friction between concrete and earth is assumed to be 25°, the resistance that may be developed to sliding equals 15,000 tan 25° = 7000 lb, which is greater than 3200 lb, the greatest push to be expected, giving a factor of safety of 2.2. The wall may be regarded as safe against sliding.

A very conservative assumption as to friction on the back of the wall is sometimes made, usually in cases where the designer doubts whether the friction can be developed. It would not be proper to carry through the analysis as was done in Ex. 10–1 and then simply neglect the vertical component of the earth pressure. The proper procedure in those cases where the back wall friction cannot be counted upon is to take $\delta = 0$.

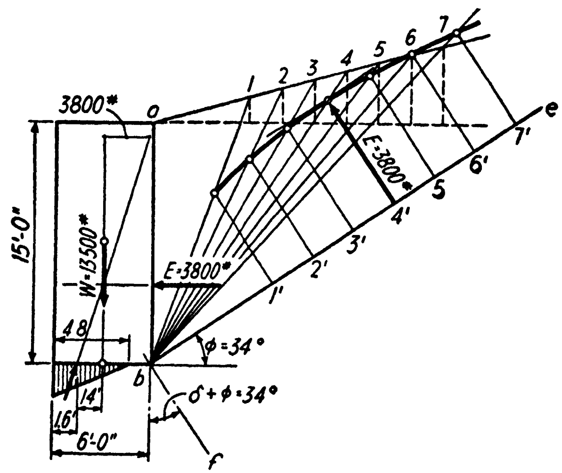


Fig. 10–5

Example 10–1a. Solve Ex. 10–1 using the same data throughout but neglecting back wall friction, i.e., $\delta = 0$. (Fig. 10–5.)

Solution. The drawing of the Culmann diagram proceeds as in Ex. 10–1 except that the reference line bf makes an angle with the vertical of only 34°; the resultant pressure is parallel to this (therefore horizontal against the wall) and equal to 3800 lb. In this case the resultant cuts the base

1.4 ft from the center; the soil pressures become triangular instead of trapezoidal. This is shown shaded in Fig. 10-5. The pressure at the toe is $13{,}500/(\frac{1}{2} \times 4.8) = 5600+$ psf. The ratio of M_r to $M_{ot} = 2.13$; and the factor of safety against sliding is $0.47 \times (13{,}500/3800) = 1.67$.

Comments. Compare the two solutions and observe that for this example the soil pressure under the toe has increased from 4400 to 5600 psf; the factor of safety against overturning has decreased from 3.1 to 2.13, and against sliding from 2.2 to 1.67.

Example 10-2. Same data as for Ex. 10-1 except that the surface of the ground back of the wall is a level storage ground with a maximum loading of 500 psf. What is the pressure on the back of the wall? (Fig. 10-6.)

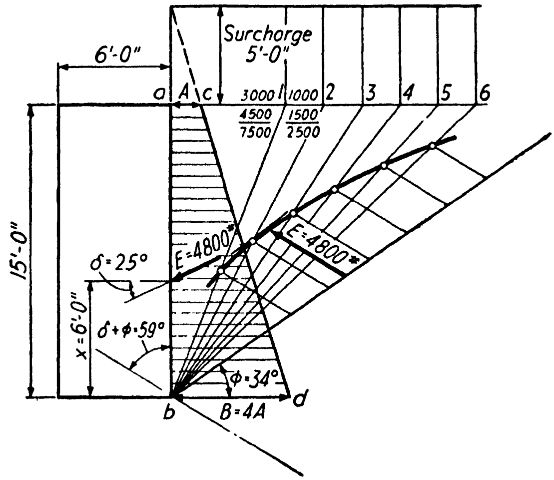


Fig. 10-6

Solution. The complete solution of this problem is shown in Fig. 10-6 and little comment is needed. The effect of the storage load on the earth pressure was taken into account by adding the superimposed weight to each trial wedge when constructing the Culmann diagram. This superimposed load causes the point of application of the earth pressure to lie above the third point, at a point determined by consideration of the hydrostatic pressure variation, represented by the trapezoid *abcd* in the figure. The proportions of this trapezoid were fixed by noting that the pressure on the top surface is one-fourth that on a horizontal plane at the level of the wall base. To locate the distance from the base *d* to the centroid (the general operation with a trapezoid such as this, a very common type in structural computations), divide the figure into two triangles and take moments about the base, giving

$$[(\tfrac{1}{2} \times 15 \times A)(\tfrac{2}{3} \times 15)] + [(\tfrac{1}{2} \times 15 \times B)(\tfrac{1}{3} \times 15)] = (\tfrac{1}{2} \times 15)(A + B)x$$

It will be noted that $\frac{1}{2} \times 15$ may be cancelled on both sides of this equation which leads to the easily remembered rule ($15 = H$)

$$x = [(A \times \tfrac{2}{3}H) + (B \times \tfrac{1}{3}H)] \div (A + B) = \frac{H}{3}\left(\frac{B + 2A}{B + A}\right)$$

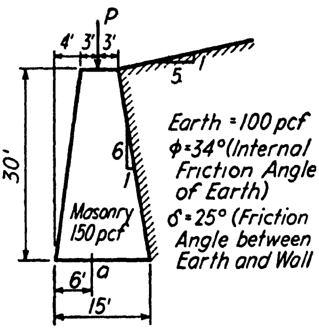
For convenience take $A = 1$; then

$$x = [10 + (4 \times 5)] \div 5 = 6.0 \text{ ft}$$

The material piled upon a backfill such as this is often considered to be replaced by an equal weight of earth of the same density as the earth back of the wall; the depth of this *surcharge*, as it is technically called, evidently equals the unit pressure on the top surface divided by the unit weight of earth. It will be noted how the sketching of this surcharge here facilitated the consideration of the trapezoid of pressure, *abdc*.

Problem 10–1. Using data from Ex. 10–2 but neglecting back wall friction, determine (*a*) magnitude, direction, and point of application of earth pressure; (*b*) where the resultant cuts the base; (*c*) soil pressures at toe and heel; (*d*) if this wall is safe against overturning; and (*e*) what you would do to make the wall stable if it is not so already?

Ans. (*a*) 5300 lb horizontal 6 ft above base; (*b*) 2.36 ft from center of wall; (*c*) 14,000 psf at toe and 0 at heel; (*d*) no, soil pressure is excessive and resultant cuts base too close to toe; (*e*) increase the wall thickness.



PROB. 10–2

Problem 10–2. The line of action of the resultant pressure acting on the base of this wall passes through *a*. What is the magnitude of the load P in pounds per foot of length of wall?

Ans. $P = 10{,}300$ lb; $E = 17{,}300$ lb.

10–4. Rankine's Theory of Earth Pressure. Rankine's analysis of the earth pressure problem assumes a uniform mass of dry granular material, without cohesion and without limit in extent. Consider the forces that must act upon the small element *abcd* shown in Fig. 10–7, whose weight W is equal to $wh\,dA$, where w is the unit weight of the material and dA is the area of a horizontal cross section. Since the mass is of infinite extent the state of stress on face *ad* must be identical with that on *bc*, and so P_1 is equal and parallel to P_2 and both cut the vertical planes on which they act at the same distance from the surface. Since the horizontal components of P_1 and P_2 are equal and since W acts vertically, the pressure R on the face *cd* can have no horizontal component and so must act vertically. Since the intensities of pressure on *cd* at *c* and *d* must be equal, this force, R, acts at the center of *cd*, and its line of action coincides with that of W. Therefore, for equilibrium, the lines of action of P_1 and P_2 must coincide, and since they both act at the same distance from the surface they are both parallel to it. So it may be concluded that on a vertical

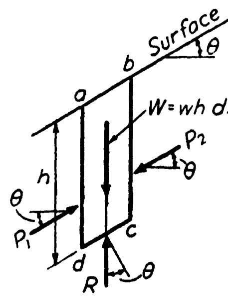


FIG. 10–7

plane through any point, as c, the resultant pressure is parallel to the surface, and on any plane parallel to the surface through the same point, the stress is vertical; that is, these are conjugate stresses and the expression for the relation between the intensities of conjugate stresses may be used:

$$\frac{p}{p_1} = \frac{\cos\theta - \sqrt{\cos^2\theta - \cos^2\phi}}{\cos\theta + \sqrt{\cos^2\theta - \cos^2\phi}}$$

where p is less than p_1. If p is greater than p_1 the relation is the same with the signs in numerator and denominator interchanged. Here θ = the common angle of obliquity of the stresses, i.e., that between the stress and the normal to the plane on which it acts (in this case equal to angle made by earth surface with the horizontal); ϕ = the maximum possible angle of obliquity (in this case the angle of internal friction of the material, generally taken as the angle of repose, the steepest angle of surface slope the loose material will maintain). The intensity of pressure upon plane cd equals

$$p_1 = \frac{W}{\text{area } cd} = \frac{wh\,dA}{\dfrac{dA}{\cos\theta}} = wh\cos\theta$$

The intensity of pressure p at the same point upon a vertical plane, then, is given by the following:

$$p = Cwh \tag{10-1}$$

where

$$C = \cos\theta\left(\frac{\cos\theta \mp \sqrt{\cos^2\theta - \cos^2\phi}}{\cos\theta \pm \sqrt{\cos^2\theta - \cos^2\phi}}\right) \tag{10-2}$$

The pressure actively exerted by earth upon a support is far less than the passive resistance that may be developed by pushing the support against the earth. Accordingly the value of C, equation 10-2, with the negative sign in the numerator and the positive in the denominator, may be considered to give the value of the active pressure and the same equation, with signs reversed, the passive resistance.

With horizontal top surface of indefinite extent, $\theta = 0$, and the expression for C takes a simpler form, for active pressure:

$$C = \frac{1 - \sin\phi}{1 + \sin\phi} = \tan^2\left(45° - \frac{\phi}{2}\right) \tag{10-2a}$$

For $\phi = 34°$, with $\theta = 0$, C has the value 0.28; that is, the lateral pressure has the same intensity as that of a liquid weighing 0.28 times

that of sand. Practicing engineers often take a liquid weighing 25 or 30 pcf as equivalent to the pressure of sand or gravel backfill on walls.

It will be found that the Coulomb and Rankine theories give identical results for all cases of pressures parallel to the free surface of the backfill against a vertical plane: the case of a level fill is a special application of this general conclusion. For cases of walls with inclined surfaces in contact with the fill and for other specialized problems, the sliding wedge theory gives results more in conformity with observed data, and by the use of graphical methods is easier to apply.

10–5. Drainage. If water is present in the earth back of a retaining wall there is a great increase in the pressure. For complete saturation, as with the water table at the back of the wall standing at its top, the wall must withstand full hydrostatic pressure plus earth pressure determined by the methods above set forth, with the weight of the sand taken as its submerged weight, with ϕ unchanged. The water does not have a lubricating effect on sand as has sometimes been supposed. Evidently it is important to have drainage provided so that there will be no standing water back of any wall not designed for it. A lesser saturation than that of standing water, however, affects the earth pressure under certain conditions. If the drainage provided is concentrated at the back of the wall a heavy rainstorm may cause as much as 35 per cent increase in pressure temporarily. This increase may be avoided by providing a thin drainage layer of coarse material sloping upward from the heel and extending completely beyond the wedge of failure. Percolation through the soil will then be vertical and wall pressure will be almost the same during and after rain.

10–6. Cantilever Retaining Wall. The rectangular beam is one of the simplest problems met in reinforced concrete design, particularly when the proportions are made such that no diagonal tension reinforcement is required. The most common form of rectangular beam is the slab, a member of great width as compared with the depth. Accordingly it is fitting to choose for the first example of actual design in this text a cantilever retaining wall of the sort shown in Fig. 10–8, consisting of the three simple elements, each a cantilever slab, the vertical stem a, the heel b, and the toe c, the last two together constituting the base. The forces acting on the stem (a, Fig. 10–9) are the earth thrust and the internal resisting shear and moment (V and M): on the heel (b, Fig. 10–9), its own weight, the downward weight of the mass of earth above, the upward pressure of the foundation bed, and the resisting shear and moment; on the toe (c, Fig. 10–9), its own weight and that of the earth above it, the upward pressure of the foundation bed and the internal stresses at the support. The downward weight of the earth above the

toe is never considered as this fill may not be in place when the wall is first loaded.

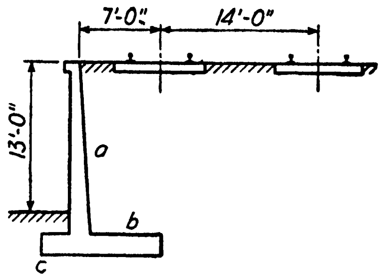


FIG. 10–8

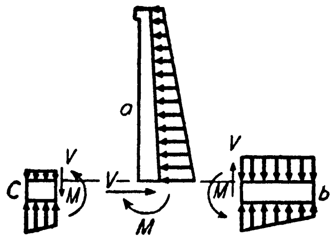


FIG. 10–9

This type of wall is economical for moderate heights, up to about 18 or 20 ft. Higher walls are generally made with brackets, called counterforts when inside, and buttresses when outside, of the vertical slab which is fastened to them. (See Fig. 10–10.)

When it is desired to build a wall close to a property line beyond which no encroachment is possible, the vertical slab is placed at the extreme end of the base giving an L-shaped wall, without a projecting toe.

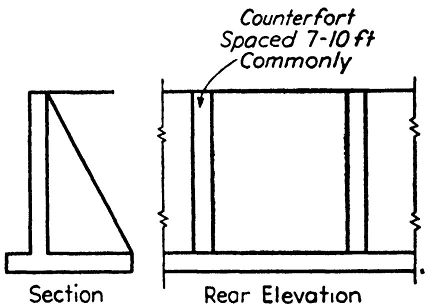


FIG. 10–10

10–7. Data for a Cantilever Wall. Detailed consideration will now be given to the design of a cantilever retaining wall to fill the situation outlined below. The necessary figures and sketches are shown together on a series of computation sheets. In the text is given a description of the various operations covering all the significant details. It is expected that the student will model his own work along the lines suggested by these sheets, making plain the various steps by clear subject headings and sketches and refraining from lengthy written descriptions. Slovenly workmanship, such as hasty sketches and crude, illegible letters and numerals, easily leads to error and results in sheets that are difficult to check. The engineering computer must always keep in mind that his work is to pass under the critical eye of a superior for check and execute it so that it will be easily understood. The student should study these computation sheets with three points in mind, best reserving the third, however, for separate consideration: (*a*) the application of the theory of earth pressure to the determination of the external forces

acting on the wall and its several parts; (*b*) the application of the theory of reinforced concrete to the proportioning of the several sections; (*c*) the precision of the computations, justifiying the approximations made and seeking others to shorten the work.

In connection with this chapter the student should read Arts. 868 to 873 of the 1940 Joint Committee Report. It is also advisable to compare all the specified fiber stresses in this example, together with other details, with the several recommendations of the Joint Committee elsewhere.

Data. Design a cantilever wall for track elevation, the details of the situation being shown in Fig. 10–8. Loading for the tracks: Cooper E-60 locomotive. Vibration or impact 15 per cent. Weight of earth fill, 100 pcf. Allowable pressure on earth, 4500 psf. Angle of repose, 1 vertical to $1\frac{1}{2}$ horizontal ($\phi = 33°40'$). Angle of friction, concrete on earth, 22° (tan 22° = 0.40). Allowable unit stresses: tension in steel, 18,000 psi; compression in concrete 800 psi; shear, no diagonal tension reinforcement being used, 40 psi; with special anchorage of tension steel 60 psi; bond, 100 psi. Ordinary concrete with a 28-day strength of 2000 psi is assumed, with $n = 15$. The steel used is structural grade, deformed bars.

10–8. First Steps in Design. (Computation Sheet W1.) The first thing for the designer to do is to assemble his information, as completely as may be necessary, on a computation sheet and make a sketch of the wall about as he judges it will appear when designed; all of which appears on Sheet W1. In making the sketch the question of the depth of foundation comes up at once. In order to prevent movement by the freezing and the thawing of the ground, the base must be set at or below the frost line, which is here assumed to be 4 ft below the surface. In the northern United States from 4 to 5 ft is the usual depth found necessary to get below the frost.

Another question to be settled is the position of the vertical stem on the base, which depends upon the limitations placed upon the line of action of the resultant foundation pressure. In this case the wall is not on rock but on compressible material and some settlement may be expected. Some engineers would require the resultant to strike near the center of the base for poor soil with the result that the base pressure is nearly uniform and there is little tendency for the base to tip with greater yielding of the soil under the toe. Probably the soil here assumed is sufficiently resistant so that the more common practice is justified. This practice keeps the resultant base pressure at or within the outer middle-third point, thus insuring the absence of lifting tendency at the heel. Comparative studies have shown that for the most eco-

CANTILEVER RETAINING WALL — Sheet W1

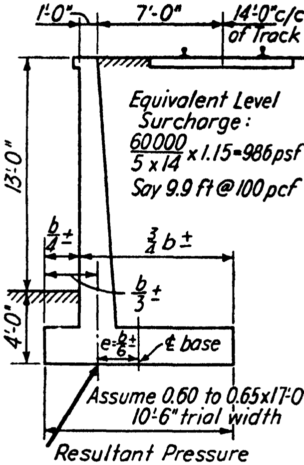
Data:

Cantilever Retaining Wall
Surcharge: Cooper E-60 Loading
Impact (Vibration) = 15%
Earth: $w_e = 100$ pcf
$\phi = 33°40' = \tan^{-1}\frac{2}{3}$ (earth on earth)
$\delta = 22°0'$ (earth on concrete)
Foundation Pressure = 4500 psf maximum
Specifications · Wt of Concrete = w_c = 150 pcf
$f'_c = 2000$ psi $n = 15$
$f_c = 800$ psi
$v_c = 40$ psi (60 psi Spec Anch)
$u = 100$ psi
$f_s = 18\,000$ psi

Design Constants: $R = \frac{M}{bd^2} = 139$ $p = 0.0089$

$K = \frac{3}{8} = 0.400$ $j = \frac{7}{8} = 0.867$

Earth Pressure: See Sheets W1, W2, W3



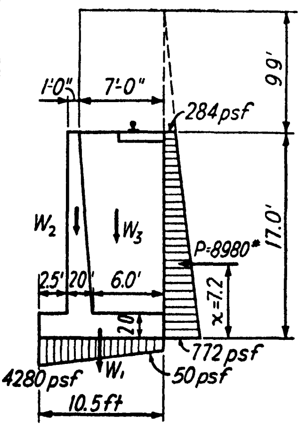
Case I - Level Surcharge Inert Prism: Omit Back Wall Friction

$$p = wh \times \frac{1-\sin\phi}{1+\sin\phi} = 100h\,\frac{1-0.554}{1+0.554} = 28.7\,h$$

$$P = \left(\frac{284+772}{2}\right)17 = 8980 \text{ lb}$$

$$x = \frac{17}{3}\left(\frac{772 + 2 \times 284}{772+284}\right) = 7.2 \text{ ft}$$

Base Width: For a rough check of trial width take moments about resultant through third point of base; $8980 \times 7.2 = \frac{3b}{4} \times 26.9 \times 100\left(\frac{2b}{3} - \frac{3b}{8}\right)$; $b = 10'\text{-}6''$



Base Pressures: Moments about toe:

$W_1 = 2 \times 10.5 \times 150$	= 3150 lb	; × 5.25 =	16500 lb-ft
$W_2 = 1\frac{1}{2} \times 15 \times 150$	= 3380	; × 3.28 =	11100
$W_3 = 6\frac{1}{2} \times 24.9 \times 100$	= 16190	; × 7.25 =	117400
	22720 lb		$145000 = M_R$
		$-8980 \times 7.2 =$	$-64700 = M_{OT}$
	22720		180300
			3.54 ft from toe

$e = 5.25 - 3.54 = 1.71$ ft

$$p = \frac{W}{A} \pm \frac{My}{I} = \frac{W}{A}\left(1 \pm \frac{6e}{d}\right) = \frac{22720}{10.5}\left(1 \pm \frac{6 \times 1.71}{10.5}\right)$$

$$= \begin{cases} 4280 \text{ psf Toe} \\ 50 \text{ psf Heel} \end{cases}$$

Resistance to Sliding:

$$\text{Factor of Safety} = \frac{W \tan\delta}{P_h} = \frac{22720 \times 0.40}{8980}$$

$= 1.01$ ∴ Must Anchor

nomical wall the stem should be placed approximately over the point where it is desired that the resultant strike the base. These considerations explain the dimensions shown on this sketch. A top width of wall of 12 in. is a common minimum to allow for easy pouring of concrete. A batter of about ½ to ¾ in. per foot of height on the back face will usually provide sufficient depth in the stem at all points to care for the shear and moment; this assumption is quickly checked as the design proceeds. A smaller batter on the front will improve appearance, particularly if slight tipping occurs.

After the sketch is made, the preliminary work is completed by determining the magnitude, direction, and point of application of the resultant earth pressure against the wall.

10–9. Earth Pressure against Wall.* Before determining the resultant earth pressure translate the assigned Cooper E-60 loading to an equivalent surcharge of earth. The 60,000-lb maximum axle load is spread over 5 ft of track (the distance between axles) and over 14 ft across the track (the distance, center to center, of tracks on a double-track line, and, in this case, double the distance from the back of wall to the center line of track). Fifteen per cent is added for the effects of vibration and impact. The sum gives the equivalent surcharge in uniform load which, divided by the weight of a cubic foot of earth, results in an added height of 9.9 ft to allow for the effect of the track. If the track were farther from the back of the wall, a line at 45° downward from the ends of the ties† would indicate approximately where the surcharge would first be felt on the wall. Transforming the surcharge to an equivalent uniform load is not necessary, only convenient; in Case IV the locomotive loading is treated as a series of concentrations.

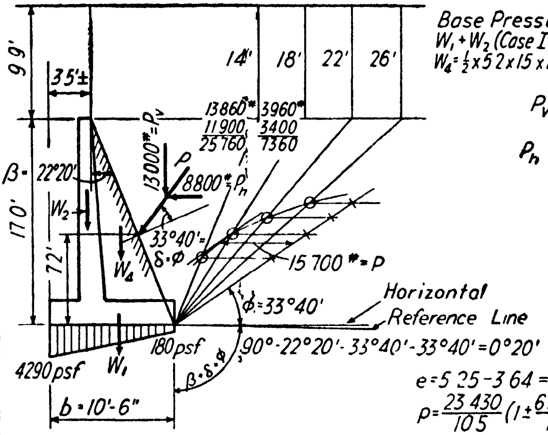
To illustrate the theories of earth pressure on walls and to give the student a comparison of the effects of different assumptions the resultant pressure will be evaluated for the following cases:

* The treatment of earth pressure here given is sufficiently complete for ordinary purposes. For the inclusion of a horizontal force (700 plf, more or less) along the top of the wall to represent possible frost pressure and for other useful suggestions see C. W. Dunham's The Theory and Practice of Reinforced Concrete, McGraw-Hill Book Co., 1939, pp. 196–244. The method there used for concentrated loads follows the tests of M. G. Spangler in University of Iowa Bulletin 140. These were correlated with the mathematical derivations of J. Boussinesq which regard earth as an elastic medium and could not consistently be combined with Coulomb's theory of the sliding wedge. The student should have in mind that the pressures obtained in this text are based on an assumed slight yielding or rotation of the wall.

† Although many engineers draw such a line at 45° with the horizontal as being conservative and easy to apply, tests show the slope to be nearer 60° with the horizontal or even at 2 vertical on 1 horizontal, which still further relieves the wall of pressure.

CANTILEVER RETAINING WALL — Sheet W2

Case II - Level Surcharge - Inert Wedge - Full Back Wall Friction



Base Pressure — Moments About Toe

$W_1 + W_2$ (Case I) $= 5530$ lb — 27600 lb-ft

$W_4 = \frac{1}{2} \times 5.2 \times 15 \times 100 = 3900 \quad \times 5.9 = 23000$

10,430 — 50600

$P_v = 13000 \quad \times 7.54 = 98020$

23430 — $148620 = M_R$

$P_h = 8800 \times 7.2 = 63360 \; M_{OT}$

$23430 \,\overline{)\,85260}$

3.64 ft from toe

$\beta = \tan^{-1}\frac{7.0}{17.0} = \tan^{-1} 0.412 = 22°20'\pm$

$e = 5.25 - 3.64 = 1.61$ ft

$$p = \frac{23430}{10.5}\left(1 \pm \frac{6 \times 1.61}{10.5}\right) = \begin{cases} 4290 \text{ psf Toe} \\ 180 \text{ psf Heel} \end{cases}$$

$$\text{Sliding Factor} = \frac{23430 \times 0.40}{8800} = 1.07 \; \therefore \text{Must Anchor}$$

Case III - Level Surcharge - Inert Prism - Full Back Wall Friction

Base Pressure — Moments about toe.

$W_1 + W_2 + W_3$ (Case I) $= 22720$ lb — 145000 lb-ft

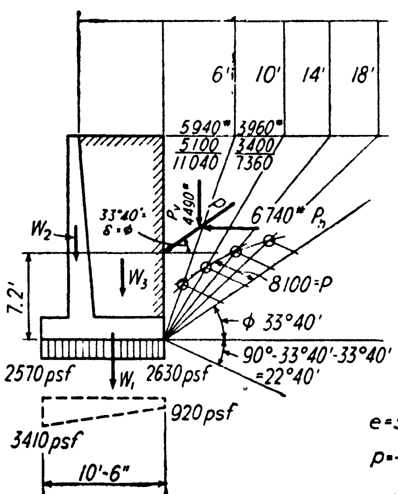
$P_v = 4490 \quad \times 10.5 = 47200$

27210 — $197200 = M_R$

$P_h = 6740 \quad \times 7.2 = 48600 = M_{OT}$

$27210 \,\overline{)\,143600}$

5.27 ft from toe



$e = 5.25 - 5.27 = -0.02$ ft

$$p = \frac{2710}{10.5}\left(1 \pm \frac{6 \times 0.02}{10.5}\right) = \begin{cases} 2570 \text{ psf Toe} \\ 2630 \text{ psf Heel} \end{cases}$$

$$\text{Sliding Factor.} \; \frac{27210 \times 0.4}{6740} = 1.62$$

If stabilizing effect of P_v is omitted from moment computations·

$W_1 + W_2 + W_3 = 22720$ lb — 145000 lb-ft

$P_h = -6740 \quad \times 7.2 = -48600$

$22720 \,\overline{)\,96400}$

4.24 ft from toe

$e = 5.25 - 4.24 = 1.01$ ft

$$p = \frac{22720}{10.5}\left(1 \pm \frac{6 \times 1.01}{10.5}\right) = \begin{cases} 3410 \text{ psf Toe} \\ 920 \text{ psf Heel} \end{cases}$$

$$\text{Sliding Factor:} \; \frac{22720 \times 0.4}{6740} = 1.35 \; \therefore \text{Must Anchor}$$

SHEET W-2

Case I: level surcharge; inert prism of backfill extending to a vertical plane through the top of the stem; no allowance for friction of earth on earth along this vertical plane.

Case II: level surcharge; inert wedge of backfill between heel of wall and top of stem; full allowance for friction along this boundary plane.

Case III: same as Case I but taking into account full friction along the vertical boundary plane.

Case IV: using rail concentrations in lieu of a uniform surcharge; inert wedge of backfill; full friction along this boundary plane.

Cases IV–A, B, and C: same as Case IV but with various combinations of loaded and unloaded tracks.

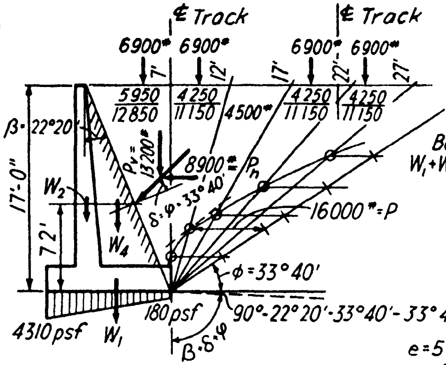
This plainly does not exhaust the logical possibilities which suggest themselves. Case I, omitting all consideration of friction on the boundary plane, is needlessly conservative; Case III, taking full friction into account, is considered by some to be too far in the other direction. Case II seems reasonable, though here, again, taking full friction into account may not be sufficiently safe. Some designers are inclined to follow the method of Case II, but use a friction angle of one-half that taken on the other side of the sliding wedge. This is reasonable as vibration in the soil might well prevent the full maximum friction from being developed on the inclined boundary plane.

Computation Sheet W1, Case I. For a level surcharge and pressure against a vertical plane, Rankine's formula gives the same result as a graphical solution by the Coulomb-Culmann method with wall friction neglected. The Rankine formula indicates that for these data the earth is equivalent to a liquid weighing 28.7 pcf; this makes the pressure 284 and 772 psf respectively at the top and bottom of the wall. The total resultant horizontal pressure is represented by the volume of the shaded prism, and acts through its center of gravity. The height x to this centroid is computed by the method outlined on page 153. By taking moments about the toe of the weights of stem, base, and superimposed earth on the heel, the restraining moment is found. Reducing this by the overturning moment and dividing by the total vertical force gives the distance from the toe to the point where the resultant cuts the base. The eccentricity from the center of the base to this intersection of the resultant force with the bottom of the wall is computed and from this the pressures under the toe and heel are obtained from the well-known

$$p = W/A \pm Mc/I = \frac{W}{A}\left(1 \pm \frac{6e}{d}\right)$$

CANTILEVER RETAINING WALL — Sheet W3

Case IV - Wheel Concentrations - Inert Wedge - Full Friction



Conc Load: $\frac{60000}{2\times5}\times 115\% = 6900$ lb

Base Pressure: Moments About Toe

$W_1+W_2+W_4$ (Case II) = 10 430 lb — 50 600 lb ft
13 200 — ×7.54 = 99 530
23 630 — 150 100 = M_R
P_h = 8 900 — ×7.2 = 64 100 = M_{OT}
23 630) 86 000
3.64 ft from toe

$e = 5.25 - 3.64 = 1.61$ ft

$$p = \frac{23630}{10.5}\left(1 \pm \frac{6\times1.61}{10.5}\right) = \begin{cases} 4310 \text{ psf Toe} \\ 180 \text{ psf Heel} \end{cases}$$

Sliding Factor $\frac{23630\times0.4}{8900} = 1.07$ Must Anchor

---

Case IV-A - Far Track Empty, Near Track Loaded
Same as Case IV as plane of rupture cuts off effect of far track

Case IV-B - Near Track Empty, Far Track Loaded.
Same as Case IV-C as plane of rupture cuts off effect of far track

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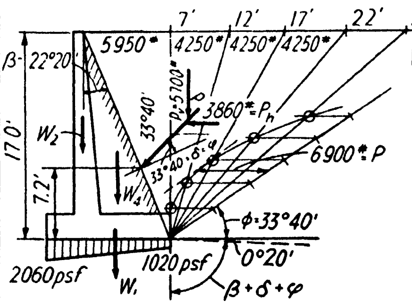
Case IV-C Both Tracks Empty



Base Pressure - Moments About Toe

$W_1+W_2+W_4$ (Case II) 10 430 lb — 50 600 lb-ft
P_v = 5 700 — ×8.17 = 46 600
16 130 — 97 200 = M_R
P_h = 3860 × 5.67 = −21 900 = M_{OT}
16130) 75 300
4.66 ft from toe

$e = 5.25 - 4.66 = 0.59$ ft

$$p = \frac{16130}{10.5}\left(1 \pm \frac{6\times0.59}{10.5}\right) = \begin{cases} 2060 \text{ psf Toe} \\ 1020 \text{ psf Heel} \end{cases}$$

Sliding Factor $\frac{16130\times0.4}{3860} = 1.67$ ∴ Must Anchor

Summary

	p Toe	p Heel	Hor Comp of Thrust	M Overturning	M Resisting	Factor $\frac{M_R}{M_{OT}}$	Sliding Factor
Case I	4280	50	8980	64700	145000	2.24	1.01
Case II	4290	180	8800	63360	148620	2.35	1.07
Case III	2510	2630	6740	48600	92200	3.96	1.62
	3470	920	6740	48600	145000	2.98	1.35
Cases IV, IV-A	4310	180	8900	64100	150100	2.34	1.07
Cases IV-B, IV-C	2060	1020	3860	21900	97200	4.44	1.67

Under each case the factor of safety against sliding is computed from $W \tan \delta / P_h$, which should be 2 or more, as explained on page 168. In Case I the frictional resistance is only very slightly greater than the horizontal thrust and a block of concrete for anchorage is necessary as shown on page 165.

Instead of the sloping back surface shown for the earth prism resting on and stabilizing the wall, some designers argue that a flatter plane should be chosen, extending upward from the extreme top corner of the base at an angle such that the obliquity of the stress thereon equals the angle of internal friction of the material. For a horizontal surface of backfill without surcharge this plane would make an angle with the horizontal equal to $45° + \phi/2$.

Sheet W2, Case II. Here the Culmann construction is applied exactly as outlined in the first part of this chapter. The various assumed planes of rupture are shown, and the vertical weight of each wedge computed. The resultant pressure is assumed to make an angle of ϕ above the normal to the sloping boundary plane of the inert wedge, so that the reference line through the heel of the wall is 0°20′ below the horizontal. The weights and moments of the various parts of the wall are obtained from Case I or by direct computation. The eccentricity of the resultant force is 1.61 ft, giving toe and heel pressures of 4290 and 180 psf respectively.

Sheet W2, Case III. Although the frictional force here assumed on the back of the earth prism may be effective in resisting overturning, there is argument as to whether it would result in a pressure distribution under the heel of the wall as shown shaded. Possibly conservative judgment would suggest omitting this vertical component of the earth thrust in taking moments about the toe of the wall, resulting in a pressure diagram as shown dotted.

Some designers argue that it is not proper to count on a rectangular prism of earth stabilizing the wall since on rupture only a much smaller triangular wedge would remain in position as failure starts.

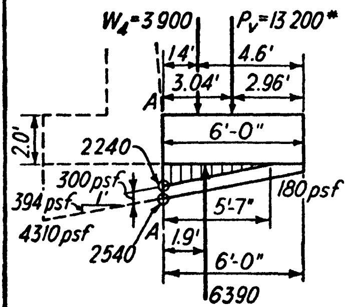
Sheet W3, Case IV. The loads of 6900 lb were obtained by dividing a 60,000-lb axle load by 2 rails and by 5-ft axle spacing and then adding 15 per cent for impact. The Culmann diagram indicates a resultant very nearly equal in magnitude and direction to that obtained in Case II. To determine the point of application the loads on the sliding wedge to the plane of rupture were divided by the base of the wedge, giving an equivalent surcharge of 9.9 ft, so x was taken as 7.2 ft for Case IV as in Case I.

Sheet W3, Tabulation of Results. Case I has long been in use and is the classical method employing the Rankine formulas. The Culmann-

CANTILEVER RETAINING WALL Sheet W4

Design of Section for Case IV

Base : Heel



Max Shear and Moment on Sect A-A

$P_V = -13\,200$ lb ; $\times 3.04 = -40\,200$ lb-ft

$W_4 = -3\,900$; $\times 1.4 = -5\,500$

Earth Pressure $+6\,390$; $\times 1.9 = +12\,200$

$V = 10\,710$ lb $\quad M = 33\,500$

$d = \frac{V}{vjb} = \frac{10\,710}{60 \times \frac{7}{8} \times 12} = 17''+ \quad d = \sqrt{\frac{M}{Rb}} \quad \sqrt{\frac{33\,500}{131}} = 16''$

Keep heel 24" thick ($d = 21''$) decrease shear

$v = \frac{V}{bjd} = \frac{10\,710}{12 \times \frac{7}{8} \times 21} = 49$ psi > 40 ∴ Spec Anch

$R = \frac{M}{bd^2} = \frac{33\,500}{(21)^2} = 76 < 139$ ∴ $f_c < 800$ psi

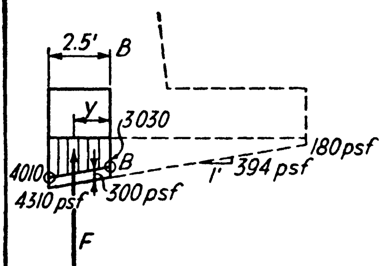
$A_s = \frac{M}{f_s jd} = \frac{33\,500}{18000 \times \frac{7}{8} \times 21} = 0.102$ □"/in. Top

$1''\phi$ @ $7\frac{1}{2}''$ c/c $= 0.106$ □"/in.

(Use $1''\phi$ - 7" c/c)

$u = \frac{vb}{\Sigma o} = \frac{49 \times 7\frac{1}{2}}{\pi} = 117$ psi > 100 ∴ Spec Anch

Base : Toe



$F = 2.5 \left(\frac{4010 + 3030}{2}\right) = 8800$#

$y = \frac{2.5}{3}\left(\frac{3030 + 2 \times 4010}{3030 + 4010}\right) = 1.31$ ft

Max Shear and Moment on Sect B-B

$V = 8800$#

$M = 8800$# $\times 1.31$ ft $= 11\,530$ lb-ft/ft or lb-in./in.

Keep toe 24" thick to match heel:

$v = \frac{V}{bjd} = \frac{8800}{12 \times \frac{7}{8} \times 21} = 40$ psi

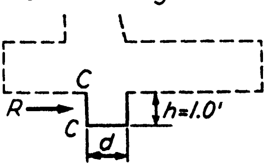
$R = \frac{M}{bd^2} = \frac{11\,530}{(21)^2} = 26 +$ (f_c is very low)

$A_s = \frac{M}{f_s jd} = \frac{11\,530}{18000 \times \frac{7}{8} \times 21} = 0.035$ □"/in. Bott

$\frac{1}{2}$□" @ 7" c/c 0.0357 □"/in.

$u = \frac{vb}{\Sigma o} = \frac{40 \times 7}{2} = 140$ psi > 100 ∴ Spec. Anch

Cut-off Wall or Anchorage



Passive Earth Pressure $= wh\left(\frac{1 + \sin 33°40'}{1 - \sin 33°40'}\right) = wh\,(3.5)$

at CC, $wh = 2900 \pm$ psf (Case IV)

Possible Resistance $= 3.5 \times 2900 = 10\,100$ psf

To Make Sliding Factor $= 2$; $R = 2 \times 8900 - 0.4 \times 23\,600$
$= 8\,450$ lb req'd

Min $h = \frac{8450}{10\,100} = 0.84$ ft. Use 1.0 ft for convenience

On Sect CC, $V = 8450 \quad M = 8450 \times 6 = 50\,700$ lb-in.

$d = \frac{V}{bjv} = \frac{8450}{12 \times \frac{7}{8} \times 40} = 20$ in.

If no tension reinforcement:

$d = \sqrt{\frac{50\,700 \times 6}{12 \times 60}} = 20.6$ in.

SHEET W-4

Coulomb construction permits easy computation of pressures on inclined surfaces and Case II seems better to fit actual conditions. In this problem the agreement between Cases I and II is considerably closer than the variation in data on earth. Case III shows for this problem a slight clockwise rotation of the wall and a higher heel than toe pressure, if the vertical component of the inclined thrust is included. Hence another computation was made, omitting the stabilizing effect of P_v but utilizing the friction on the inert prism to reduce P_h, a procedure used by many engineers although it is not strictly logical. Case IV for this problem produces the same thrust as Case II. It was included to show how easily the graphical construction handles concentrations, such as might occur with an adjoining structure resting on the earth.

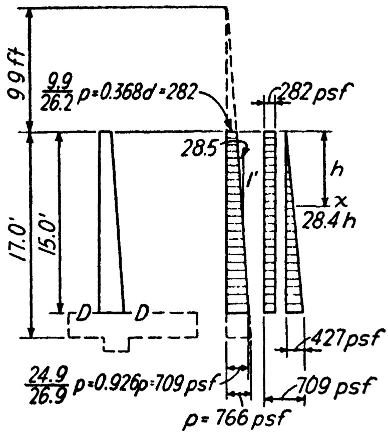
In view of the excellent agreement the balance of the wall design will be based on the assumptions of Case IV. The absence of load on either or both tracks would reduce the horizontal component of the earth pressure, and thus reduce the bearing at heel and toe of wall and the stresses in the vertical stem. It is not likely that any condition of partial loading could be a factor in the design but Cases IV–A, B, and C have been included to show the methods involved as well as the desirability of investigating all possible conditions.

10–10. Base Width. (Computation Sheets W1, W2, W3.) Designers endeavor to tell as much of the story of their work by sketches as possible, and use a minimum of written description.* The first step in retaining wall design is to assume a width of base that shall be as narrow as possible for economy yet sufficiently wide for stability against overturning. This can be approximated by taking 0.40 to 0.45 of the overall height of wall if there is no surcharge and 0.60 to 0.65 of the height for a fairly heavy surcharge. Such arbitrary rules are, of course, very rough approximations to be verified by computation. The designer must keep an open mind and not fix values too rapidly until they can be checked. He should not hesitate to go back and change any section that proves to be unnecessarily strong or too weak. If figures are kept

* In studying any section of these computations first read the sketch and check the calculations made on it. Values that appear without explanation are either repeated from an earlier sketch or are calculated in the accompanying computations or follow so directly from the data shown that details were not considered necessary. The computations on Sheet W2 illustrate this: the wall weights W_1 and W_2 in Cases II and III are repeated from the computations for the same items in Case I; the pressures under the toe and heel of the walls in Cases II and III are worked out to the right of the diagrams and are repeated under the sketches for ready reference; the vertical and horizontal components of the inclined resultant earth pressure were obtained by multiplying by the sine and cosine of 56° respectively, which is such a simple slide-rule operation that the result was recorded without explanation.

CANTILEVER RETAINING WALL — Sheet W 5

Vertical Stem



From Case IV $P_h = 8900^{\#} = \frac{17}{2}(p+0.368p)$ $\quad p = 766$ psf

Max Shear and Moment on Sect D-D

$v = \frac{15}{2}(282+709) = 7450$ lb

$d = \frac{V}{bjv} = \frac{7450}{12\times\frac{7}{8}\times 40} = 17.8$ in.

$M = 282\times 15\times 7\frac{1}{2} + 427\times\frac{15}{2}\times\frac{15}{3}$

$= 31\,700 + 16\,000$

$= 47\,700$ lb-ft/ft or lb-in/in.

$d = \sqrt{\frac{M}{Rb}} = \sqrt{\frac{47\,700}{139}} = 18.5$ in.

Make Stem 12" at top, 24" at bottom

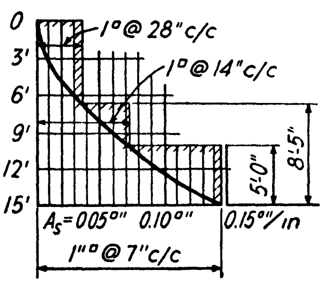
At any level:

$V_x = 282\,h + 14.2\,h^2$

$M_x = \frac{282h^2}{2} + \frac{14.2h^3}{3} = 141h^2 + 4.73h^3$

Stem Schedule

h	V	d	$v=\frac{V}{bjd}$	M	$R=\frac{M}{bd^2}$	A_s	Rods
3	980	11.4	8+	1400	11	0.0075	See Sketch Below
6	2200	13.8	13+	6100	32	0.028	
9	3560	16.2	21+	14 900	57	0.058	
12	5430	18.6	28	28 500	83	0.098	
15	7450	21	34	47 700	108	0.145	1□@7"c/c=0.143



Cutting off Steel

$u = \frac{vb}{\Sigma o} = \frac{34\times 7}{4} = 60$ psi

Embedment: $L = \frac{18\,000}{4\times 100} = 45$ in (Hook Bottom)

Temperature Steel: $0.002\times 12\times 18$ average $= 0.44$ sq in.

$\left\{\begin{matrix}\frac{1}{2}^{\phi}@\,9''\,c/c\\ \text{Front}\end{matrix}\right\} + \left\{\begin{matrix}\frac{1}{2}^{\phi}@\,12''\,c/c\\ \text{Back}\end{matrix}\right\} = 0.46$ sq in.

as suggested on the accompanying design sheets, neatly scratching out a few values and substituting revised ones will be all that is necessary.

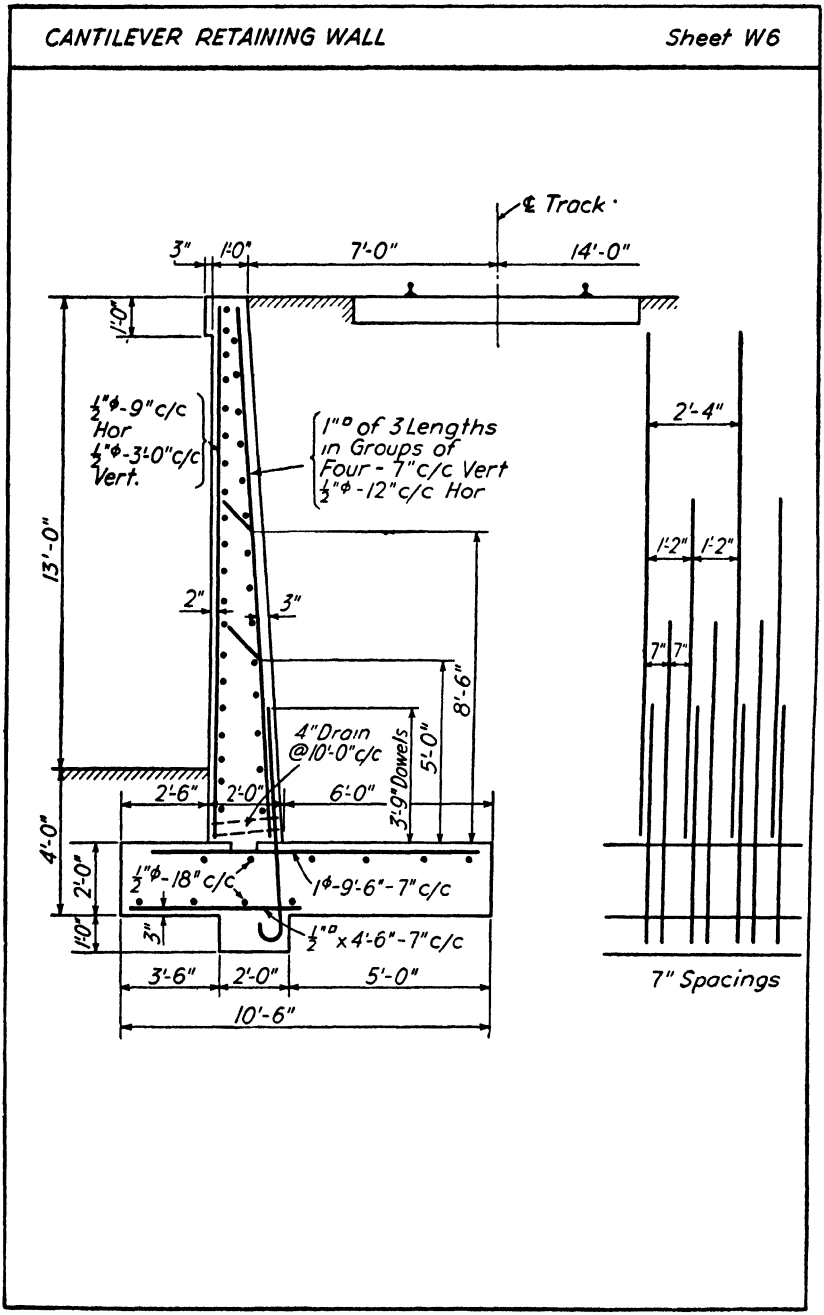
As a further check on the base width the computation in the middle of Sheet W1 was made as soon as P was found in Case I. By transforming the wall into a block of earth and by taking moments about the middle-third point of the base, a fairly good estimate of the width needed for these particular conditions is obtained.

A glance at Cases IV–A, B, and C indicates that no increase in base width would be necessary for any partial loading condition considered.

10–11. Stem Thickness. (Computation Sheet W5.) The complete design of any element of this wall is impossible until the others are also designed, so a trial computation was made to find out how thick the vertical stem must be made at the bottom, assuming a base 2 ft thick. The stem thickness must be sufficient to make it safe in bending and in shear, shear being a measure of the diagonal tension. Since no stirrups or bent rods are to be used a low shear stress has been set. This preliminary check on the stem was roughly made on a piece of scratch paper before making the moment computations under Case I. It was subsequently perfected into the form shown on Sheet W5 after the pressure computations were completed. For the first approximation it is sufficiently accurate to take the external shear at the bottom of the stem as somewhat less than P_h in Case I, allowing for the decrease due to the thickness of the base. Let V be taken about 8000 lb. The moment can be roughly estimated as 8000 (7.2–2.0) = 41,600 lb-ft/ft or lb-in./in. Using equation 7–1 with $v = 40$ psi: $v = V/bjd$; $d = V/bjv = 8000/(12 \times \frac{7}{8} \times 40) = 19$ in. Using equation 9–7, $R = M/bd^2$; $d = \sqrt{M/Rb} = \sqrt{41{,}600/131} = 18$ in. To the greater of these, 3 in. must be added as covering for the reinforcement resulting in a minimum stem thickness of at least 22 in. It will be satisfactory to proceed on the assumption of a 2-ft stem thickness, possibly adjusting it an inch or two as the design proceeds.

10–12. Base Pressures and Sliding. (Computation Sheets W1, W2, W3.) The final determination of the intensities of base pressure are now possible since variations in the base thickness produce entirely negligible differences. The computations follow closely the model set in Ex. 10–1 and require little explanation. They are worked out separately under each case of earth pressure considered.

The earth in front of the toe cannot be counted on to resist sliding not only because it may be absent when needed but also because after it is in place it may shrink and draw away from contact with the concrete. The factor of safety against sliding is obtained from $W \tan \delta/P_h$ and should be 2 or more for adequate safety. Computations of this


CANTILEVER RETAINING WALL
Sheet W6
℄ Track
3"
1'-0"
7'-0"
14'-0"
1'-0"
13'-0"
½"ϕ-9"c/c Hor
½"ϕ-3'-0"c/c Vert.
1"▫ of 3 Lengths in Groups of Four - 7"c/c Vert
½"ϕ-12"c/c Hor
2"
3"
8'-6"
5'-0"
4"Drain @10'-0"c/c
3'-9" Dowels
2'-6"
2'-0"
6'-0"
4'-0"
2'-0"
½"ϕ-18"c/c
1ϕ-9'-6"-7"c/c
3"
1'-0"
½"▫x4'-6"-7"c/c
3'-6"
2'-0"
5'-0"
10'-6"
2'-4"
1'-2"
1'-2"
7"
7"
7" Spacings

SHEET W-6

factor are made for each case of earth pressure considered and each time some form of anchorage is indicated, such as the cut-off wall shown at the bottom of Sheet W4.

10–13. Design of Base. (Computation Sheet W4.) In this design the simplest form of wall is being used with a base of uniform thickness throughout and without fillets at the junction of stem and base. The heel appears first on the sheets simply because it was judged that probably it would prove the critical portion, and this turned out to be so. Three inches of cover is assumed over the reinforcing steel to protect it from the earth.

Note the record made of the slope of the pressure line on the diagram of the heel: 394 psf variation per lineal foot of base width. This ratio was used in obtaining the intermediate intensities in preference to setting up a proportion from the similar triangles involved. This manner of dealing with sloping lines facilitates checking.

The loading on the heel includes the weight of the inert wedge previously designated W, and the vertical component of the inclined earth thrust P_v, together with the upward pressure on the base taken from the computations of Case IV. These upward pressures include an allowance of 300 psf to support the dead weight of the heel itself. As the heel rests directly on the soil at uniform bearing, this 300 psf need not enter into the moment computations and so has been deducted from the upward earth pressure. The shear and moment on Section AA at the junction of heel and stem are computed as shown. The bending moment is expressed in pound-feet per lineal foot of wall, which is numerically the same as pound-inches per lineal inch of wall. This latter set of units simplifies the determination of steel area somewhat as explained below.

Just as in the preliminary design of the stem, the minimum thickness of heel to resist shear is obtained from $d = V/bjv$, using 60 psi for v because the tension steel will have to be anchored. The minimum thickness to resist bending is obtained from $d = \sqrt{M/Rb}$. These values of 17+ and 16 in. are both less than the 21 in. assumed, but rather than reduce the thickness of base it was decided to decrease the intensity of shear. The advantage of keeping the bending moment in pound-inches per inch of width is now apparent; since A_s is obtained in square inches per inch of length of wall, the required spacing of rods to provide this area is obtainable directly by dividing the area of a single bar by the area required per inch. The steel area required is obtained with 1-in. round rods at 7½-in. centers. The note to use a 7-in. spacing was added when the stem was completed and it was found that the other reinforcing steel was spaced 7 in. apart. Keeping the heel rods at that

same spacing would prevent conflict of bars where they pass each other.

The steel area was determined by equation 9-7, using an average value $j = 7/8$. The degree of approximation involved is of interest. Here $p = (0.102)/21 = 0.0049$, $n = 15$, and from Fig. A-1, $j = 0.888$; then $A_s = 33{,}500/(18{,}000 \times 0.888 \times 21) = 0.100$ sq in. per in. width — a negligible difference. Bond stress was checked at the junction of heel and stem from equation 7-2, $u = vb/\Sigma o$, using the value of v already found, and for b taking the distance center to center of rods. It is more than the 100 psi allowed without special anchorage, so a note is added calling for special anchorage of the heel reinforcement.

The design of the toe is very similar to that of the heel. Here the only force acting is the upward earth pressure of varying intensity reduced by the 300 psf of direct pressure due to the bearing of the toe slab on the earth. Conservative custom requires that no account be taken of the overburden of earth on top of the toe, as already explained.

The cut-off wall or key is designed at the same time as the rest of the base. The resistance offered to sliding by the cut-off wall may be estimated by assuming that all or part of the passive resistance of the earth in front of it is available. As shown on Sheet W4, the intensity of this passive resistance is 3.5 times the intensity of vertical pressure at any point. Theoretically the intensity of passive resistance under the stem of the wall is 10,100 psf, so that a 1-ft projection would develop 10,100 lb of horizontal resistance as against 8450 lb of thrust required after deducting the total frictional resistance from double the horizontal earth thrust. To insure this passive resistance in any reasonable degree the earth must be thoroughly compacted in front of the key wall or, if the natural soil is firm and hard, it must be left undisturbed and the key wall poured and compacted in a natural trench without forms. Evidently the efficiency of this arrangement depends upon the amount of movement that takes place in developing the necessary resistance. No large movement can take place except as some 3.5 ft of earth in front of the projection is pushed along ahead of it, shearing or sliding along a horizontal plane. The resistance offered to shear (earth sliding on earth) is very considerable, approximately $(1/1.5) \times 3.5 \times [(4310 + 2900)/2] = 8400$ lb (tangent of angle of friction $= 1/1.5$). The shear in the cut-off wall is computed at its junction with the base; its width is computed at this same section and made sufficient to keep the shear and tension in plain concrete less than 60 psi.

10-14. Design of Stem. (Computation Sheet W5.) By referring to Case IV the total P_h is found as 8900 lb, and since the distribution is taken as hydrostatic, the triangular stress variation indicated on Sheet

W5 permits the establishing of ordinates at top and bottom of the stem. The shear and moment at the bottom are next computed and the minimum thicknesses of stem to carry each of these are ascertained. It is then decided not to reduce the concrete size by the couple of inches possible, but to have lower stresses in the stem. Any material variation in thickness of members would affect the weight of wall and its resistance to overturning and would necessitate refiguring the moments and possibly changing the base width, which should be done if any economy can be obtained. However, underreinforced sections are less expensive than a condition of balanced reinforcement, so in this case a 2-ft stem is satisfactory.

The bending moment drops off very rapidly, being a cubic function of the depth, so computations are made at 3-ft lifts and the steel area varied accordingly. The steel areas are plotted and points located where different ones of the vertical bars may stop. Reference to the sketch on Sheet W6 will make this clearer. As shown there, at the theoretical stopping point the vertical rods are bent across the stem into the compression face for anchorage.

In order to prevent vertical cracks due to shrinkage and to temperature stresses, horizontal reinforcement is required, the usual amount being 0.002 to 0.003 of the concrete area. This is not entirely sufficient to accomplish the end desired and long walls should be built in sections of 60 ft or less, separated by expansion joints of elastic material, the walls being doweled together with short rods encased in tubes which hold the walls in alignment but permit their moving slightly.

10–15. Conclusion. (Computation Sheet W6.) A sketch showing the complete design is shown on Sheet W6 and is made in enough detail to give all the information needed by a draftsman preparing the working drawings.

Certain additional steel is here shown which has not previously been mentioned: ½-in. rounds vertically in the stem to support the horizontal temperature steel and ½-in. rounds longitudinally in the base to tie the whole together and insure that it acts as a unit.

In addition to the drains shown, a layer of porous material should be laid as suggested on page 156. This will assist in relieving the wall of any hydrostatic pressure.

A construction joint is shown between the stem and base, with a tongue, say 1⅝ by 5⅝ in., made by burying a 2 by 6 in. plank with slightly beveled edges in the base when it is poured. The width of this key must be sufficient to carry the total shear at a limit of about 125 to 150 psi for its unit value.

For ease in placing, the vertical wall rods are spliced at the construc-

tion joint between stem and base. Dowels are embedded in the base hooked into the cut-off wall, and made to project 3 ft 9 in. above the top of the base. This amount of lap comes from $L = (f_s d/4u) = [18{,}000/(4 \times 100)]d = 45d$ or 45 in. for 1-in. rods. The shortest group of verticals need not be spliced as they extend less than 6 ft above the base.

When the designer prepares the rough sketch he checks to see that the bars are the proper length for bond, and watches similar details. On short, stubby beams bond is often a deciding factor. The heel rods must run 45 diameters past the back of the stem for anchorage, and the toe rods must run 45 diameters past the face of the stem. In this instance, giving the length of the bars is the best way to insure proper embedment.

Although ½-in. square bars have been freely used in this design and also later in the text it should be noted that the rolling of this bar was discontinued in the summer of 1942 as an item of war economy ("Simplified Practice Recommendation R 26–30").

The student should realize that in this chapter we have been dealing with matters where experience often overrules theory, sometimes with good results and sometimes diasastrously. For example, one of our large railroads has decided, on the basis of long experience, that with the methods of earth pressure determination in use in their offices it is not necessary to use a surcharge greater than 600 psf, 6 ft of earth, to cover the effect of a loaded track paralleling a wall. The highway department of one of our large eastern states replaces the middle-third criterion by the middle-half, thus allowing the line of pressure to strike at or within the outer quarter-point. To estimate the effect of these or other rules of practice upon the final design it should be remembered that it is necessary to scrutinize the entire design procedure.

CHAPTER XI

HIGHWAY BRIDGES

11–1. Highway bridges of reinforced concrete are built as arches, rigid frames, cantilevers, and as continuous and non-continuous beams. This chapter deals with only the most common of these forms, the simple span, of which there are three general types: the slab bridge, used with economy for spans up to about 20 ft; the half-through and the deck-beam bridge, the half-through and the deck-girder bridge. In a beam bridge the load of the road slab is carried by beams which rest on the abutments; in a girder bridge, floor beams, with or without stringers, carry the floor slab, and in turn are supported by the main girders. A girder is usually defined as a beam that receives its principal load from other beams.* The ordinary limit of span for beam bridges ranges from 40 to 60 ft. For longer spans girder bridges are the rule on account of their greater rigidity. One of the longest spans on record for a girder bridge is the 224-ft center span of the three-span continuous bridge over the Rio de Peixe in Brazil.

In this chapter are given the design computations for a slab and a deck-beam bridge. A half-through beam bridge is similar in cross section to the slab bridge shown on page 179, differing from it in that the light balustrade is replaced by heavy beams carrying the roadway slab, which spans across from side beam to side beam. This type cannot be used economically for roadways wider than about 20 ft.

11–2. Concentrated Loads on Slabs. Highway bridge slabs must be designed to carry the heavy concentrations brought upon them by the wheels of modern motor trucks. The question of the width of slab which supports any given concentrated load arises at once. This is illustrated in Fig. 11–1, where a wheel is shown resting at the center of a wide slab, supported along the edges *ab* and *cd*. The strip of slab of width W immediately beneath this wheel cannot deflect under the load without at the same time causing the deflection of the adjacent strips; in this way the effect of the load is distributed over an indefinite width. This is one of the situations so common in structural design in which it is important to note carefully the shape into which the

*** A floor beam is a transverse member at right angles to the axis of the bridge; stringers are longitudinal beams spanning between floor beams. The terms " beam " and " girder " are often used interchangeably.**

element is forced by the load. Many slabs such as this have been designed as though they changed from a plane to a cylindrical surface under load. A more accurate picture is that of a saucer-shaped depression under the wheel. It is plain, then, that there is an elongation of

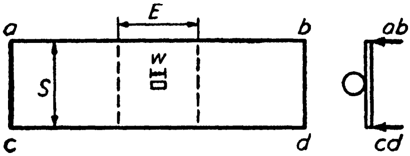


FIG. 11–1

the bottom fibers parallel with the supports as well as in the direction of the span. This phenomenon explains the modern requirement of heavy distribution reinforcement in the bottom of bridge slabs placed normal to the main steel. Several experimenters have studied the problem and on the basis of their data various rules have been proposed for use in design.* The ruling of the American Association of State Highway Officials, "Standard Specifications for Highway Bridges," third edition, 1941, is reproduced in part below and its use is illustrated in the succeeding design computations.

3.3.2 Distribution of Loads and Design of Concrete Slabs. *Bending Moment.* Bending moment shall be calculated according to methods given under Cases A, B, C and D.

* The spreading or crosswise distribution of a concentrated load in an elastic slab of infinite or finite width on rigid or yielding supports, with simple or continuous spans, and with the edges plain or stiffened with beams or ribs, is a problem that is of considerable importance and has received a good deal of attention.

For theoretical studies see: A. Nadai, Die elastischen Platten, Julius Springer, Berlin, 1925; H. M. Westergaard, "Computation of Stresses in Bridge Slabs due to Wheel Loads," *Public Roads*, Vol. 11, No. 1, March, 1930.

For practical applications and test data see: H. M. Westergaard and W. A. Slater, "Moments and Stresses in Slabs," Proceedings, A.C.I., Vol., 17, 1921; D. L. Holl, "Analysis of Thin Rectangular Plates Supported on Opposite Edges," Iowa Eng. Exp. Sta., Bul. 129, 1936; H. R. Erps, A. L. Googins, and J. L. Parker, "Distribution of Wheel Loads and Design of Reinforced Concrete Bridge Floor Slabs," *Public Roads*, Vol. 18, No. 8, Oct., 1937; F. E. Richart and R. W. Kluge, "Tests of Reinforced Concrete Slabs Subjected to Concentrated Loads," Univ. of Ill., Bul. 314, June, 1939; N. M. Newmark, "A Distribution Procedure for the Analysis of Slabs Continuous over Flexible Beams," Univ. of Ill., Bul. 304, June, 1938; V. P. Jensen, "Solutions for Certain Rectangular Slabs Continuous over Flexible Supports," Univ. of Ill., Bul. 303, June, 1938; V. P. Jensen, "Moments in Simple Span Bridge Slabs with Stiffened Edges," Univ. of Ill., Bul. 315, Aug., 1939; C. W. Dunham, Theory and Practice of Reinforced Concrete, McGraw-Hill Book Co., pp. 271–281; N. M. Newmark, "What Do We Know About Concrete Slabs?" *Civil Engineering*, Sept., 1940.

The above are for the reader's reference. In the design problems in this chapter the 1941 A.A.S.H.O. recommendations are followed throughout.

Case A. Main Reinforcement Perpendicular to Traffic. Slabs shall be designed for standard H or H–S truck* loadings.

Distribution of Wheel Loads		*Formula for Moment* Freely supported spans	Continuous spans
Spans 2 to 7 ft:	$E = 0.6S + 2.5$	$+0.25 \frac{P}{E} S$	$\pm 0.2 \frac{P}{E} S$
Spans over 7 ft:	$E = 0.4S + 3.75$	$+0.25 \frac{P}{E} S$	$\pm 0.2 \frac{P}{E} S$

Case B. Main Reinforcement Parallel to Traffic.
Spans 2 to 12 feet; standard H or H–S truck loadings.
Distribution, $E = 0.175S + 3.2$

Moment, freely supported spans $= +0.25 \frac{P}{E} S$

Continuous spans $= \pm 0.2 \frac{P}{E} S$

The formulas for distribution and moment, Cases A and B, include the effect of all wheel loads placed in positions to produce maximum moments. Continuous spans shall be designed in accordance with the above formula unless moments are calculated by more exact methods which may permit a greater reduction.

Case C. Main Reinforcement Parallel to Traffic. H Loading.
Spans over 12 ft.

The slab shall be designed for the loading, truck or lane, which produces maximum moment. Loads shall be distributed as follows:

(*a*) Wheel loads: $E = \dfrac{10N + W}{4N}$ Load per foot of slab $= \dfrac{P}{E}$

(*b*) Lane loads:

Uniform load $= \dfrac{NQ}{0.5W + 5N}$ per square foot of slab

Concentrated load $= \dfrac{NP'}{0.5W + 5N}$ per foot width of slab

* The standard H truck loading is of dimension and weight distribution shown on the design sheets in this chapter (SB1 and B1), three weights being specified, 10, 15, and 20 tons. The H–S truck loading consists of a 20- or 15-ton truck as above, with a semi-trailer following, a pair of wheels of the same total weight as the rear truck wheels, following at 14-ft distance. When long loaded lengths are involved lane loadings replace truck loading, a lane loading representing a train of trucks. The lane loading for H–20 is shown on design sheet B1 for H–15 and H–10 weights are in proportion. The lane loading for the H–S system is the same as that of the corresponding H–20 or H–15 except that the concentrated load is increased 78 per cent for moment and 54 per cent for shear; it is to be used for loaded lengths over 40 ft.

Case D. Main Reinforcement Parallel to Traffic. H-S Loading. Spans over 12 feet.

The truck loading shall be used for loaded lengths over 12 feet and up to and including 40 feet. Lane loading shall be used for loaded lengths over 40 feet. The wheel loads and lane loads shall be distributed as specified in items (*a*) and (*b*), Case C.

In Cases A, B, C and D:

S = effective span length as defined under "Span Lengths"
E = width of slab over which a wheel load is distributed
N = maximum number of lanes of traffic permissible on bridge
W = width of roadway between curbs on bridges
W = width of graded roadway across culverts
Q = uniform lane load per linear foot of lane
P = load on one wheel
P' = concentrated lane load per lane

The moment for slabs over 12 feet, Cases C and D, shall be calculated as follows: the loads per foot of slab shall be determined according to the method given for distribution of loads. The loads thus determined shall be placed on the span or spans in position to cause maximum positive or negative moments. The moment shall be calculated in accordance with standard practice for design of simple and continuous spans.

Edge Beams (Longitudinal). Edge beams shall be provided for all slabs having main reinforcement parallel to traffic. The beam may consist of the curb section reinforced, of a beam support or of additional slab width. It shall be designed to resist a live load moment of $0.01PS^2$ where P = the wheel load and S = span length.

The moment as stated is for a freely supported span. It may be reduced 20 per cent for continuous spans unless a greater reduction results from an exact analysis.

Distribution Reinforcement. Reinforcement amounting to 50 per cent of the main steel required for positive moment shall be placed in the bottom of all slabs, normal to the main steel, in the middle half of slab only, to provide for lateral distribution of the loads.

Shear. Slabs designed for bending moment in accordance with the foregoing rules shall be considered adequate without shear reinforcement.

Unsupported Edges, Transverse. The design assumptions of this article do not provide for the effect of loads near unsupported edges. Therefore, at the ends of the bridge and at intermediate points where the continuity of the slab is broken, the edges shall be supported by diaphragms or other suitable means. The diaphragms shall be designed to resist the full moment and shear produced by the wheel loads which can come on them.

The reader is referred to the complete specifications for comprehensive detailed general information concerning the design of modern highway bridges.

11–3. Design of a Slab Bridge. The example chosen is the design of a slab bridge to carry a 26-ft macadam roadway across a 16-ft clear opening. The loads are those usual for heavy traffic and are shown with other necessary data on the first computation sheet. The working stresses are those designated by the A.A.S.H.O. (1941) for the quality of concrete here used. The designer completes the assembly of data on this first sheet with a cross-sectional view that shows the general features of the structure. The balustrade and curb will be poured after the slab has set and so the deeper section at the edge will be of no assistance in carrying dead load; its live load utility will be neglected. The slab, then, acts as a rectangular beam about 29 ft wide with supports about 17 ft apart center to center.*

The A.A.S.H.O. formula was used (see Computation Sheet SB1) to determine the width of slab supporting one wheel. The moment and shear were computed for a strip of slab 12 in. wide, maximum moment occurring with the heavy wheel at the center of the span and maximum shear with it at the end, bringing the lighter wheel 3 ft on the bridge. Equations 9–8 and 7–1 were used to determine the depth required by the given stresses in bending and to check the shear intensity, the value of R being taken from Fig. A–2 in the Appendix. To this depth $1\frac{1}{2}$ in. were added to give the total thickness, thus providing about 1 in. of concrete below the steel to protect it from corrosion. A.A.S.H.O. 3.7.7. permits 1 in. cover in slabs.

The actual value of d taken for the slab is so near to that theoretically necessary for balanced design that the steel area was calculated without discernible error by equation 9–7 (see Computation Sheet SB2), using the approximate value for j of $\frac{7}{8}$. Actually $j = 0.881$ so this assumption is very close. The operation of choosing the steel and its spacing consists in dividing a number of bar areas by the area required per inch, thus determining the spacing for each size.

In this design a computation of shear intensity, based upon spreading a wheel concentration over a reasonable width of slab, confirms the judgment of the specification that slabs designed for bending moment by its rules do not require diagonal tension reinforcement.

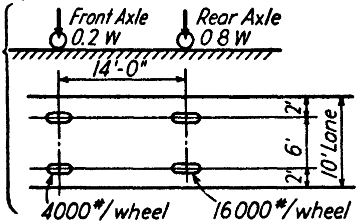
Even when no diagonal tension steel is needed, as in this case, it is customary always to bend up a portion of the main reinforcement, as shown on Computation Sheet SB2, to make the beam more dependable. This bent steel is arranged so that a few sloping bars cross the vertical

* Although simple span slab bridges are built, it should be pointed out that it is usually more economical to develop the negative moments around the corners at the abutments, using the horizontal earth pressure on the walls to balance the vertical load on the slab, and vice versa. See p. 232.

COMPUTATIONS FOR SLAB BRIDGE — Sheet SB1

Data

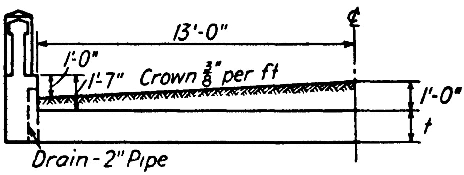
Roadway: 26'-0" Wide
Span: 16'-0" Clear (17'-0" c/c bearings, assumed)
Wearing Surface: Ordinary macadam surface carried across bridge
Loading: H-20 Truck



Impact: $I = \frac{50}{L+125} = \frac{50}{142} = 0.352$ Tire Width = 20 in.
$= 35.2\%$

Materials: Concrete, $f_c' = 3000$ psi; $n = 10$
Steel: Structural grade deformed bars

Stresses: $f_c = 1000$ psi
$f_s = 18{,}000$ psi
$v = 60$ psi if longitudinal bars not anchored; 90 psi if anchored (no web reinforcement)
$u = 100$ psi without anchorage
$= 150$ psi with anchorage

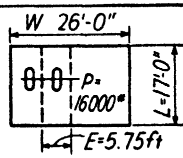


Slab

Width of slab supporting wheel load

$$E = \frac{10N + W}{4N} \quad \text{where } N = \text{no. of lanes} = 2$$

$$= \frac{20+26}{8} = 5.75'$$



For 12" wide strip

		Moment	Shear
Live:	$\frac{16000}{5.75} = 2780^{\#}$	$\times \frac{17}{4} = 11{,}820$ pf	$2780^{\#}$
	$\frac{4000}{5.75} = 695^{\#}$		$\times \frac{3}{17} = 130$
Impact: 35%		4140	1020
Dead: 12" Roadway = 130 psf			
Try 15" Slab = 188	318	$\times \frac{17^2}{8} = 11{,}490$	$\times \frac{17}{2} = 2700$
		27,450 pf	$6630^{\#}$

M & V

Depth

$$d = \sqrt{\frac{M}{Rb}} = \sqrt{\frac{27450}{157}} = 13.2''$$

1.5" Cover
15" Slab $\quad d = 15 - 1\tfrac{1}{2} = 13\tfrac{1}{2}''$

$$\text{Check: } v = \frac{V}{bjd} = \frac{6630}{12 \times \frac{7}{8} \times 13\frac{1}{2}} = 47 \text{ psi} < 60 \quad \therefore \text{O.K}$$

plane through the slab above the face of the abutment, thus lessening the danger of any vertical crack at this section, such as might result from shrinkage or temperature restraint.

The cross reinforcement resists the tension parallel to the supports as already described and so binds the whole slab together and prevents longitudinal cracking. This insures the integrity of the section, which is necessary if the effect of a concentrated load is to be distributed over a width of slab greater than the width of the area under bearing. This explains the term "distribution reinforcement." These bars are also useful in construction in holding the main reinforcement in place, the two sets of rods being wired together. The amount of this cross steel is specified only for the middle portion of the slab. In the outside quarters, adjacent to the supports, the usual rule for temperature reinforcement was used ($p = 0.2\%$) although, of course, the resistance to transverse contraction and expansion offered by the abutments is very much less than would be provided by rigid restraint along the sides of the span. A common practice is to use ½ in. square or ⅝ in. round bars, spaced 12 in. on centers, which furnish about the same area as that here used.

With the short spans of slab bridges it is not necessary to take elaborate precautions to prevent temperature stresses due to the restraining action of the supports. A double layer of heavy tar paper at one end prevents adhesion of the bridge slab to the abutment. It is not uncommon with short spans to provide for movement at both ends in this same way.

The following is ordinary reinforcement for a hand rail such as that here used: vertically, ½ in. square bars 12 to 18 in. on centers, with hooked ends well anchored in the slab: longitudinally, two ½ in. square bars in the top and nominal temperature reinforcement below.

In order that no water shall stand on the bridge deck, weep-holes or drains should be provided as shown. For pleasing appearance a slight camber, $\frac{1}{80}$ in. per ft, a total of about ⅜ in., should be given the structure. A slight sag below the true straight level of the supports due, for example, to deflection of the formwork on pouring the bridge, is very unsightly.

11-4. Design of a Deck-Beam Bridge. (Computation Sheets B1 to B5.) For this example a deck-beam bridge was chosen with a clear span of 40 ft. The general data as to roadway, loads, and stresses are the same as in the previous article (Sheet B1), except that a 2-in. bituminous wearing surface is used.

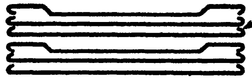
11-5. Slab. (Computation Sheet B1.) The slab is treated as a rectangular beam, continuous over the several longitudinal supporting

COMPUTATIONS OF SLAB BRIDGE — Sheet SB2

Steel

$A_s = \frac{M}{f_s jd} = \frac{27\,450}{18\,000 \times \frac{7}{8} \times 13\frac{1}{2}} = 0.128$ sq in./in. • $1''\phi$ @ $6''$ c/c $= 0.13$ sq in./in.

Assume Steel thus:



$u = \frac{vb}{o} = \frac{47 \times 9}{\pi} = 134$ psi

$> 100 < 150$

$\therefore$ Anchor

Overload

Moment: $L = 2 \times 11\,820 = 23\,640$

$I = 2 \times 4140 = 8\,280$

$D = \underline{11\,490}$

$43\,410$

$$\begin{cases} R = \frac{M}{bd^2} = \frac{43\,410}{(13\frac{1}{2})^2} = 238 \\ p = \frac{0.79}{13 \times 6} = 0.0101 \end{cases}$$

From Fig A-2: $\left.\begin{array}{l} f_s = 26\,000 \text{ psi} < 18\,000 \times 1.5 \\ f_c = 1500 \text{ psi} = 1\,000 \times 1.5 \end{array}\right\}$ O. K.

Slab Cross Steel

$\left.\begin{array}{l} \text{End Quarters: } 0.002 \times 12 \times 13\frac{1}{2} = 0.324 \text{ sq in /ft} \\ \frac{5}{8}\phi \text{ @ } 12'' \text{ c/c} = 0.31 \text{ '' '' ''} \\ \text{Middle Half: } \frac{1}{2} \text{ of } 1\phi \text{ @ } 6'' \text{ c/c} = 1\phi \text{ @ } 12'' \text{ c/c} \end{array}\right\}$ All in bottom of slab

beams, and with a width equal to the length of the bridge. For simplicity a typical strip 1 ft wide is used in the computation. The design, based largely on the A.A.S.H.O. specification (1941) resulted in a 7-in. thickness. It is good practice to set the limit of minimum thickness for bridge slabs at 6 in.

Since the floor slab is continuous over the several longitudinal beams that support it, the bending moment was reduced 20 per cent (see A.A.S.H.O. specification). The reason for this is discussed in Chapter XII. It was assumed that the maximum positive moment at midspan equals the maximum negative moment over the supports, which is probably reasonably near the truth.

Perhaps the most peculiar feature in these figures is that the maximum shear is not computed. This is because shearing stress is used as a measure of diagonal tension, which will be within allowable limits when the moment is distributed by the A.A.S.H.O. recommendations.

Equal areas of tension steel are required in the bottom of the slab between beams and in the top over the beams. It is possible to furnish the required area of top steel satisfactorily by a combination of straight and bent rods, various arrangements being common. The use of an equal number of straight rods in the top and in the bottom is believed to give a better solution of the problem. When two trucks are passing on the bridge there will be need of tension steel in the top across the whole width of the middle panel. A single truck at mid-panel anywhere on the bridge will cause tension in the top of the slab in the adjacent panels for the whole width also. There should be steel in the slabs placed to carry these stresses.

11–6. Interior Beams. (Computation Sheets B2, B3.) (*a*) *Stresses.* The A.A.S.H.O. specification gives the fraction of a wheel load on one stringer as the stringer spacing divided by 5, 1.17 in this case. Accordingly, the maximum shear and moment due to placing a front and rear wheel directly on the beam must be multiplied by that factor. The specification states that this factor applies definitely to wheel loads and so it is not here applied to the lane loading, the distribution of which is taken as proportional to tributary widths. The lane loading is supposed to be equivalent to a truck train consisting of a single H–20 in the midst of a line of H–15 trucks.

The sketch for the maximum moment due to a truck shows a front and rear wheel placed on the span in accordance with the well-known theorem which states that for maximum moment the center of the span should lie midway between the resultant and an adjacent load, in this case plainly the heavier of the two.

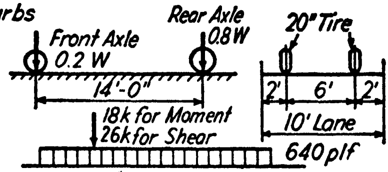
Before the dead load stresses are known the size of the beam stem must

COMPUTATIONS FOR BEAM BRIDGE — **Sheet B1**

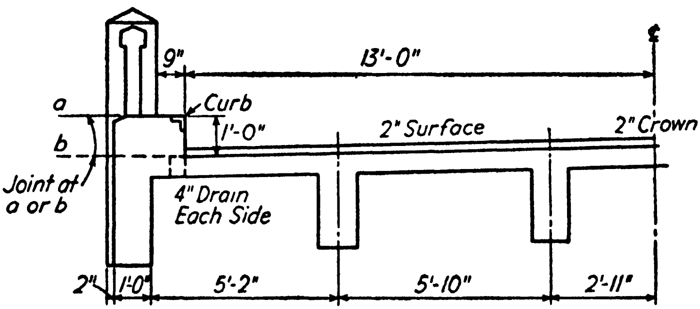
Data

Roadway : 26'-0" between curbs
Span : 40'-0" clear
Loading : H-20 Truck
$W = 40\,000^{\#}$:
or
Lane Loading:

Impact : $I = \dfrac{50}{L+125}$



Material : Concrete, $f'_c = 3000$ psi, $n = 10$
Steel : Structural grade deformed bars
Stresses : $f_s = 18\,000$ psi
$= 16\,000$ psi in diagonal tension reinforcement
$f_c = 1\,000$ "
$v = 60$ " longitudinal bars not anchored } No web
$= 90$ " " " anchored } reinforcement
$= 140$ " " " not anchored } with web
$= 180$ " " " anchored } reinforcement
$u = 100$ " " " not anchored
$= 150$ " " " anchored



Slab

Assume 12" beams, clear span of slab = 4'-10"
Effective slab width : $E = 0.6S + 2.5 = 0.6 \times 4.83 + 2.5 = 5.4'$

Moment : Live $= \pm 0.2\dfrac{P}{E}S = 0.2 \times \dfrac{16\,000}{5.4} \times 4.83 \qquad = 2\,860$ pf/f

Impact $= \dfrac{50}{4.83 + 125} = 0.385 \qquad = 1\,100$

Dead : 2" Surface = 25 psf
7" Slab $= \underline{88}$ " assumed
$113 \times (4.83)^2 \times \frac{1}{10} = \dfrac{265}{4225}$ pf/f or pi/i

M

Depth : $d = \sqrt{\dfrac{M}{Rb}} = \sqrt{\dfrac{4225}{157}} = 5.2"$
$\underline{1.5}$" Cover
$6.7"$ Use 7" Slab, $d = 5\frac{1}{2}"$

Depth

Steel : $A_s = \dfrac{M}{f_s jd} = \dfrac{4225}{18000 \times \frac{7}{8} \times 5\frac{1}{2}} = 0.049$ sq in./in.
Use : $\frac{1}{2}"\phi$ @ 4" c/c = 0.05 sq in./in.

Longitudinal Steel : Use 6-$\frac{1}{2}"\phi$ Space $\begin{cases} 4 \text{ @ } 8" \text{ in middle portion} \\ 1 \text{ in each outer quarter} \end{cases}$

Sheet B-1

be fixed. This may be done approximately by computations on scratch paper, giving a trial size to be used as the basis of further investigation. The stem of a simple tee-beam such as this must be large enough to provide ample resistance to the diagonal tension stresses, measured by shearing stress, and also large enough to allow the longitudinal reinforcement to be placed with proper clearances. It is also desirable that the proportions be such that the design is economical. The preliminary scratch computations whose results are used on Sheet B2 were based on the need of providing a beam with sufficient shearing (diagonal tension) strength, and the recorded calculations of the required cross-section area show that the assumed size (and weight) is satisfactory for that purpose. The proportions shown in the sketch are within the limits set by the common rule of making the depth between two and three times the breadth, but the depth is somewhat shallower than that set by another equally common rule which makes the depth in inches about equal to the span in feet.

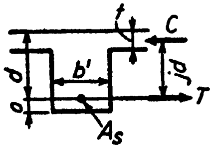
(*b*) *Proportions.* In this simple situation a formula for economic proportions has some justification. The following formula,*

$$d = \sqrt{\frac{rM}{f_s b' j}}$$

was devised by Professors Turneaure and Maurer in their book, Principles of Reinforced Concrete Construction (fourth edition, page 141). The term r is the ratio of the cost per unit volume of concrete and steel. It is suggested that the student apply this formula here. Its use will show that for economy a considerable increase should be made in the depth of the 12-in. beam, the total becoming $39.3 + 5 = 44.3$ in.

(*c*) *Steel.* Choice was made of nine 1¼ in. square bars to give the required steel area (Sheet B2). It is customary to make the beam symmetrical in every respect about the vertical axis. Placing the bars in two layers would require a disproportionately wide beam stem. The requirements as to bar clearances are illustrated by the sketch on this sheet. (Compare Art. 504 and Art. 506) of the J.C. Report, 1940. It is obvious that there must be a sufficient mass of concrete about each rod to transmit the shearing stresses set up. (See the derivation of

* The derivation is as follows, using the data of the figure. It is assumed that



no appreciable difference is made in the cost by the small difference in formwork involved. Let c = unit cost of the concrete and rc = unit cost of the steel. Let a = distance from center of steel to bottom of beam. Then the total cost of the stem per unit length is $C = c[(d + a - t)b' + (rM/f_s jd)]$. Differentiating with regard to the variable d and equating the result to zero, gives for minimum cost, $d = \sqrt{rM/f_s b'j}$.

COMPUTATIONS FOR BEAM BRIDGE — Sheet B2

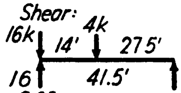
Interior Beams

Assume 18" Abutments: Span $40+1.5=41.5'$ c/c

Proportion of Wheel Load Carried by One Beam (1941 A.A.S.H.O. 3.3.1) $= \frac{S}{5} = \frac{5.83}{5} = 1.17$

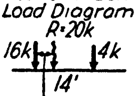
Live Load Moments & Shears - Truck -

Shear:



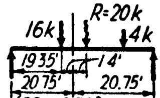
$V = 18.65 \times 1.17 = 21.8k$

Load Diagram.



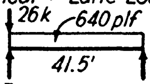
$\frac{4 \times 14}{20} = 2.8'$

Moment:



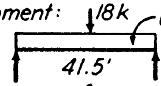
$M = \left(\frac{20 \times 19.35^2}{41.5} - 180.4\right) \times 1.17 = 211\ kf$

Shear: Lane Loads:



$V = 39.3 \times \frac{5.83}{10} = 22.9\ k$

Moment:



$\frac{0.64 \times (41.5)^2}{8} = 138\ kf$

$\frac{18 \times 41.5}{4} = \underline{187}$

$M = 325 \times \frac{5.83}{10} = 189\ kf$

Dead Load Moments & Shears

Wearing Surface = 25 psf
7" Slab = 88

$113 \times 5.83 = 659$ plf
Stem 12×24 = 300
959 plf

Shear:
$V = \frac{1}{2} \times 959 \times 41.5 = 19\,900\#$

Moment.
$M = 959 \times \frac{41.5^2}{8} = 206\,500$ pf

Moments & Shears

	Shear	Moment
Live Load	29 900#	211 000 pf
Impact $= \frac{50}{41.5+125} = 0.30$	6 900	63 300
Dead	19 900	206 500
Total	49 700#	480 800 pf

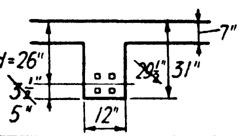
Stem

$b'd = \frac{V}{jv} = \frac{49\,700}{\frac{7}{8} \times 180} = 316$ sq in.

$b' = 12"$ $d = 26.4"$ Try $\longrightarrow d = 26"$

($j = 0.9$ Probably)

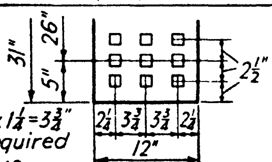
Size Assumed for D L O.K.



Steel

$A_s = \frac{M}{f_s jd} = \frac{480\,800 \times 12}{18000 \times 0.90 \times 26} = 13.7$ sq in

$9 - 1\frac{1}{4}" \square = 14.04$ sq in.



Check of f_c:

$$b = \begin{cases} \frac{L}{4} = 41.5 \times 12/4 = 124.5" \\ S = 70 \longleftarrow \\ 6b' = 6 \times 12 = 72 \\ b' + 12t = 12 + 12 \times 7 = 96 \end{cases}$$

$3 \times 1\frac{1}{4} = 3\frac{3}{4}"$ Required

$pn = \frac{14.04 \times 10}{70 \times 26} = 0.077$ (Fig. A-6)

Check for $k.j.A_s$ (Fig. A-4) $t/d = 7/26 = 0.27$ $C_c = 0.142$

$k = 0.33$ $j = 0.898$
$\therefore A_s = O.K.$

$f_c = \frac{M}{C_c bd^2} = \frac{480\,800 \times 12}{0.142 \times 70 \times 26^2} = 860\ psi < 1000\ psi$

Bond

Bond: Bend up top layer and leave straight the two bottom layers:

$u = \frac{V}{jd\Sigma o} = \frac{49\,700}{\frac{7}{8} \times 26 \times 6 \times 4 \times 1\frac{1}{4}} = 73 < 140$ psi

O.K. without anchorage

equation 7-2; Art. 7-13.) It would be an interminable task to calculate this embedment afresh for each beam designed, and an unnecessary one, as the standard spacing rules give safe results for ordinary beams. Horizontally the clear spacing is limited to 1½ times the diameter of a round rod, or 2 times the size of a square rod, or 1¼ times the maximum size of aggregate, with a minimum of 1 in. Vertically the clear spacing is limited to 1 in. A more conservative rule limits the vertical spacing to 1 in. or to the diameter of the largest bar. This will allow mortar to work freely around the rods even if the maximum aggregate cannot pass. The use of one of the several varieties of bar supports and spacers now on the market is strongly urged to insure the proper placing of this steel.

These intermediate beams are tee-beams and the limits to the width of slab that may be assumed to act integrally with the stem are indicated on Sheet B2, using the rules of the 1941 A.A.S.H.O. specification. The assumed value of j was checked from Fig. A-4, and the steel area recomputed. The flexural compression in the extreme concrete fiber was found from Fig. A-6. It is safely within the 1000 psi maximum allowable.

It was assumed that the two lower layers of bars are left straight in the bottom the length of the beam, and bond stress was calculated accordingly.

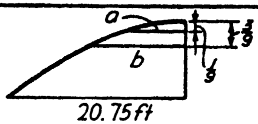
The assumption was made that the maximum possible bending moment at any section of the beam is equal to the ordinate at that section to a parabola drawn with the maximum moment already computed as the center ordinate. This would be exact were the live load either a uniform or a single moving concentrated load and, as it is, gives results reasonably close to the truth. On this basis the limiting positions of the points of bend were found. It was decided to arrange the bent steel so that a pair of bent rods cuts the vertical plane at the support, with the third rod close enough to the first pair so that no stirrups are required between the points of bend. The 1941 A.A.S.H.O. specification requires the first diagonals to cross the neutral axis at one-quarter the beam depth from the face of support. The distance between bends was made 20 in., about $\frac{3}{4}d$ and, since the sloping rods can carry the stress in that distance, no stirrups are needed in addition. For stirrup design it is sufficiently accurate to use the clear span, taking the shear at the edge of the support as that at the center (182 psi), with a straight line variation of shearing stress down to the maximum intensity at the center of the beam. Any possible shear curve due to any loading will fall within the curve thus drawn, and accordingly the reinforcement proportioned by its use will be adequate.

COMPUTATIONS FOR BEAM BRIDGE	Sheet B3

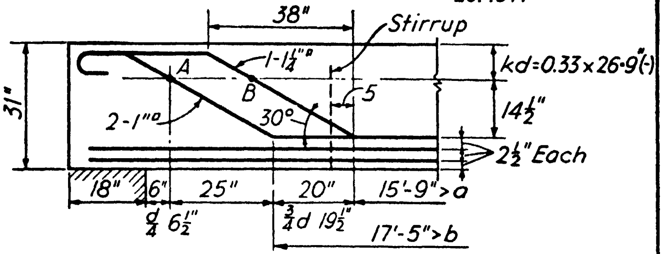
Bending Up of Steel — *Int Beam Bend Up Steel*

$a = 20.75\sqrt{\frac{1}{9}} = 6.9$ ft

$b = 20.75\sqrt{\frac{3}{9}} = 12.0$ ft

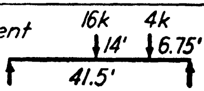


Scale Sketch



Diagonal Tension Reinforcement — *Diag Tens Reinf*

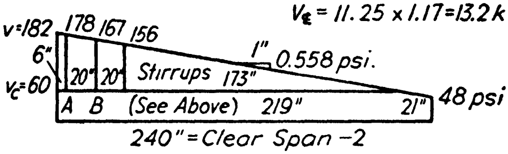
Max V at ℄



$$\text{Max } v_{end} = \frac{V}{bjd} = \frac{49\,700}{12 \times \frac{7}{8} \times 26} = 182 \text{ psi}$$

	8.0
	0.65
L =	8.65
30% I =	2.60
D =	0

$V_℄ = 11.25 \times 1.17 = 13.2$ k $\quad v_℄ = 48+$ psi



Stress in Bent Rods $\left(A_v = \frac{V' \cdot s \cdot \sin\alpha}{f_v \cdot j \cdot d}\right)$ A.A.S.H.O. 1941)

$A: f_v = \frac{v'bs \sin\alpha}{A_v} = \frac{118 \times 12 \times 20 \times 0.5}{2 \times 1.56} = 4600(-)\text{psi} < 16\,000$

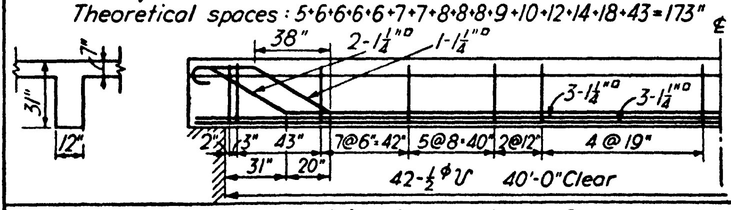
$B: \therefore$ O.K.

Vertical Stirrups: Total area required at each end

$A_v = \frac{v'bs}{f_v} = \frac{\frac{1}{2} \times 96 \times 12 \times 173}{16\,000} = 6.23$ sq in. $\quad$ Req'd · 16-$\frac{1}{2}$"φ U = 6.40 sq in.

Theoretical spaces: 5+6+6+6+6+7+7+8+8+8+9+10+12+14+18+43 = 173" ℄

Partial Summary of Design (For Convenience of Detailer Only)



Outside Beams

Stress: Live: Assume wheel riding directly on beam. Stresses then are those of interior beam ÷ 1.17

Dead · Weight of parapet and curb above slab	= 500 plf
Slab and wearing surface $\frac{1}{2} \times 4.7 \times 113$	= 266
Stem including slab: Try 12×36	= 450
	1220

$V = \frac{1}{2} \times 1220 \times 41.5 = 25\,300$ #

$M = \frac{1}{8} \times 1220 \times (41.5)^2 = 263\,000$ '#

The intensity of stress in the pair of bent rods nearer the support was computed by the requirement of the 1941 A.A.S.H.O. specification, assuming that the concrete carries diagonal tension to an amount measured by a unit shear of 60 psi, and is very small. The stress in the inner rod is also well within the allowable.

Although stirrups are not needed in the 6½ in. at the end of the beam, by strict rule interpretation, ½ in. round stirrups will be used 2 in. and 5 in. from the face of the support. The necessary stirrups in the 173 in. inside the bent rods were placed by making enough spaces about equal to the minimum computed to reach to the section where the shear permits change to the next practical spacing, and so on. If the sketch is to scale and the rate of change of shear per foot is shown, it is simple to do this mentally. Stirrups should be used throughout the beam for the purpose of tying the stem and flange together even though not required by shear. Many designers would place the stirrups in this beam without counting at all upon the bent rods.

The reason for the use of ½-in. stirrups appears as soon as the anchorage requirements are investigated. A standard hook must develop a unit stress of 10,000 psi in the bar at normal bond stress, here 100 psi. This requires dimension A, Fig. 11-2, to be 25 diameters, 12.5 in. The additional 6000 psi of stress may be at the higher rate of 150 psi, requiring an additional 10 diameters, 5 in., dimension B. This brings dimension C to about 11.5 in., theoretically one-half of the depth, d, usually about one-third of the depth as here assumed for the diagonal tension computation. The difference was ignored as meaningless in this range and the ½-in. stirrup was taken as securely anchored. A ⅝-in. stirrup would require for its development considerably greater depth of beam than here used.

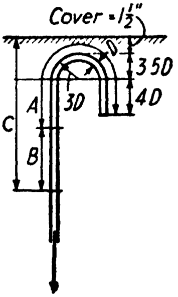


FIG. 11-2

Problem 11-1. Compute the maximum total moment at the quarter-point of an interior beam and compare with the value given by the assumption of parabolic variation of moment made on Computation Sheet B3.

Ans. 336,700 ft-lb. ¾ × 438,200 = 329,000 ft-lb.

Problem 11-2. Compute the maximum intensity of shear at the quarter-point of an interior beam and compare with the value given by the assumed shear curve on Computation Sheet B3.

Ans. 100.5 psi. 101 psi.

11-7. Outside Beams. (Computation Sheets B4, B5.) The A.A.S.H.O. specification permits the live load on the outside stringer to be computed as the reaction from the truck wheel, assuming the slab

COMPUTATIONS FOR BEAM BRIDGE — Sheet B4

Outside Beams

Summary

	Shear	Moment
Live	19 700#	180 500'#
Impact	5 900#	54 200'#
Dead	25 300#	263 000'#
	50 900#	497 700'#

Void — See 2nd Trial

Trial Section:

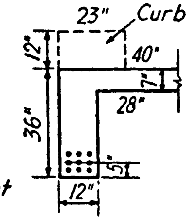
$$v=\frac{V}{bjd}=\frac{50\,900}{12\times\frac{7}{8}\times 31}=157>140\,psi$$

To avoid special anchorage of steel make b=13"

$$A_s=\frac{M}{f_s jd}=\frac{497\,700\times 12}{18\,000\times 0.90\times 31}=11.9\ sq\ in.$$

$9\text{-}1\frac{1}{8}"\square = 11.34$ sq in.

(f_s=18900 say O.K. on account uncertainties of loading)



Check of f_c (Fig. A-6) $pn=\frac{11.34\times 10}{40\times 31}=0.0915$, $t/d=7/31=0.226$; j=0.903, C_c=0.142, Fig. A-4

$$f_c=\frac{M}{C_c bd^2}=\frac{497\,700\times 12}{0.142\times 40\times (31)^2}=1070>1000\,psi$$

O.K. with conservative loading assumed

Second Trial

Second Trial: Assume curb cast with stem. Rectangular Beam (see sketch above) b=23" d=44"

$$R=\frac{M}{bd^2}=\frac{497\,700\times 12}{23\times (44)^2}=134$$

p=0.0085 with f_c=900 ; f_s=18 000 psi (Fig A-4)

Use ←

$A_s=pbd=0.0085\times 23\times 44=8.60$ sq in

$4\text{-}1\frac{1}{4}"\square + 2\text{-}1\frac{1}{8}"\square = 8.78$ sq in.

Check of k (Fig. A-1), k=0.33 kd=15"<19" available

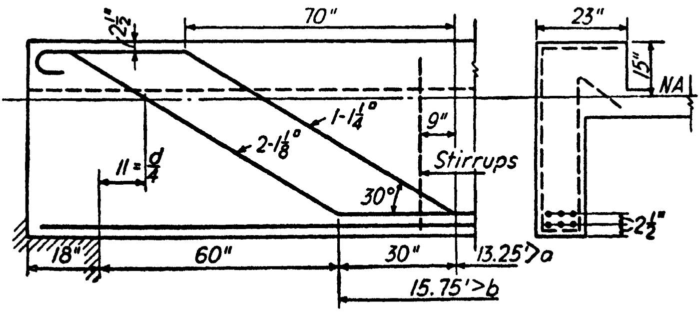
Bond: $v=\frac{V}{bjd}=\frac{50\,900}{12\times\frac{7}{8}\times 44}=110\ psi<140\ psi$ OK

$$u=\frac{vb}{\Sigma o}=\frac{110\times 12}{3\times 4\times 1\frac{1}{4}}=88<100\ psi\ O.K.$$

Bending of Steel, See Interior Beam:

$$a=20.75\sqrt{\frac{1.56}{8.78}}=8.75\ ft$$

$$b=20.75\sqrt{\frac{4.10}{8.78}}=14.2\ ft$$



to act as a simple beam between stringers. Other rulings are more severe and require at least that an outside stringer carry a full wheel load (half a truck weight). This attempts to make provision for accident and a truck mounting the curb. This second rule was followed here.

Two trial sections were investigated; the first assuming a construction joint at the top of the slab, making the shape of the outside stringer that of an inverted *L*; the second assuming the joint at the top of the curb, making the outside stringer an unsymmetrical rectangular beam, disregarding the slab entirely. In this case the neutral plane is in the slab and narrowing of the stem below the slab affects only the shear and bond computations. Here the full width of 23 in. was used although the presence of the steel angle curb-guard, extending the length of the bridge, interferes with this functioning. In addition to the neglected slab we may note that the width of 23 in. is more than required: from $b = M/Rd^2$ we obtain 19.5 in.

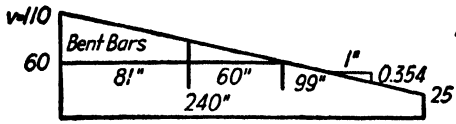
11–8. Expansion Provision. (Computation Sheet B5.) To prevent high temperature stresses and undesirable cracking in bridge and abutments, provision must be made for free expansion and contraction of the structure with temperature changes. The manner of making this provision varies with the length of span and no definite rule of practice is standard. "Specifications for Design of Highway Structures," released by the Department of Highways of the State of Ohio in July, 1940, recommends:

> **Art. 93.** For concrete structures with a total span of 30 ft or less . . . no allowance for temperature need be made.
>
> For statically determinate concrete structures of more than 30 ft span, provision shall be made for expansion and contraction due to a temperature variation of 40°F. above and 80° below temperature at time of construction.
>
> Sliding bearing plates (preferably of bronze or steel on bronze) are permitted up to a span of 60 ft for a concrete beam bridge . . .
>
> For concrete slab bridges requiring provision for movement at the end, the sliding shall be facilitated by providing bronze sliding plates or a very smooth concrete bridge seat and by separating the slab from the seat by sheet asbestos packing or a similar lubricant.
>
> Where plates are used they shall be provided in pairs with an efficient and durable lubricant between them.
>
> Sliding surfaces, except for the area occupied by plates, shall be effectively separated by an open space if practicable. Otherwise, by expansion joint filler.
>
> Plates should be located at least 3″ from the outer edge of the bridge seat to prevent spalling.

COMPUTATIONS FOR BEAM BRIDGE	Sheet B5

Outside Beams

Diagonal Tension Reinforcement (See Interior Beam)



At ℄, V=11.25k
v=25 psi

Bent Bar Stress O.K. by Inspection : cf Interior Beam

Vertical Stirrups $A_v = \frac{v'bs}{f_v} = \frac{\frac{1}{2} \times 21 \times 12 \times 60}{16\,000} = 0.47$ sq in.

$1\text{-}\frac{3}{8}\phi$ U-0.22 sq in.

$$\text{Spacing} = \frac{A_v f_v}{v'b} = \frac{0.22 \times 16\,000}{21 \times 12} = 14''$$

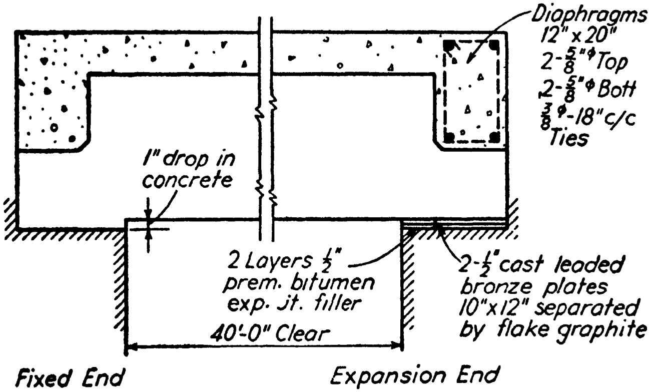
See Sketch for bending steel

Place one stirrup 3" from support

Place three stirrups at 15" spacing starting 9" outside of inner bend point as shown

Space stirrups at about 20" c/c to ℄ of beam

End Bearings



BEARINGS FOR INTERIOR BEAMS

Diaphragm	V = 49 700 lb
	1 300±
	51 000

$$\text{Bearing Pressure} = \frac{51\,000}{10 \times 12} = 425 \text{ psi} < 500$$

For Outside Beams use same detail as above to standardize plate sizes even though load is smaller.

BEARINGS FOR OUTSIDE BEAMS

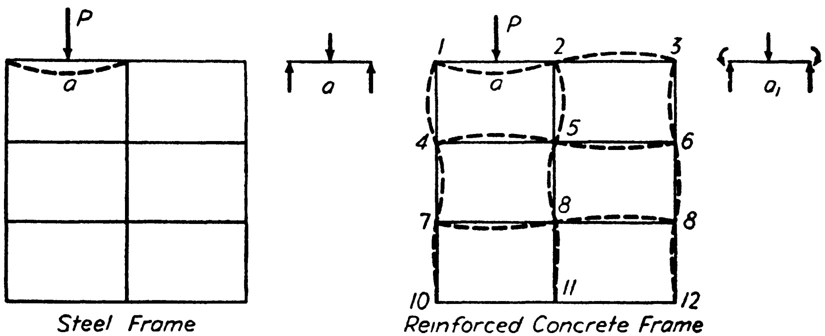
For all other cases where the design contemplates free movement of superstructure on bridge seat, steel rockers or rollers shall be provided.

In this design, expansion is provided for by supporting one end of each beam on a combination of phosphor bronze plates proportioned to bring the bearing stresses on the concrete within the given limit of 500 psi.

CHAPTER XII

CONTINUOUS BEAMS AND RIGID FRAMES

12–1. A reinforced concrete structure is, in effect, a monolith, a unit in itself, and not merely an assemblage of individual beams and columns. Any load causes stress, not only in the members immediately supporting it, but also in every other member of the frame. The magnitude of this effect rapidly decreases with the distance of the member from the load. In ordinary steel construction the joints are not generally rigid enough to develop large bending resistance. The only parts of the steel frame shown in Fig. 12–1 stressed by the load P, are the beam beneath



Shape Under Load Shown by Dotted Lines

FIG. 12–1

the load and the two supporting columns. In the similar reinforced concrete frame Fig. 12–1, the rigidity of the joints causes every member to deform under the load. Accordingly the steel beams are designed as simple end-supported members, and the columns as members carrying direct stress only.* In a reinforced concrete frame the beams are restrained at the ends by their rigid connections, and these end moments must be considered in design. The columns likewise are subject to both direct stress and bending.

* The steel frames of tall buildings exposed to the horizontal force of the wind are made rigid enough to carry the resultant stresses by rigid construction at the joints. The columns and beams are proportioned to resist the resulting combination of direct stress and bending. Every steel frame must be made stable by proper bracing.

The perfect continuity of reinforced concrete structures is weakened by construction joints, formed when fresh concrete is poured in contact with concrete that has already set. Commonly, reinforcing steel crosses such joints and the break in continuity is not regarded. To make such a joint capable of transmitting shear as well as bending, the surfaces of contact must be keyed together, either by a formed mortise or by roughening the surface and using projecting stones. If the joint is smooth with no steel crossing it, neither shear nor bending can be considered as transmitted.

It is evident that the moments and shears in any beam of a rigid frame cannot be determined by the principles of statics alone, since there are two unknown end moments, two unknown vertical reactions and possibly two unknown horizontal reactions, a total of six unknowns, three more in number than the conditions of equilibrium of a non-concurrent coplanar force system. The exact determination of the maximum moment, shear, and direct stress in any member of a monolithic structure of many members is a very complicated matter and, in consequence, for ordinary structures simple rules and methods have been devised for obtaining these stresses quickly that give results close enough to the truth for safe and economical design. The three most important basic methods are those of *least work, slope deflection, and moment distribution.* The most used simplified method is an abbreviated moment distribution.

12–2. Continuous Beams, Theorem of Three Moments. A simple case of a continuous beam of reinforced concrete was furnished by the floor slab of the beam bridge designed in Chapter XI. Here the slab is to be poured at the same time as the beam stems and so forms a rectangular beam, rigidly attached to its supports and continuous over several spans. The *theorem of three moments* offers a convenient means of studying the stress in such a case. This theorem is an equation expressing the relation that exists between the bending moments in a continuous beam at any three consecutive supports, and for a beam of uniform moment of inertia, supported on knife edges, all either at the same level or at the proper elevation to fit the unloaded beam, it takes this form (see Fig. 12–2):

$$M_1L_1 + 2M_2(L_1 + L_2) + M_3L_2 = \frac{-w_1L_1^3}{4} - \frac{w_2L_2^3}{4} - \Sigma P_1L_1^2(k_1 - k_1^3) - \Sigma P_2L_2^2(k_2 - k_2^3) \quad [12\text{–}1]$$

When the moments of inertia in the adjacent spans differ, and when there is settlement of supports so that the middle support lies a distance

m below that on the left and a distance n below that on the right, the equation takes this form:

$$\frac{M_1L_1}{I_1} + 2M_2\left(\frac{L_1}{I_1} + \frac{L_2}{I_2}\right) + M_3\frac{L_2}{I_2} = \frac{-w_1L_1^3}{4I_1} - \frac{w_2L_2^3}{4I_2}$$
$$- \frac{1}{I_1}\Sigma P_1L_1^2(k_1 - k_1^3) - \frac{1}{I_2}\Sigma P_2L_2^2(k_2 - k_2^3)$$
$$+ 6E\left(\frac{m}{L_1} + \frac{n}{L_2}\right) \qquad [12\text{–}1a]$$

where M_1 = the bending moment at support 1 in pound-feet
M_2 = the bending moment at support 2 in pound-feet
M_3 = the bending moment at support 3 in pound-feet
L_1 = the length of the left-hand span, 1–2, in feet
L_2 = the length of the right-hand span, 2–3, in feet
I_1 = moment of inertia of beam 1–2
I_2 = moment of inertia of beam 2–3
m = differential settlement of supports 1 and 2, positive when support 2 is below 1
n = same for supports 3–2, positive for 2 below 3
w_1 = the uniform load on the left-hand span in pounds per linear foot
w_2 = the uniform load on the right-hand span in pounds per linear foot
P_1 = any concentrated load on span 1–2, a distance k_1L_1 from 1
P_2 = any concentrated load on span 2–3 a distance k_2L_2 from 3.

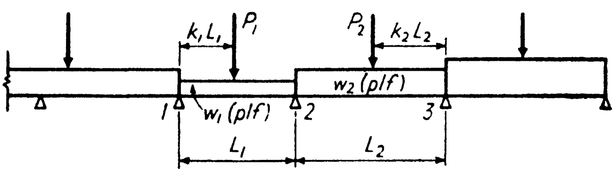


FIG. 12–2

The signs used with this equation are those relating to beams, where negative moment is that which causes tension in the top fiber.

The equation can be extended to care for a uniform load covering only a part of a span by substituting, in the term giving the effect of a concentrated load in the given span, wdx for P, x for kL, and replacing the summation sign by that for integration.

The following derivation of the theorem of three moments for uniform loading is given to make possible a ready review. For more general

statements of the theorem and its derivation the reader should consult a text on structural theory.

In Fig. 12-3 are shown any two consecutive spans of a continuous beam of uniform cross-section and level supports, with a loading that is uniform over any one span. By use of the equation for the elastic curve, $M = EI(d^2y/dx^2)$, the slope of the tangent at any support, as b, may be expressed, first, in terms of the load, etc., to the right, and then in terms of the similar factors to the left. These two expressions are for the same quantity and so are equal to each other. Their combination gives the *three moment equation.*

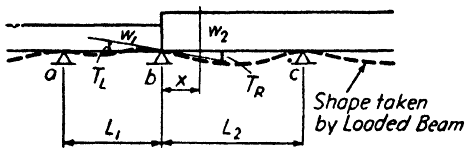


FIG. 12-3

The moment at any section in the right span equals

$$M = EI\frac{d^2y}{dx^2} = M_b + S_r x - \tfrac{1}{2}w_2x^2$$

where S_r is the shear to the right of support b. Integrating

$$EI\frac{dy}{dx} = M_b x + \tfrac{1}{2}S_r x^2 - \tfrac{1}{6}w_2x^3 + (\text{constant} = T_R EI) \qquad [1]$$

Here the constant of integration is expressed as the product of three constants and T_R is chosen to represent the tangent at b expressed in terms of the right span factors. Integrating a second time,

$$EIy = \tfrac{1}{2}M_b x^2 + \tfrac{1}{6}S_r x^3 - \tfrac{1}{24}w_2x^4 + T_R EIx + (\text{constant} = 0) \qquad [2]$$

Placing in equation (2) L_2 for x, $y = 0$, and substituting the value of S_r obtained from the expression for the moment at c ($M_c = M_b + S_r L_2 - \tfrac{1}{2}w_2L_2^2$) gives the following expression for the slope of the tangent to the curve of the beam at b in terms of the right-hand span elements:

$$T_R = \frac{1}{EI}(-\tfrac{1}{3}M_bL_2 - \tfrac{1}{6}M_cL_2 - \tfrac{1}{24}w_2L_2^3)$$

A similar expression for the slope at b in terms of the left-hand factors may be written by analogy from the expression for that at $c(T_c)$, obtained by writing equation (1) with $x = L_2$, substituting the value of S_r as before,

$$T_L = \frac{1}{EI}(\tfrac{1}{6}M_aL_1 + \tfrac{1}{3}M_bL_1 + \tfrac{1}{24}w_1L_1^3)$$

Equating $T_R = T_L$ gives

$$M_aL_1 + 2M_b(L_1 + L_2) + M_cL_2 = -\tfrac{1}{4}w_1L_1^3 - \tfrac{1}{4}w_2L_2^3$$

The use of the theorem is illustrated by the following examples.

Example 12–1. What is the maximum moment at b in the beam shown in Fig. 12–4? Dead load in pounds per foot $= w_1$; live load $= w_2 = 3w_1$.

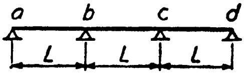


FIG. 12–4

Solution. Note that $M_a = M_d = 0$, and for dead load $M_b = M_c$, the beam being symmetrical. For dead load the three moment equation takes this form:

$$M_aL + 2M_b(L + L) + M_cL = -\tfrac{1}{4}w_1L^3 - \tfrac{1}{4}w_1L^3$$

Solving:

$$5M_b = -\tfrac{1}{2}w_1L^2$$

$$M_b = -\tfrac{1}{10}w_1L^2, \text{ the dead load moment}$$

In order to determine the spans which should be loaded to produce negative moment at b, place a single load on each of the three spans in turn (Figs. 12–5a, b, and c) and consider the resulting deflection of the beam. For example: a load on ab, with supports c and d removed, brings the beam to the position

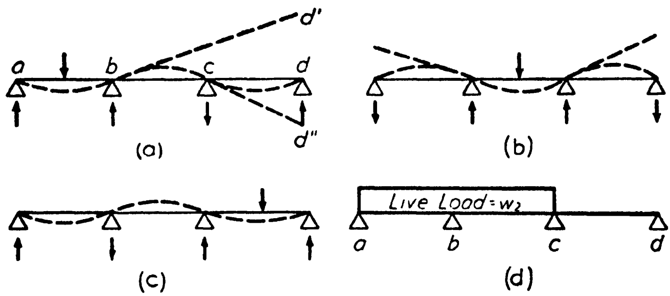


FIG. 12–5

abd'. The action of the c support is to pull the beam down into the position $abcd''$, causing tension in the top of the beam at b and thus negative moment at that point. The subsequent pushing up of the beam into place by support d does not change this. Loads anywhere on spans ab and bc cause negative moment at b, and so both these spans should be covered with the live load for a maximum value of that function. For this loading (Fig. 12–5d) there are two unknowns, M_b and $M_c (M_a = M_d = 0)$, and two simultaneous equations involving these two unknowns may be written by applying the theorem of three moments twice, first to the three supports abc and then to the three bcd:

$$M_aL + 2M_b(L + L) + M_cL = -\tfrac{1}{4}w_2L^3 - \tfrac{1}{4}w_2L^3$$

and

$$M_bL + 2M_c(L + L) + M_dL = -\tfrac{1}{4}w_2L^3$$

Solving,

$$M_b = -\tfrac{7}{60}w_2L^2 \text{ for live load moment}$$

The maximum moment equals the sum of the dead and live load moments

and would be expressed in terms of the total load per foot, $w = w_1 + w_2 = w_1 + 3w_1 = 4w_1$, giving

$$M_b = -\tfrac{1}{10} \cdot \frac{w}{4} \cdot L^2 - \tfrac{7}{80} \cdot \frac{3w}{4} \cdot L^2 = -\tfrac{9}{80}wL^2$$

In expressions for moment the multiplier of wL^2 is known as the *moment factor* or *moment coefficient.*

Example 12-2. What is the shear at a section 2 ft. from the right end of the beam shown in Fig. 12-6*a*?

Discussion. The difficulty introduced by the fixed end at c can be met as follows. Imagine the support at that point changed to a knife edge and the beam to be continued to the right, a distance L, to another support d.

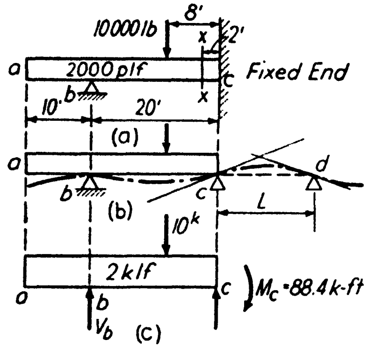


Fig. 12-6

The beam with its extension deflects more than the original beam because the restraint offered by the span cd at c is not sufficient to make the tangent to the elastic curve horizontal at that point. However, the shorter the distance L, the stiffer the span cd is and the flatter the tangents at c and d are. When $L = 0$ these tangents coincide and are horizontal, giving a condition of fixity at c. Accordingly the theorem of three moments can be applied to this problem by replacing the fixed end with an unloaded span of zero length, supported on knife edges.

Solution. The beam with its added span is continuous over three supports, at which there is only one unknown moment, that at c, (M_c): $M_d = 0$: $M_b = -2 \times 10 \times 5 = -100$ k-ft. (A kip is 1000 lb.)

Writing the equation of three moments:

$$-(100 \times 20) + 2M_c \times 20 + 0 = -(\tfrac{1}{4} \times 2 \times \overline{20}^3) - [10 \times \overline{20}^2(0.6 - \overline{0.6}^3)]$$

$$M_c = -88.4 \text{ k-ft}$$

The negative sign of this moment indicates that it causes tension in the top fiber and therefore it acts in a clockwise direction upon the beam at c as shown in part c of Fig. 12-6. Either of the unknown vertical reactions may be found by applying the condition of equilibrium, $\Sigma M = 0$. In using this condition it is customary to name a clockwise moment positive. *It is important to distinguish clearly between these two conventions of signs: that for bending moment in beams and that for moment in force systems acting on (assumed) rigid bodies*

$$\Sigma M_c = 0$$

$$(V_b \times 20) - (2 \times 30 \times 15) - (10 \times 8) + 88.4 = 0$$

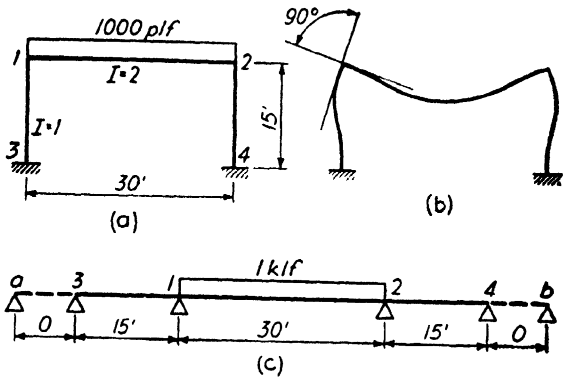
$$V_b = 44.6 \text{ k}$$

The positive sign of the result indicates that the assumed upward direction is correct. The desired shear is:

$$44.6 - 2 \times 28 - 10 = -21{,}400 \text{ lb}$$

Problem 12–1. Compute the significant bending moments for this loaded frame.

Discussion. The deformed frame is shown in (*b*) of the figure. Frame and load being symmetrical, it is evident that points 1 and 2 will remain fixed in location,



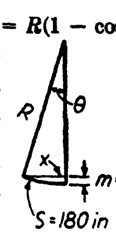
PROB. 12–1

the change in length of members due to compression and to bending being neglected, as is universal in this type of problem. (*Query.* A 30-ft straight bar is bent into an arc with a rise of 1 in. Compute the length of the chord connecting the ends of the bent bar. *Ans.* Approximately 29.9994 ft.)* The jointure of columns and beam is assumed to be rigid; that is, the tangents to the elastic curves of a horizontal beam and a vertical column at a joint remain 90° apart at all stages of loading. The amount of resistance to rotation at joint 1 or 2 developed by the connecting column evidently depends upon the stiffness of the column and not at all upon the angle between it and the beam. Accordingly the bent may be considered as represented by the continuous beam shown in (*c*) of the figure, the fixed supports being replaced by spans of zero length; that is, the rotation of the joints 1 and 2 of the beam will equal those of the bent.

Ans. $M_1 = -50$ k-ft; maximum positive moment in the beam = 62.5 k-ft.

Query. Evidently it would be economical so to proportion this frame that the positive and negative moments in the beam are equal. How can we effect this? By making the beam still stiffer? Compute the moment with the relative value of the beam moment of inertia 3 instead of 2; 1 instead of 2. How can we effect this, keeping the beam unchanged and varying the column stiffness?

* Suggestion: Inspection of the figure shows that $x = R \sin \theta$, $m = R(1 - \cos \theta)$, and $\theta = s/R$. Using the first two terms of the usual McLaurin expansion for the sine and cosine, we get $x = s(1 - 2m^2/3s^2)$. Since θ is very small we are justified in neglecting further terms of the expansions. The student will find it illuminating to consider other methods of attack upon this problem, using mathematical tables. Is more required than that we demonstrate that the error involved in assuming chord and arc of equal length in this situation is considerably too small to affect slide-rule computations?



12–3. Moment Factors for Continuous Beams and Girders. In Ex. 12–1 is illustrated the process of finding the moment factors to use in designing a given continuous beam carrying uniform loads. Applying the same methods to beams of a varying number of equal spans the following table was constructed:

Coefficients[1] for Maximum Moments in Continuous Beams (x)

$$M = xwL^2$$

Number of Spans	Intermediate Spans and Supports				End Span and Second Support			
	At center (+)		At support (−)		At center (+)		At support (−)	
	Dead	Live	Dead	Live	Dead	Live	Dead	Live
Two					0.070	0.095	0.125	0.125
Three	0.025	0.075			**0.080**	**0.100**	0.100	**0.117**
Four	0.036	0.081	0.071	**0.107**	0.071	0.098	**0.107**	0.120 (0.115)[2]
Five	**0.046**	**0.086**	0.079	0.111 (0.106)[2]	0.072	0.099	0.105	0.120 (0.116)[2]
Six	0.043	0.084	**0.086**	0.116 (0.106)[2]	0.072	0.099	0.106	0.120 (0.116)[2]
Seven	0.044	0.084	0.085	0.114 (0.106)[2]	0.072	0.099	0.106	0.120 (0.116)[2]

[1] This table and the following one are reproduced, with some variation, from Principles of Reinforced Concrete Construction, by Turneaure and Maurer, John Wiley & Sons, Inc.

[2] Where two adjacent spans only are loaded.

The columns headed "dead" loads give the maximum values of the positive and negative moment to be found in the beam due to the dead load, which of course covers the whole length. The columns headed "live" loads give the maximum moments, positive and negative, that can be caused anywhere in the beam by the live load, by placing it only on those spans where its effect is to increase the moment under consideration. Reasonably maximum values of all those given in the table are indicated in bold-face type, the larger values being rejected as involving unreasonable assumptions as to the position of the live loads. In the table on the following page these maximum values found above for live and dead loads are combined into a single term for various ratios of live and dead, 1 to 2 being the usual range of that ratio.

On the basis of this study the following conclusions are usually drawn (compare Art. 708, Reinforced Concrete Building Regulations A.C.I.

COEFFICIENTS FOR MAXIMUM MOMENTS IN CONTINUOUS BEAMS OF THREE OR MORE EQUAL SPANS DUE TO COMBINED DEAD AND LIVE LOADS (x)

$$M = xwL^2$$

Ratio of Live : Dead	Intermediate Spans		End Spans	
	At center	At support	At center	At support
1 : 1	0.066	0.097	0.090	0.112
2 : 1	0.073	0 099	0.093	0.114
5 : 1	0.079	0.104	0 097	0.115

1928): that for continuous beams of three or more equal spans, carrying uniform loads, the maximum positive and negative moments to be expected in interior spans may be taken as $wL^2/12$, the width and rigidity of the supports serving to reduce the theoretical values; for end spans the maximum positive and negative moments to be expected are $wL^2/10$. For two span beams the maximum positive moment is taken as $wL^2/10$ and the negative moment as $wL^2/8$. The revised A.C.I. Code reduces these values somewhat. The Joint Committee, 1940, does not recognize the use of moment factors except for slabs.

Certain of the above factors are somewhat less when the outside supporting columns or walls offer considerable resistance to bending. The A.C.I. Code Art. 709, makes recommendations covering the action of supporting columns, based on studies made by more elaborate methods than that just outlined.

For girders, that is, for beams carrying concentrated loads, it formerly was considered sufficiently accurate to compute the maximum positive moment as though it were a simple non-continuous member, and then modify this result in the same ratio as for the same section in a uniformly loaded beam of the same number of spans. More accurate methods are now requisite.

Example 12-3. Estimate the maximum positive and negative moments in the center span of a girder continuous over three equal spans with a possible single load, P, at the center of each. Neglect the dead weight of the girder.

Solution. For a similar beam with uniform load the maximum positive moment at the center of the interior span is $wL^2/12$, and the maximum negative moment $wL^2/10$. The maximum positive moment for a simple girder with center load is $PL/4$; for a simple beam, uniformly loaded, $wL^2/8$. Since $wL^2/12 = wL^2/8 \times 8/12$, the maximum positive moment in the center of the girder is $PL/4 \times 8/12 = PL/6$; the maximum negative moment is $PL/4 \times 8/10 = PL/5$. (By the three moment equation $+M = 7PL/40$; $-M = 7PL/40$.)

Problem 12–2. A girder which is continuous over two equal spans carries equal loads at the center of each span. Compute the maximum moment at the center support (*a*) by means of the theorem of three moments, (*b*) by the approximate method described above. Neglect girder weight.
Ans. (*a*) $-\frac{3}{16}PL$. (*b*) $-\frac{1}{4}PL$.

12–4. Method of Least Work. The stresses in statically indeterminate structures, such as those formed of beams and columns meeting in rigid (hingeless) joints, may be determined by the *method of least work.* The theorem of least work states that the internal stresses in any statically indeterminate structure are such that the total work done by them, as the structure deforms under load, is a minimum.* It is possible, therefore, to obtain exact solutions of rigid frames by writing an expression for the internal work in the structure in terms of one or more unknowns (moments, shears, and thrusts, the number being the excess of the total number of unknowns over those that may be obtained by application of the principles of statics); place the several partial derivatives of this expression equal to zero, and solve.

Expressions for the work done by internal fiber stresses are obtained as follows. Let

P = total axial stress in a piece
A = cross-sectional area of piece
L = length of piece
E = modulus of elasticity of the material = f/e
f = unit stress = P/A for column or tie = Mc/I for beam
e = strain or deformation per unit length = f/E
I = moment of inertia
y = distance from neutral axis of a beam to the stressed fiber.

In a member subjected to axial stress the internal work equals the average force acting during deformation, multiplied by the total deformation,

$$W = \frac{P}{2} \times eL = \frac{P}{2} \times \frac{PL}{AE} = \frac{P^2L}{2AE} \qquad [12\text{–}2]$$

A beam is simply an assemblage of a vast number of elementary columns and ties, such as that shown in Fig. 12–7 with dimensions

* This theorem is known as Castigliano's second law. It is a special case of Castigliano's first law, which states that the deflection under any load of a structure, loaded with gradually applied loads, equals the first partial derivative of the total internal work in the structure with respect to the stated load. If this deflection is zero, as for a point of support, this derivative equals zero, the condition for a minimum value of the work function. The significance of the theorem for a structure with redundant reactions is thus established.

$b \times dx \times dy$. The work done in any such prism equals

$$W = \frac{P^2L}{2AE} = \frac{1}{2}\left(\frac{M^2y^2}{I^2} \times \overline{bdy}^2\right)\left(\frac{dx}{bdyE}\right)$$

and the total work in the beam equals (noting $\int y^2bdy = I$)

$$W = \int \frac{M^2dx}{2EI} \qquad [12\text{-}3]$$

This expression can be written directly by aid of the moment-area theorem which states that the angle between two tangents to the elastic curve equals the area of the M/EI diagram between the points of tangency; accordingly, the work done in the distance dx equals the average moment $(M/2)$ in the distance, multiplied by the angular rotation (Mdx/EI). Inspection of the expression shows that the integral may be evaluated by taking the moment of the bending moment diagram about its base and dividing by EI.

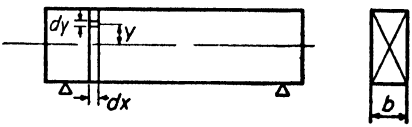


FIG 12-7

Example 12-4. What is the moment at the end of the beam ab of the rigid frame a-b-c-d (shown in Fig. 12-8)? The section is uniform in size and material throughout.

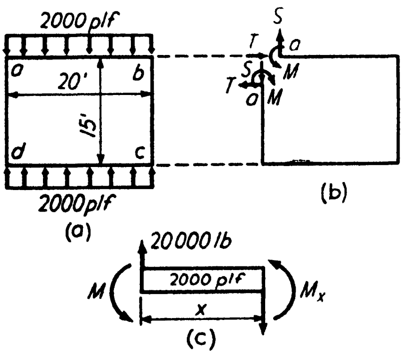


FIG. 12-8

Solution. The values of E, I, and A are the same for all members. Cut the frame at any convenient point, as at a, and represent by arrows the unknown thrust, moment, and shear there acting. From the symmetry of the structure it is plain that the shear in ab at a equals one-half the load; $S =$ 20,000 lb. It is also plain that the thrust in ab must be the same in amount and direction as that in dc. This thrust, T, then must equal zero, as otherwise there would be two equal horizontal forces acting on the vertical ad in the same direction, which is not consistent with equilibrium. The moment M_x, in ab at any distance x from a, equals:

$$M_x = -M - \frac{wx^2}{2} + Sx = -M - \frac{2x^2}{2} + 20x$$

The total work in the structure equals

$$W = 2\int_0^{20} \frac{(-M - x^2 + 20x)^2dx}{2EI} + 2\int_0^{15} \frac{M^2dx}{2EI} + 2\left(\frac{20^2 \times 15}{2AE}\right)$$

Differentiating in respect to M and placing the expression equal to zero,

$$\frac{dW}{dM} = 2\int_0^{20} \frac{2(-M - x^2 + 20x)(-1)dx}{2EI} + 2\int_0^{15} \frac{2Mdx}{2EI} = 0$$

$$\left(Mx + \frac{x^3}{3} - 10x^2\right)\Big|_0^{20} + Mx\Big|_0^{15} = 0$$

$$20M + \frac{8000}{3} - 4000 + 15M = 0$$

$$M = +38{,}000 \text{ lb-ft}$$

The positive sign indicates that the assumed direction for the moment at a happened to be correct, i.e , negative moment, causing tension in the top fiber. The moment factor here equals 1/21.1.

Problem 12–3. A 24-in. steel beam, American standard, weighing 79.9 plf, is 40 ft long and is simply supported at each end (with the possibility of either an upward or downward reaction but no moment) and at the middle. A load of 5 kips per ft extends over one 20-ft span. Draw the curve of bending moment for this loading, using the method of least work, and checking by the three moment equation. *Ans.* Moment at middle support = −125 k-ft.

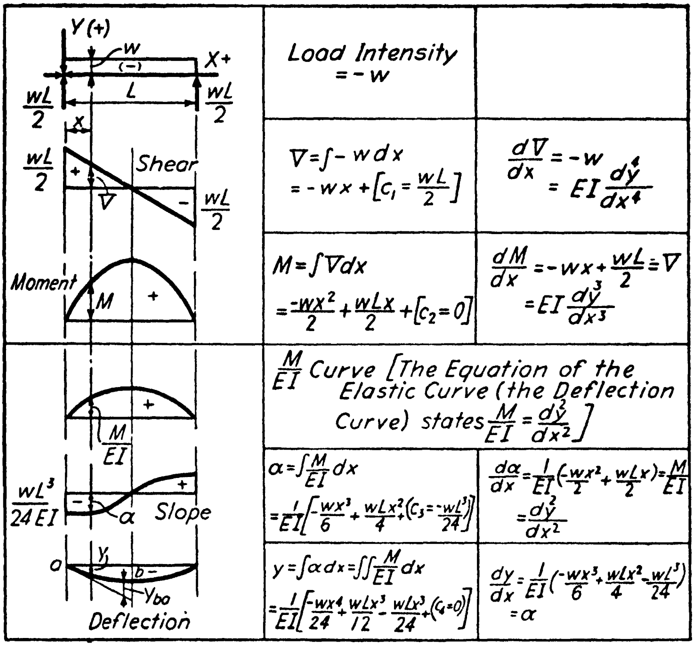
12–5. Method of Slope Deflection.* The basic method of analysis of rigid frames is that of slope deflection, which in general is simpler of application than least work. Briefly described the method is as follows. A series of simultaneous equations is set up and solved, each equation expressing a relation between certain of the bending moments at the ends of the several members of the structure under consideration, and giving as a result the values of all these end moments. The relation most commonly used in setting up these equations is the equilibrium existing between all the end moments at any one of the rigid joints. The determination of the shears and direct stresses, to complete the solution of the structure, is a simple matter once these moments are determined.

It should be noted that this method is an approximate one, as no account is taken by it of the change in length of any member due to axial stress. Its use gives results identical with those of the method of least work when the work due to axial stress is disregarded.

Mastery of both the slope deflection and the moment distribution methods requires familiarity with the theorems relating to beam deflection, and in order to facilitate the student's review the basic relationships are here shown in tabular form, a uniformly loaded simple beam being chosen for the illustration for sake of simplicity. The student should note carefully the sequence of repeated integrations and, con-

* Presented by Professor George A. Maney, 1915.

versely, the repeated differentiations, giving the six-curve sequence or the two three-curve sequences of load-shear-moment — M/EI-slope-deflection curves. Considering either sequence, note that the ordinate of any curve equals the slope of the following curve both in sign and magnitude, that the difference between two successive ordinates of any curve equals the area under the preceding curve between the same ordi-



nates. The first of these relationships is of great practical importance as aid in drawing these curves for any particular case; the second relationship applied to the second set of curves leads at once to the first theorem of area-moments: *The angle between any two tangents to the elastic curve (deflection curve) of a loaded beam equals the area under the* M/EI *curve between their points of tangency.* The second theorem states that *the deflection (offset) of any point on the elastic curve from the tangent at another point (for example,* y_{ba}*) equals the moment about the first point of the area under the* M/EI *curve between the two points.* This follows since $y_{ba} = \int (x_b - x)d\alpha = \int (x_b - x)\frac{Mdx}{EI}$.

The conjugate beam method of determining beam slopes and deflections (frequently called the third and fourth area-moment theorems) follows from the analogy existing between the two sets of curves in the

sequence. It is observed that the same mathematical relationships hold between the load, shear, and moment curves on the one hand and the M/EI, slope, and deflection curves on the other. It follows that the mathematical operations performed for the derivation of the shear and moment curves from the load curve may also be employed for deriving the slope and deflection curves from that of M/EI. In the table those operations were repeated integrations but usually we obtain the shear curve from that of loads by direct application of the definition of shear, the algebraic sum of the loads on one side of the section in question; the moment curve from that of load by direct application of the definition for moment, the moment of the loads on one side of the section about a point (neutral level of the beam section) in the section. (Or, more conveniently, frequently by the second relationship noted in the preceding paragraph.) From the identity of mathematical relationship the conclusion, emphasized by the preceding sentences, at once follows, that *shear and moment curves constructed for the* M/EI *curve as load are respectively the actual slope and deflection curves. The beam carrying the* M/EI *loading is called the conjugate beam* as against the actual beam whose slope and deflection are desired.

In applying the conjugate beam method it is essential that the support and continuity of the conjugate agree with the compelling conditions of the actual beam. Since at a simple support there is slope but no deflection for the actual beam, the corresponding support of the conjugate beam must provide shear but permit of no moment, a condition satisfied by another simple support. A fixed support of the actual beam is attended by absence of slope and deflection, that is, by absence of shear and moment for the conjugate. This demands a free or floating end. The conjugate for a fixed-ended beam is a free-ended (floating) beam of the same span; this will be in equilibrium statically under the positive and negative M/EI loading areas. The interior support of a continuous beam is accompanied by slope without deflection; accordingly, there must be shear without moment in the conjugate, a condition established by a hinge. *Query.* What is the construction for the conjugate at the point corresponding to a hinge in the actual beam?

12-6. The General Slope Deflection Equation: Straight Members of Uniform Cross Section. Let *ab*, Fig. 12-9, represent a straight member of uniform cross section in a loaded rigid frame, a portion of the frame being here shown, the member *ab* itself and the ends of the other members which are continuous with it at the joints *a* and *b*. Under the action of all the loads on the structure, that shown and those not shown, applied to other members, the member *ab* takes some position such as that represented in exaggerated fashion in (*b*) of the figure; first, each

joint, a and b, has rotated through a small angle; second, one joint has moved laterally from its original position relative to the other; third, between the joints the beam has sagged under the load immediately applied. The stress effect of each of these three motions or actions can be easily visualized. Consider Fig. 12–10a where member ab is again represented, this time with only one deforming action shown, a counterclockwise twist to joint a, the couple causing this twist being such as is indicated by the feathered arrows. Evidently this action will cause tension in the top of ab adjacent to the joint whereas at the farther end the

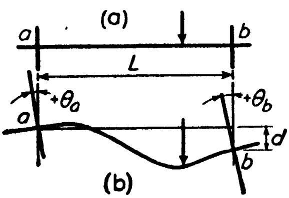


FIG. 12–9

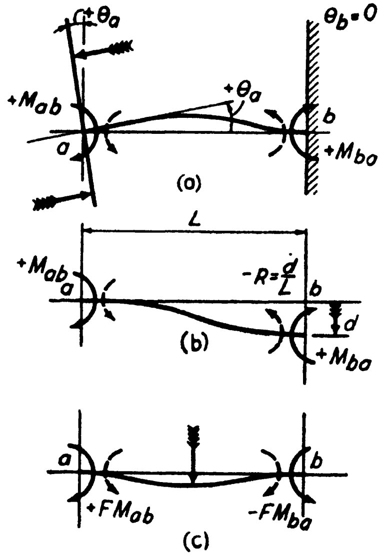


FIG. 12–10

resistance of joint b will cause compression in the top. At each end of the member are shown two small curved arrows, the one toward the joint (in solid lines) representing the moment applied by the beam to the joint, the other (dotted) the moment applied by the joint to the end of the beam. It will be seen that these moments cause stress in the top of the beam as described. In Fig. 12–10b the only deformation is a lateral movement in the plane of bending of joint b relative to the original beam position, which sets up the end moments shown. In Fig. 12–10c the only deformation is the sagging under a gravity load on the beam, the two end joints remaining fixed. It is a very easy matter to evaluate the end moments of Figs. 12–10, a and b in terms of the angle changes at the joints, θ_a and θ_b, and the ratio of lateral movement to span, d/L; to these end moments must be added those due to the loads on a fixed-ended beam (Fig. 12–10c).

In considering deformation effects of this character it is to be noted that the curvatures are so small that the curved length of the member is always assumed as equal to the straight length. Visualization of this sort is exceedingly important in the general estimation of stress effects

as well as in analysis. A correct estimation of the significant deformations of a beam requires the careful noting of where the deforming action originates and a consideration of the resistances set up in opposition.

In order to evaluate the end moments caused by twisting of the joints consider Fig. 12–11 where is shown the bent shape of ab caused by a counterclockwise rotation of each end. The twisting of joint a applies the moment M_{ab} (shown dotted) to the end of the member; the twisting of joint b applies moment M_{ba} (shown dotted). In order to express these moments in terms of the angles of twist and the stiffness elements of the member we will use the deflection theorem (conjugate beam or elastic weights) which states that the slope of the tangent to the elastic curve of a loaded beam is equal to the shear at that point in a conjugate beam (that is, an imaginary twin beam of the same length loaded with the M/EI diagram of the actual beam). Since the end slopes are to be evaluated we write the expressions given below for the end supports of the conjugate beam shown in Fig. 12–11b loaded with the M/EI curve for the loading shown in (a) of the figure, the curve for each end moment being drawn separately for convenience of computation. It will be seen that while the counterclockwise moment at a causes the counterclockwise rotation there shown, the counterclockwise moment at b would of itself cause a clockwise rotation at a. For the deflection theorem these two moments must be of opposite sign as shown.

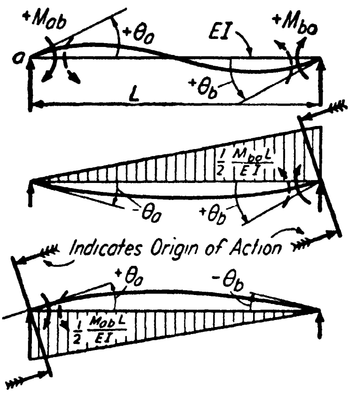


FIG. 12–11

$$\theta_a = \frac{2}{3} \times \frac{1}{2} \times \frac{M_{ab}L}{EI} - \frac{1}{3} \times \frac{1}{2} \times \frac{M_{ba}L}{EI}$$

$$\theta_b = \frac{2}{3} \times \frac{1}{2} \times \frac{M_{ba}L}{EI} - \frac{1}{3} \times \frac{1}{2} \times \frac{M_{ab}L}{EI}$$

The solution of these equations gives the following, letting the fraction I/L be represented by K,

$$\left.\begin{aligned} M_{ab} &= 2EK(2\theta_a + \theta_b) \\ M_{ba} &= 2EK(2\theta_b + \theta_a) \end{aligned}\right\} \qquad (a)$$

The insertion of the actual values of the θ's in equation (a), *with positive signs for counterclockwise twisting*, will lead to the values of the corre-

sponding moments, which will act in a counterclockwise direction on the ends of the member (shown dotted in Fig. 12–11*a*) *and in a clockwise direction on the joints* (shown by solid lines). In using the slope deflection method the attention is for the most part directed at the moments acting on the joints and *positive moment at the end of a beam will be taken as that which acts clockwise on the joint. Correspondingly, counterclockwise rotation of the end tangents of a member is considered positive.*

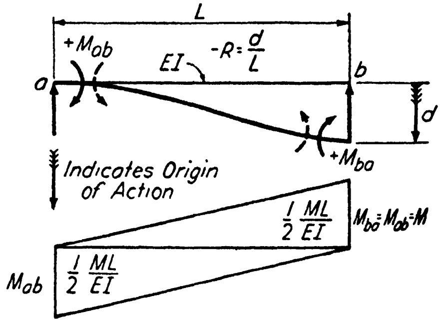


FIG. 12–12

The only deformation given to the beam of Fig. 12–12 is a lateral movement of joint b with resulting end moments as shown, moments which are plainly equal. Proceeding as before, we get

$$\left.\begin{aligned} M = M_{ab} = M_{ba} &= 2EK(-3d/L) \\ &= 2EK(-3R) \end{aligned}\right\} \qquad (b)$$

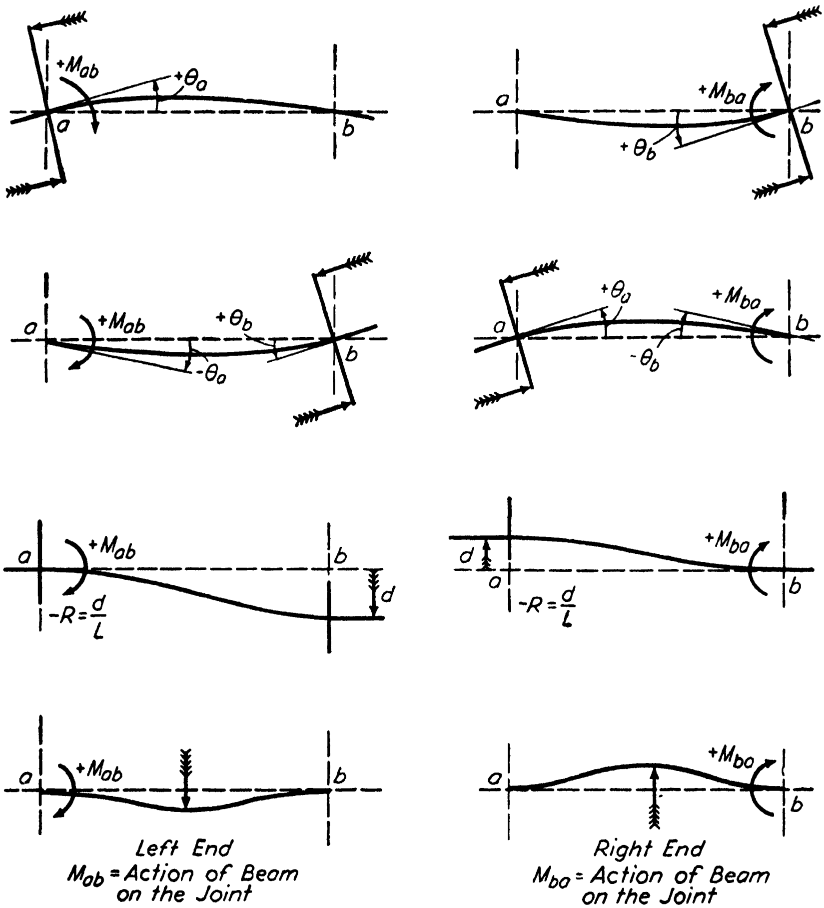
where R replaces the ratio d/L. Note that R represents the angle through which the line joining the ends a and b rotates; that a clockwise (negative) rotation results in positive end moments by the convention adopted. Signs are summarized in Fig. 12–13.

The moments set up at the end of a fixed-ended beam by transverse loading may be found by use of the three moment equation as described in Ex. 12–2, Discussion, p. 198. Tables of the values for common types of loading are given in several texts and handbooks.* Note (Fig. 12–10) that a gravity load on a beam causes a positive moment at the left end and a negative moment at the right end, using the convention of the previous paragraphs. This end moment will henceforth be designated as FM_{ab}, FM_{ba}, etc.

For any straight member of uniform cross section in a loaded frame with rigid joints (that is, one where the angles between the several members there meeting remain unchanged by frame distortion), the end moments can be expressed in terms of the three elements just considered, joint rotation, relative displacement of joints laterally in the plane of bending, and fixed-ended moment due to loads applied between joints. This results in the following general equation, the end of the

* For instance, in Bulletin 108, Univ. of Ill., "Analysis of Statically Indeterminate Structures by the Slope Deflection Method," by Wilson, Richart, and Weiss.

All Actions Shown Produce Positive Valves for the Moments Indicated Reverse of these Actions would Produce $-M$ and Reverse the Signs for θ and R



Indicates Type of Origin of Action.
Action of Joint on End of Beam is Reverse of Moment Shown.

Both Ends Framed in: $M_{near} = 2EK\,(2\theta_{near} + \theta_{far} - 3R) \pm FM_{near}$

For End Hinged: $M_{near} = 2EK\,(\frac{3}{2}\,\theta_{near} - \frac{3}{2}R) \pm H_{near}$

FIG. 12-13

beam for which the moment is written being designated the near end:

$$M_{\text{near}} = 2EK(2\theta_{\text{near}} + \theta_{\text{far}} - 3R) \pm FM_{\text{near}} \qquad [12\text{-}4]$$

where $K = I/L$

E = modulus of elasticity

$R = d/L$

I = moment of inertia of section

d = deflection of one end of the member normal to the original axis

FM_{near} = the moment caused at the near end of an identical fixed-ended beam carrying the same load as the given beam.

Note that in writing equation 12–4, and also 12–5 below, θ_a, θ_b, and R are considered unknown in magnitude and direction and so the general equation is written automatically with its given signs. The FM directions are known, being fixed by the loading, and so their proper signs must be inserted. For example, in writing this equation for the moment at the left end of the beam in Fig. 12–8, the FM term is positive, and for the right end, negative.

For slope deflection problems it is desirable to have conveniently at hand a single sheet giving the general equation with explanation of signs and notation (such as Fig. 12–13). Such a sheet should also show a special form of the general equation to use when one end of a member is hinged. This is derived as follows (Fig. 12–14):

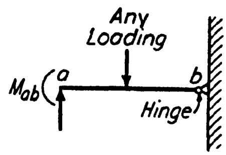


FIG. 12–14

$$M_{ab} = 2EK(2\theta_a + \theta_b - 3R) + FM_{ab} \qquad (a)$$

$$M_{ba} = 0 = 2EK(2\theta_b + \theta_a - 3R) - FM_{ba} \qquad (b)$$

Solving (b) for θ_b

$$\theta_b = \frac{1}{2}\left(\frac{FM_{ba}}{2EK} - \theta_a + 3R\right)$$

Substituting in (a)

$$M_{ab} = 2EK\left(2\theta_a + \frac{FM_{ba}}{4EK} - \frac{\theta_a}{2} + \frac{3R}{2} - 3R\right) + FM_{ab}$$

$$= 2EK(1.5\theta_a - 1.5R) + (FM_{ab} + \tfrac{1}{2}FM_{ba}) \qquad (c)$$

The form of the general equation for far end hinged is:

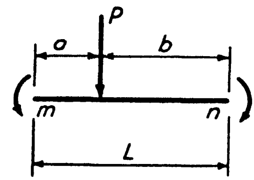
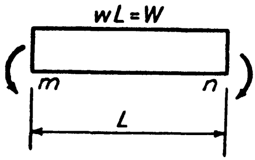
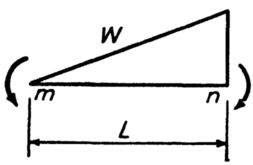
$$M_{\text{near}} = 2EK(1.5\theta_{\text{near}} - 1.5R) \pm H_{\text{near}} \qquad [12\text{-}5]$$

where H_{near} = the arithmetical sum ($FM_{\text{near}} + \tfrac{1}{2}FM_{\text{far}}$), the sign being as for FM_{near}, as determined by the loading. The values of the

constants *FM* and *H* for certain simple loadings are given in Table 12-1.

TABLE 12-1

VALUES OF CONSTANTS *FM* AND *H*

Loading	FM_{mn}	FM_{nm}	H_{mn}	H_{nm}
	$\frac{Pab^2}{L^2}$	$\frac{Pba^2}{L^2}$	$\frac{Pab}{2L^2}(L + b)$	$\frac{Pab}{2L^2}(L + a)$
	$\frac{WL}{12}$	$\frac{WL}{12}$	$\frac{WL}{8}$	$\frac{WL}{8}$
	$\frac{WL}{15}$	$\frac{WL}{10}$	$\frac{7}{60}WL$	$\frac{2}{15}WL$

Example 12-5. Compute the reactions for the three-column bent shown. The product of the modulus of elasticity by the ratio of moment of inertia to length ($I/L = K$) is taken as EK for the girders; for the outside columns, nEK; for the middle column, mEK. (Fig. 12-15.)

Solution. Since the joints are rigid, all the tangents to the members meeting at any joint rotate equally with the deformation of the structure; that is, θ_a for member ab equals θ_a for member af. By use of the general slope deflection equation the end moments for all members may be expressed as follows:

$$\begin{cases} M_{ab} = 2EK(2\theta_a + \theta_b) \\ M_{af} = 2EK(2n\theta_a - 3nR) \end{cases}$$

$$\begin{cases} M_{ba} = 2EK(2\theta_b + \theta_a) \\ M_{be} = 2EK(2m\theta_b - 3mR) \\ M_{bc} = 2EK(2\theta_b + \theta_c) \end{cases}$$

$$\begin{cases} M_{cb} = 2EK(2\theta_c + \theta_b) \\ M_{cd} = 2EK(2n\theta_c - 3nR) \end{cases}$$

$$\begin{cases} M_{fa} = 2EK(n\theta_a - 3nR) \\ M_{eb} = 2EK(m\theta_b - 3mR) \\ M_{dc} = 2EK(n\theta_c - 3nR) \end{cases}$$

It will be noted that in writing these expressions the value of $R = d/h$ was taken as equal for all three columns, the shortening of the girders under direct load being neglected. Similarly it was assumed that points a, b, and c remained on the same level as regards each other; that is $R_{ab} = 0$, $R_{bc} = 0$. Also $\theta_f = \theta_e = \theta_d = 0$.

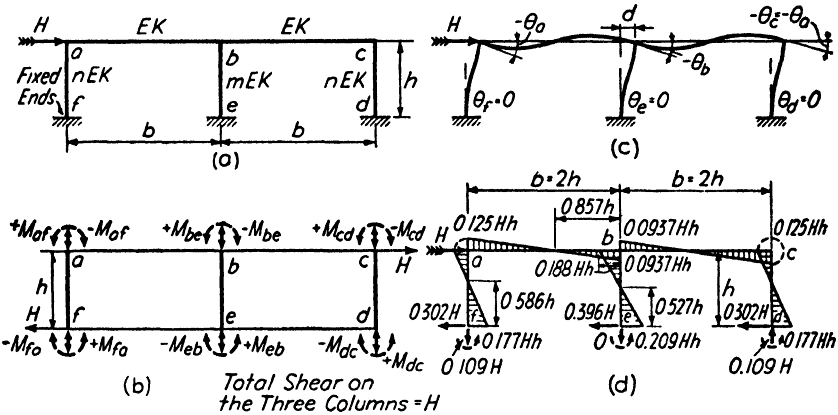


FIG. 12–15

In this series of expressions for moment there occur four unknown quantities, θ_a, θ_b, θ_c, $R = d/h$, the evaluating of which will make it possible to determine the values of all the end moments and so lead to a complete solution of the structure. Four independent equations are required for the finding of these four unknowns. Three of these equations are found at once from the condition of equilibrium at each joint.

$$M_{ab} + M_{af} = 0 \tag{1}$$

$$M_{ba} + M_{bc} + M_{be} = 0 \tag{2}$$

$$M_{cb} + M_{cd} = 0 \tag{3}$$

The fourth is given by consideration of the three columns isolated from the rest of the structure by sections just below the connecting girder and just above the supports. Here the unknown direct stresses and end moments acting on the columns are represented by double-headed arrows, and the total shear on the three columns (the distribution of which is unknown) by the horizontal arrows marked H. This system of forces is in equilibrium and taking moments about any point gives (counterclockwise moments on the columns being taken as positive):

$$M_{af} + M_{fa} + M_{be} + M_{eb} + M_{cd} + M_{dc} - Hh = 0 \tag{4}$$

In these equations the unknown end moments are assumed to be positive, as always with this method. The moment of the shear is negative, being clockwise for any section. The direct stresses balance and do not appear in the moment equation. The first of these equations, when rewritten with the two moments expressed in terms of slope and deflection, takes this form:

$$2EK(2\theta_a + \theta_b + 2n\theta_a - 3nR) = 0 \tag{1'}$$

The solution of four simultaneous equations written out in this form would be inconvenient and so the four equations are tabulated as shown in the following table. A general solution is cumbersome and accordingly the table continues with the equations as they appear for $n = m = 1$, for which values a solution is obtained.

No.	Operation	Coefficients of θ_a, θ_b, etc.				Right Side of Equation $\frac{1}{2EK}$	Check
		θ_a	θ_b	θ_c	R		
1		$2 + 2n$	1		$-3n$	0	
2	General	1	$4 + 2m$	1	$-3m$	0	
3	Form		1	$2 + 2n$	$-3n$	0	
4		$3n$	$3m$	$3n$	$-12n - 6m$	$+Hh$	
1	For	4	1		-3	0	2
2	$n = m$	1	6	1	-3	0	5
3	$= 1$		1	4	-3	0	2
4		3	3	3	-18	$+Hh$	-9
1′	#1 ÷ 4	1	0.25		-0.75	0	0.5
4′	#4 ÷ 3	1	1	1	-6	$+0.333Hh$	-3.0
5	#2 − #1′		5.75	1	-2.25	0	4.5
6	#2 − #4′		5		$+3.00$	$-0.333Hh$	8
5′	#5 ÷ 5.75		1	0.174	-0.391	0	0.783
6′	#6 ÷ 5		1		$+0.600$	$-0.0667Hh$	1.600
7	#3 − #5′			3.826	-2.609	0	1 217
8	#3 − #6′			4	-3.600	$+0.0667Hh$	0.400
7′	#7 ÷ 3.83			1	-0.683	0	0.317
8′	#8 ÷ 4			1	-0.900	$+0.0167Hh$	0.100
9	#8′ − #7′				-0.217	$+0.0167Hh$	-0.217
9′					1	$-0.0767Hh$	
7′				1		$-0.0523Hh$	
6″			1			$-0.0207Hh$	
1″		1				$-0.0523Hh$	

The work of solving these four simultaneous equations could have been done more easily by taking advantage of the various unit coefficients instead of working systematically across from left to right, eliminating the unknowns in regular order. In many problems this systematic procedure is the easier method and it is followed here for sake of illustration. Each step of the work is checked in the right-hand column where is listed the total of the coefficients

on the left-hand side of each equation. The operation performed on the individual coefficients results in a total that always should equal the result of the same operation performed on the total in the right-hand column.

The final figures in the right-hand column of the table equal $2EK$ times the value of θ_a, θ_b, θ_c, and R. Substituting the proper values in the equation for M_{ab} gives

$$M_{ab} = 2EK\left(-\frac{2 \times 0.0523Hh}{2EK} - \frac{0.0207Hh}{2EK}\right) = -0.125Hh$$

Note that the $2EK$ terms cancel.* Proceeding in similar fashion, the other moments are found to have these values:

$$M_{ba} = -0.0937Hh = M_{bc}$$
$$M_{be} = +0.188Hh$$
$$M_{fa} = +0.177Hh = M_{dc}$$
$$M_{eb} = +0.209Hh$$
$$M_{ab} = -0.125Hh = -M_{cd} = +M_{cb} = -M_{af}$$

in which Hh has only numerical significance, the algebraic signs being as here called for.

The moments at the tops and bottoms of the columns are positive that is, acting upon the column bases and top joints in a clockwise direction. The desired reactions are shown in (*d*) of the figure, with moment curves for the several members, the values being for the case where the girder span equals twice the column height.

Attention should be called to the necessity of checking the computation carefully at each stage of the work. It is very easy to make slips in setting up the equations, in preparing the table for the solution of the simultaneous equations, and in making that solution. The values of θ should not be used until they have been inserted in the original equations and found to satisfy every one of them. The best way to avoid mistakes in the calculation of reactions is to make liberal use of careful sketches.

* Since $2EK$ does not enter into the result it is plain that the moments depend on the proportional stiffness of the several parts and not upon the properties of the actual material or sections. Also there is a definite relation between stress and deformation within the elastic limit. Because of these two facts it is possible to analyze complicated indeterminate structures with ease and certainty by study of the deformations of cardboard or celluloid models which preserve the same ratios of stiffness throughout as the original structures. Professor George E. Beggs of Princeton University was the originator of this method, which he described in the Proceedings of the A.C.I., 1922 and 1923. His method has been used with success and economy for research and for actual design. Professor Beggs's achievement is a great step forward in an important field, as indeterminate structures are being used more and more. The difficulties of mathematical analyses are generally tremendous and their results are often regarded with suspicion as based upon uncertain assumptions. This direct method is free from these objections.

Example 12-6. Calculate the moments at the ends of the slabs making up the culvert section shown in Fig. 12-16.

Data. For top of culvert $I = 1000$ in.^{4} for a 1-ft strip. For walls of culvert $I = 800$ in.^{4} for a 1-ft strip. E is constant for all members.

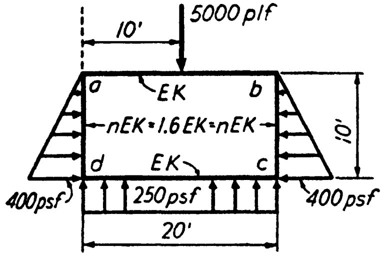


FIG. 12-16

Solution. The only new elements presented by this problem are the transverse loads and the numerical values for the sections.

For the top and bottom $K = 1000/240$; for the sides $K = 800/120$. The ratio of these two values, called n on the figure, is 1.6.

For the top $FM_{ab} = PL/8 = +150{,}000$ lb-in. acting clockwise on joint a. At the other end of the member the moment on b is counterclockwise, so $FM_{ba} = -150{,}000$ lb-in. Similarly, $FM_{dc} = WL/12 = -100{,}000$ lb-in.; $FM_{ad} = WL/15 = -16{,}000$ lb-in.; $FM_{da} = WL/10 = +24{,}000$ lb-in.

Certain information is given by symmetry: $\theta_a = -\theta_b$; $\theta_d = -\theta_c$. There are no deflections to consider, so $R = 0$ in all cases.

The following expressions are written for the moments in the left half of the structure, using the general slope deflection equation 12-4:

$$M_{ab} = 2EK(2\theta_a + \theta_b - 3R) \pm FM_{ab}$$

$$= 2EK\theta_a + 150{,}000$$

$$M_{ad} = 2EK(2n\theta_a + n\theta_d) - 16{,}000$$

$$= 2EK(3.2\theta_a + 1.6\theta_d) - 16{,}000$$

$$M_{da} = 2EK(3.2\theta_d + 1.6\theta_a) + 24{,}000$$

$$M_{dc} = 2EK\theta_d - 100{,}000$$

The corresponding moments in the right half are equal to these numerically and opposite in sign. These expressions contain two unknowns, θ_a and θ_d, and two equations for finding these unknowns are obtained by the condition of equilibrium existing at each joint.

$$M_{ab} + M_{ad} = 0$$

$$M_{da} + M_{dc} = 0$$

Substituting the values found gives:

$$4.2\theta_a + 1.6\theta_d = -134{,}000 \div 2EK$$

$$1.6\theta_a + 4.2\theta_d = +\ 76{,}000 \div 2EK$$

Solution of these equations and of the expressions for end moment gives the required information:

$$M_{ab} = +104{,}600 \text{ lb-in.} \qquad M_{dc} = -64{,}600 \text{ lb-in.}$$

The directions indicated by these signs agree with evident facts.

Problem 12-4. Solve the following examples and problems by the slope deflection method:

(*a*) Ex. 12-1; (*b*) Ex. 12-2; (*c*) Prob. 12-1; (*d*) Ex. 12-3; (*e*) Prob. 12-2; (*f*) Ex. 12-4; (*g*) Prob. 12-3.

12-7. Moment Distribution. In 1929 Professor Hardy Cross introduced the *method of moment distribution** for the analysis of continuous structures, the most generally useful of all methods since it permits results to be obtained with any desired precision, being a method of successive approximations, and since it focuses the attention upon the structural action of the frame under load and does not involve a maze of mathematics. The direct application of the method is to a frame with joint rotation under load but no joint translation, and involves the following steps: (*a*) the determination of the moments at the ends of all members on the assumption that all joints are fixed, that is, locked against rotation; (*b*) the successive unlocking of each joint in turn and the recording of the redistribution of end moments affected by each unlocking; (*c*) the addition of end moments at all joints when the desired precision has been attained. The extension of the method to structures with joint translation will be considered later.

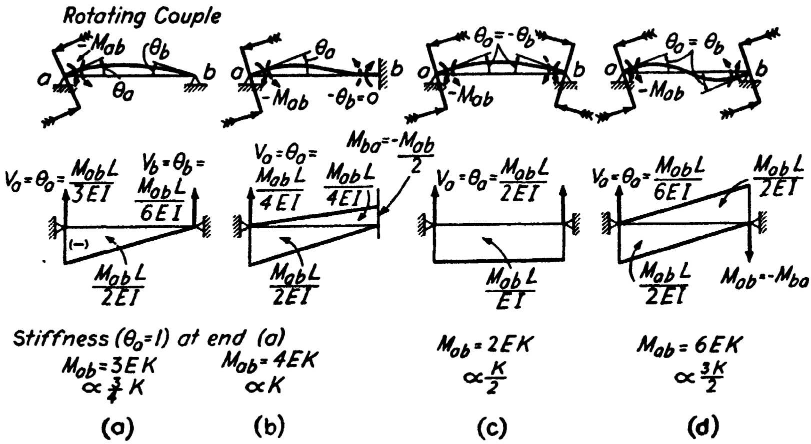
The determination of the end moments for a loaded fixed-ended prismatic beam has been considered on page 209, and the common cases are covered by the table on page 212.

When all the fixed-end moments have been recorded for any beam or frame it will be found generally that the end moments brought to any joint by the several members there meeting do not balance; that is, the clockwise moments will not be equal to the counterclockwise. The resultant moment acting on any joint is called, accordingly, the *unbalanced moment.* This unbalanced moment stands on the record only because of the imaginary operation of locking the joints against rotation. When any single joint is unlocked this unbalanced moment acts instantly to rotate it and the movement at once develops a resistance to that rotation on the part of every member meeting at the joint: the rotation ceases when this resistance equals the initial unbalanced moment. The operation of unlocking a joint consists in recording the resistance, the bending moment, developed at the end of each member meeting at the unlocked joint, opposite the unbalanced moment in direction. When the operation is completed the joint is at rest; in the language of the method, the unbalanced moment has been *distributed* to the several

* "Continuity as a Factor in Reinforced Concrete Design," Proc., A.C.I., 1929. Of the many references available the student should consult Cross and Morgan, Continuous Frames of Reinforced Concrete, John Wiley & Sons, Inc., 1932, and the paper with discussion "Analysis of Continuous Frames by Distributing Fixed Ended Moments," Hardy Cross, Trans., A.S.C.E., Vol. 96, 1932.

members: the operation is spoken of as *distributing the unbalanced moment.*

When a joint is rotated, as by the action of the unbalanced moment, the ends of all the members there meeting turn through the same angle. A stiff member will offer large resistance to rotation, a flexible member small resistance. Rotational resistance is proportional to stiffness where stiffness is defined as the bending moment which, when applied to one end of a beam, causes unit rotation of that end.* An expression for stiffness may be derived by aid of the conjugate beam theorems, as carried through in Fig. 12–17, which considers the four common cases,



CK	
Member	Value
(a)	3K
(b)	4K
(c)	2K
(d)	6K

Fig. 12–17

the stiffness of end a being found when end b is freely supported (case a), is fixed (case b), receives an impressed rotation $\theta_a = -\theta_b$ (case c), $\theta_a = \theta_b$ (case d). In each case the conjugate beam is loaded with the M/EI curve of the actual beam and the left-hand reaction computed ($V_a = \theta_a$): the stiffness equals the value of M_{ab} when $\theta_a = 1$. Only one case requires any discussion — case b with end b fixed. Since in the actual beam there is no deflection at a there is no moment at a in the conjugate: accordingly, the magnitude of the positive (M_{ba}/EI)

* It is customary to add the stipulation that the far end of the beam is fixed against rotation. It is believed to be simpler to proceed as here.

loading must be half that of the negative (M_{ab}/EI) loading since its lever arm as regards a is twice that of the negative. For a prismatic member stiffness is proportional to $K = (I/L)$ and is a function of the condition at the far end.* Note that the sign convention used in Fig. 12–17 is that used with beams; positive moment accompanies compression in the top fiber.

In Fig. 12–18 there is shown a moment M, positive, that is clockwise on the joint, applied to joint a where four members meet. This may be considered as the resultant of the four fixed-end moments brought to a by reason of transverse loads on the four members (not shown) with

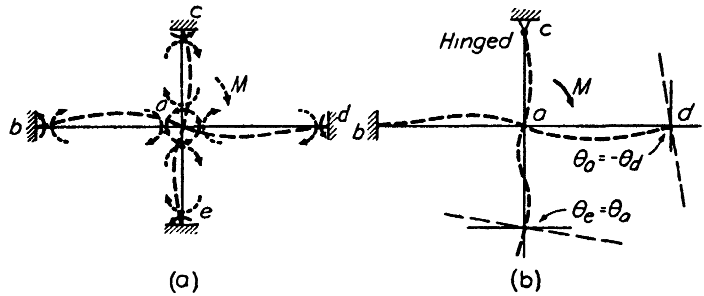


FIG. 12–18

joint a locked against rotation. When the joint is unlocked this moment will cause the rotation shown, which will come to rest when the resisting moments shown have been built to equal M in magnitude. (As before, the full arrows next to a joint indicate the moment applied to the joint by the beam; the dotted arrows the moment applied to the beam by the joint.) From the concept and definition of stiffness it is plain that each beam resists this rotation in proportion to its stiffness; this enables us to express the resistance of any one of them as follows: $M_{ab} = -M$ (stiffness of ab/total stiffness of the four members) $= -M(K_{ab}/\Sigma K)$. In this case (Fig. 12–18a) the far ends are fixed and so stiffness is proportional to K throughout. In Fig. 12–18b the conditions of the far ends vary as already considered in the previous figure. In this case we have

$$M_{ab} = -M \frac{4K_{ab}}{(4K_{ab} + 3K_{ac} + 2K_{ad} + 6K_{ae})}$$

$$M_{ac} = -M \frac{3K_{ac}}{(4K_{ab} + 3K_{ac} + 2K_{ad} + 6K_{ae})}, \qquad \text{etc.}$$

* When stiffness is defined in terms of the far end fixed (case b), the resistance to rotation at end a for the other three cases is spoken of as measured by a "modified stiffness" or a "modified K": the modifying factor is $\frac{3}{4}$ for case a, $\frac{1}{2}$ for c, and $\frac{3}{2}$ for d.

Commonly all far ends are fixed and it is more convenient to record stiffness as measured by K rather than $4K$. Accordingly, when far end conditions vary from fixity it is more convenient to consider stiffness as measured by $\frac{3}{4}K$, $\frac{1}{2}K$, and $\frac{3}{2}K$ for cases a, c, and d, Fig. 12-17.

In carrying through an analysis by moment distribution, when any joint is unlocked, usually all the other joints are considered to be locked. The exception obviously is in the case of imposed rotation at the far end of a member, as already provided for and which will be illustrated later. Consequently, when a joint is unlocked and the unbalanced moment is distributed at that joint, there are also set up moments at the far ends of all members there meeting when these far ends are locked. Inspection of Fig. 12-17*b* shows that this moment induced at the far end has a magnitude equal to one-half that set up in the rotating end of the beam, and that it acts upon the far joint in the same direction as the induced resistance at the near end, clockwise in the case shown. This moment at the far end is named the *carry-over moment* and its ratio to the near end moment is called the *carry-over factor*, one-half in the case of prismatic members.

Two sets of signs are common in moment distribution operation, that common for beams, as used up to now in this article, and that now becoming standard for slope deflection; positive moment is that which acts in a clockwise direction on a joint, and will be used henceforth. The carry-over factor, accordingly, is $+\frac{1}{2}$.

When any joint is unlocked in carrying through a moment distribution analysis it is evident that the carry-over moments cause new unbalanced moments, which are progressively smaller as the sequence of unlocking proceeds.

Problem 12-5. Verify the several values for end moments for Fig. 12-18*b* here given, using the stiffness relationship and checking by the slope deflection equation. Note that $\Sigma K = K_1 + (3K_2/4) + (1K_3/2) + (3K_4/2)$. Here positive moment is taken as that acting clockwise on a joint.

$$M_{ab} = -\frac{MK_1}{\Sigma K} \quad M_{ac} = -M\left(\frac{3K_2}{4\Sigma K}\right) \quad M_{ad} = -M\left(\frac{K_3}{2\Sigma K}\right) \quad M_{ae} = -M\left(\frac{3K_4}{2\Sigma K}\right)$$

$$M_{ba} = \frac{M_{ab}}{2} \qquad M_{ca} = 0 \qquad M_{da} = -M_{ad} \qquad M_{ea} = M_{ae}$$

The two examples which follow demonstrate the detailed application of moment distribution and the special terminology used to designate the several operations. The second example considers a case of joint translation combined with rotation.

Example 12-7. Compute the bending moments developed at the supports of this continuous beam due to the load shown in Fig. 12-19. The beam is of uniform section over its entire length.

Explanation. Preliminary to solution it was necessary to compute the relative stiffnesses of the three spans and the proportion of the total resistance to bending developed at each side of a joint, that is, $K/\Sigma K$ for joint c directly, the similar relationship at b involving a modified K for member ba since a is hinged. Next the fixed-ended moment, $wL^2/12$, was computed and recorded for the loaded span, a clockwise, or positive, moment on joint b, counterclockwise on joint c. From here on two procedures are shown, A and B, the second of which is used when an abbreviated solution is desired, one not carried to the maximum of precision possible.

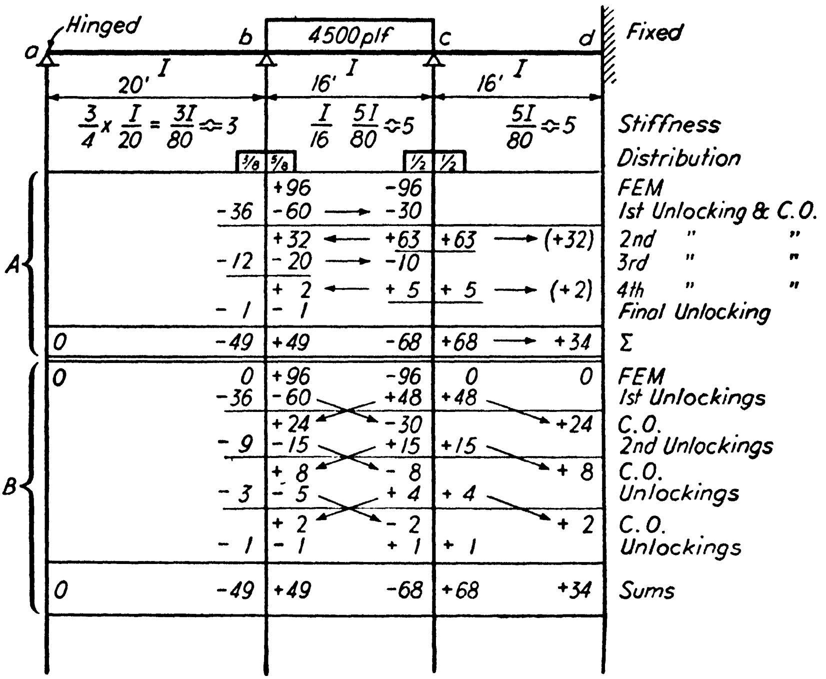


FIG. 12–19

Solution A. With the fixed-ended moments (henceforth FEM) written, the unlocking starts with joint b where the unbalanced moment is 96 k-ft clockwise. The ratio of distribution or resistance is $\frac{3}{8}$ to the left of b and $\frac{5}{8}$ to the right, 36 and 60 k-ft respectively, both counterclockwise or negative. Combining these values with the original FEM gives us -36 to the left end $+36$ to the right, a state of balance for joint b, a status indicated by the horizontal line drawn below the distributed moments.

The application of a clockwise moment of 60 k-ft at the b end of beam bc, c being fixed (locked), tends to rotate the joint c in a counterclockwise direction, the moment there induced equaling half of that at b, -30 k-ft as recorded. In the common terminology of the method, the effect of the distribution of the FEM at b *carries over* to the adjacent joints when they are locked; here -30 is the " carry-over " and the fraction $\frac{1}{2}$ is the " carry-over factor."

The carry-over effect from b to c increased the unbalanced moment at c, that is, added to the FEM already there. The total FEM to be distributed is thus -126, the resistance being equal on both sides, giving $+63$ to right and left. The carry-over to b, which was considered to be locked during this operation, is $+63/2$ or $+32$, fractions of kip-feet being avoided for simplicity. Joint b now being unbalanced, a new distribution and carry-over follow; and so on until the carry-over is too small to need consideration. The totals are then summed to give the desired moments at b and c, and the carry-over to d is written in one operation instead of being recorded following the second and fourth distributions (unlockings) as, of course, may be done if preferred.

Solution B. In this variation of method, joint b is first unlocked and the distribution noted, and then joint c likewise, the carry-overs not being recorded; a horizontal line is drawn and the balance of the joints checked as precaution against error. The carry-overs are then recorded below the horizontal since their magnitudes are too large to ignore, and a second distribution (unlocking) is carried through, again omitting carry-overs. In some situations this would be sufficiently precise (the loading being such that the magnitude of the moment at d evidently is not of interest as part of this particular solution) and the summation would be taken. In this case the work is carried two steps farther in order to check solution A.

Example 12–8. Compute the reactions at d and c of this rigid frame induced by the load shown in Fig. 12–20.

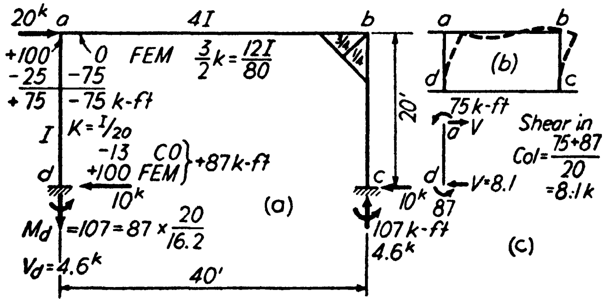


FIG. 12–20

Explanation. Inspection of the loaded frame shows the rotation at joint a will be equal to that at joint b in direction and magnitude. The stiffness of ab, accordingly, will be proportional to $3K/2$. The distribution of resistance at joints a and b is recorded at b only, ¾ in the girder and ¼ in the column.

The next step assumes that the frame is forced over to the right an indeterminate distance by a force applied at a or b, no rotation of joints a and b being permitted. This action sets up positive end moments (clockwise on joints) in the columns and a convenient value of 100 k-ft is ascribed to each end moment, and recorded. Next, joints a and b are unlocked together, whereupon each rotates the same amount and a redistribution of the FEM results as recorded at joint a with CO (carry-over) to d. The summation of end moments shown must be those consistent with the force finally required to maintain the side lurch first impressed on the frame. This force is easily

found: the sum of the shears in the columns, as worked out in Fig. 12-20c, a horizontal force of 16.2 kips acting to the right at a. The actual end moments for the given force of 20 kips at a must be in proportion, as shown at c and d in Fig. 12-20a.

If the given force had been applied to some intermediate point of column ad the deformation of the frame would not have been as shown in (b) and it would have been necessary to lock and unlock joints a and b separately. In this situation the stiffness of the girder would have been proportional to K directly.

Problem 12-6. Carry through a solution of Ex. 12-8 neglecting symmetry, locking and unlocking a and b in succession, the stiffness of ab being proportional to K.

Problem 12-7. Solve the following examples and problems by the method of moment distribution.

(a) Ex. 12-1; (b) Ex. 12-2; (c) Prob. 12-1; (d) Ex. 12-3; (e) Prob. 12-2; (f) Ex. 12-4; (g) Prob. 12-3; (h) Ex. 12-5; (j) Ex. 12-6.

Problem 12-8. The frame of Ex. 12-8 carries a single gravity load of 40 kips applied at the left quarter-point of the girder. Draw the shear and bending moments curves.

Discussion. This eccentric load will cause side sway. Neglecting this effect, we obtain moments ($M_{ad} = -92$, $M_{bc} = +56$, etc.) which give unequal shears in the two columns; a result equivalent to the effect of the given load and an additional horizontal load of 2.7 kips acting to the left at a or b. In order to obtain the correct result, accordingly, there must be added the moments produced by an equal horizontal force acting to the right at a or b, which may be found by proportion from the results of Ex. 12-8.

Ans. $M_{da} = -32$; $M_{cb} = +42$ k-ft.

12-8. Moments in Building Frames.

The development of practicable methods of rigid frame analysis has led to the abandonment of the use of moment factors except in certain restricted situations. The indiscriminate use of moment factors in the past proved unsatisfactory, and resulted sometimes in unsafe and sometimes in uneconomical designs. The general nature of the problem met with in building construction is illustrated in Fig. 12-21 for the case of maximum positive moment in an interior span, cd. It is obvious that for this stress the movable load should cover span cd, resulting in a deformation of the whole frame somewhat as shown, only the members adjacent to the load being greatly affected.

Inspection of these deformations indicates that the positive moment in cd will be increased by loading alternate spans on both the same floor and the floors above and below (ab, ef, $b'c'$, etc.). For maximum negative moment at c required placement of live load is easily visualized similarly, to cover spans bc and cd, and after that alternate spans on the same and adjacent levels.

The exact solution of the large number of moments which must be considered in a frame of any size is a laborious matter and fortunately

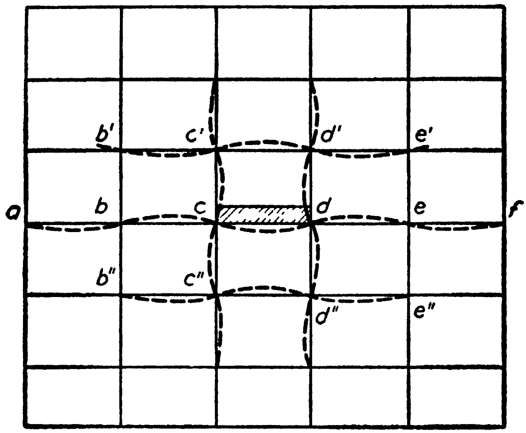


FIG. 12–21

not necessary. Sufficiently accurate results can be obtained by considering only parts of the frames; for example, that shown in Fig. 12–22*a* for maximum positive moment in an interior beam, the beam itself only being loaded; and that shown in Fig. 12–22*b* for maximum negative moment at point 1. It is customary to consider the far ends of the adjacent beams and columns as fixed but more conservative results can be obtained by assuming them as having an impressed rotation

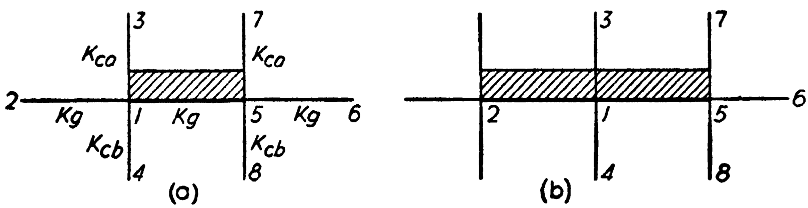


FIG. 12–22

of the character consistent with the presence of other loads which add to the magnitude of that being studied.

A great saving of time in office practice may be effected by using the tables of maximum moment coefficients prepared by Messrs. A. J. Boase and J. T. Howell* and printed in the Appendix of Reinforced Concrete Design Handbook of the American Concrete Institute, a very useful compilation of design data.†

* First presented in the *Journal, A.C.I.*, Sept., 1939, "Design Coefficients for Building Frames."

† Published cooperatively by the American Concrete Institute, Portland Cement Association, Concrete Reinforcing Steel Institute, and the Rail Steel Bar Association; obtainable at any of their offices.

Solution by Slope Deflection. The slope deflection equation enables us to set down a very simple expression for the negative moment at the end of the beam 15 in Fig. 12-22*a*, which leads to an expeditious solution for both this and more complicated cases. Considering ends 2, 3, 4, 6, 7, 8 to be fixed and the slope at 5 to be equal and opposite to that at 1, we may evaluate the bending moments at joint 1 as follows:

$$M_{15} = 2EK_g(2\theta_1 - \theta_1) + \text{FEM}_{15} \quad \text{and also}$$

$$= -(M_{12} + M_{13} + M_{14})$$

$$M_{12} = 2EK_g(2\theta_1) \qquad M_{13} = 2EK_{ca}(2\theta_1) \qquad M_{14} = 2EK_{cb}(2\theta_1)$$

The sum of the moments at the joint equals zero; from this we obtain

$$M_{12} = -\text{FEM}_{15}\left(\frac{4K_g}{6K_g + 4K_{ca} + 4K_{cb}}\right), \quad \text{etc.}$$

$$M_{15} = -\text{FEM}_{15}\left(\frac{4K_g + 4K_{ca} + 4K_{cb}}{6K_g + 4K_{ca} + 4K_{cb}}\right) \tag{A}$$

If the far ends of the adjacent members are not fixed their stiffnesses are no longer proportional to K but require the addition of a multiplier as explained in connection with Fig. 12-18*b*. Designating by the letter C these multipliers of K, it is possible to rewrite equation A in a more general form to cover the situation when the far ends (1, 2, 3, etc.) are not fixed but either are hinged or have an impressed rotation.

$$M_{15} = -\text{FEM}_{15}\left(\frac{C_2K_g + C_3K_{ca} + C_4K_{cb}}{(2 + C_2)K_g + C_3K_{ca} + C_4K_{cb}}\right) \tag{B}$$

By way of summary, note that the C terms have these values:

Far end fixed	$C = 4$
Far end hinged	$C = 3$
Far end rotated same as near end	$C = 6$
Far end rotated equally but opposite to near end	$C = 2$

If two adjacent spans are loaded the equations (A or B) may be used for each and the results combined. Formidable as these equations appear they are surprisingly easy to use as they stick in the memory very quickly without effort. However, the method of moment distribution is still easier of application and much more easily tabulated.

Solution by Moment Distribution. The statement of the problem already made suffices to indicate its solution by moment distribution.

The 1940 J.C.* report outlines the solution in a general formula the derivation of which is made clear by the example below. It is noted that this comprises only two cycles of distribution; this is sufficient "for preliminary designs and also for final designs in the ordinary cases" (J.C.).

Example 12–9. Compute, by moment distribution, the negative bending moment to the left of joint 2 for the loaded frame shown in Fig. 12–23.

Discussion. The far ends of all members are assumed to be fixed and all stiffnesses are proportional to the K values accordingly. The operation of

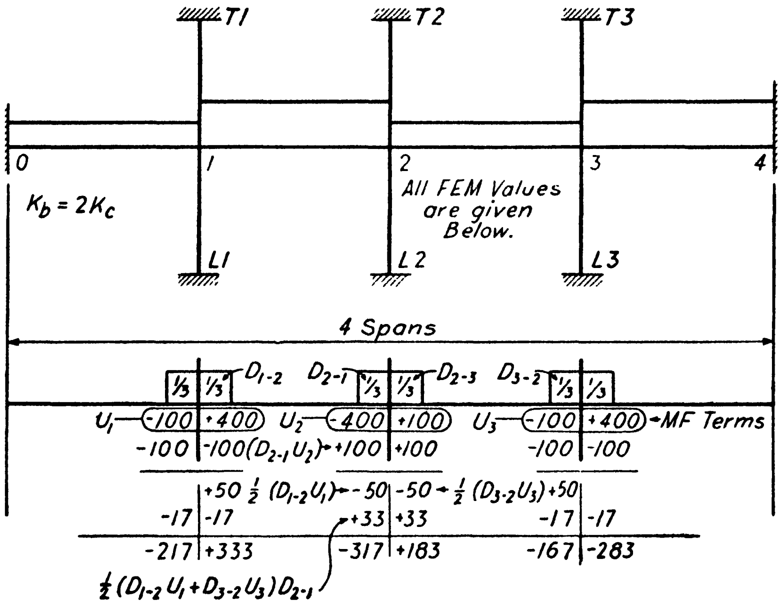


FIG. 12–23

the two cycles of distribution requires no comment. The sign convention here used is that moments clockwise on a joint are positive and so carry-over moments have the same sign as their originating moments.

The required moment, −317 k-ft, is the sum of four terms, each of which has been given a letter designation so that the sum may be identified as that indicated by Formula 1, J.C. Report, Appendix 2; the distribution ratios are called D, with subscript to indicate joint and span; the sum of the FEM (or MF per J.C.) terms are lettered U with a subscript naming the joint.

The student should note that this is a general situation, chosen to illustrate a given formula and not intended to result in any given moment maximum.

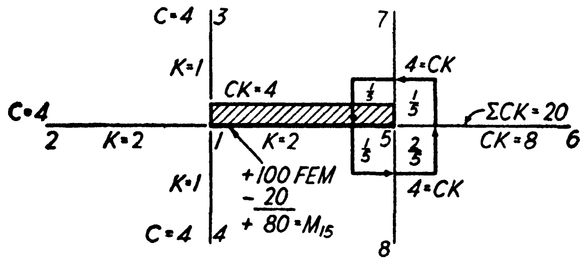
* Appendix 2, J.C. Report, 1940.

Problem 12–9. Complete the determination of M_{2-1} started in Ex. 12–9, working in integral kips.

Discussion. Two more cycles only are possible if fractional kip values are not used.

Ans. -321 k-ft.

Problem 12–10. Compute the positive moment at the center of span 15, Fig. 12–22a in terms of wL^2, when FEM $= wL^2/12 = 100$, $K_{ca} = K_{cb} = 1$; $K_g = 2$, for two cases: (a) far ends (2, 3, 4) fixed, (b) far ends rotating equally but opposite to joint 1



(c) Compute also the moment factor for positive moment for combined live and dead load where the live load intensity equals three times that of the dead load, assuming that the beam acts as though fixed-ended as far as dead load is concerned. The current codes use a factor of $1/16$ for this case.

Ans. (a) $0.058wL^2$; (b) $0.069wL^2$; (c) 0.054 for case a, 0.062 for case b.

The moment distribution solution for the negative moment in part (a) is shown in the figure herewith completely; the distribution of moment at joint 1 is shown at 5 and a single unlocking completes the operation.

12–9. Frames with Non-Prismatic Members.

Considerations of economy and of architectural fitness often lead to the use of non-prismatic members in bridge and building frames; haunches are introduced at the ends of a beam where the stress is highest, tapered beams and beams with a curved soffit (under part) are in common use. But for the fact that tables and diagrams are available which give the necessary constants, the stress computation of such structures would be exceedingly onerous, although no new principle is brought into operation by the variability of section. Once these constants have been found, analysis, either by slope deflection or by moment distribution, proceeds in the same manner and with the same ease as with prismatic sections.

The nature of the problem presented by varying moment of inertia is given in Fig. 12–24* which shows (a) one span of a continuous girder of variable cross section, carrying a single concentrated load, P. Assuming

* The problem here portrayed is worked out with numerical values completely in G. E. Large and C. T. Morris, "The Moment Distribution Method of Structural Analysis Extended to Lateral Loads and Members of Variable Section," Engineering Experiment Station, Ohio Univ., Bul. 66, 1931, page 13.

the span fixed at A and B, the first problem is to compute the bending moments there set up. For writing the two equations needful for the two unknowns we may use the moment area theorems; here the angle between the tangents to the elastic curve at A and B is zero, and the offset of each point of support from the tangent to the other support is also zero. Accordingly the area of the M/EI diagram between A and B equals zero (equal positive and negative areas), and also the moment of this diagram about either support equals zero. Since E is a constant it may be omitted from the calculations. The work of solution consists in computing the moment of inertia at a series of sections and recording these values in convenient fashion, as in the "I" curve (b): the bending moment diagram is drawn in three sections, for the load alone and for an assumed value of each end moment alone (c). Integration being impracticable here, the span is divided into a convenient number of sections and the three M/I curves plotted. The determination of the centroids of total area here shown is not necessary but is convenient for discussion of detail.

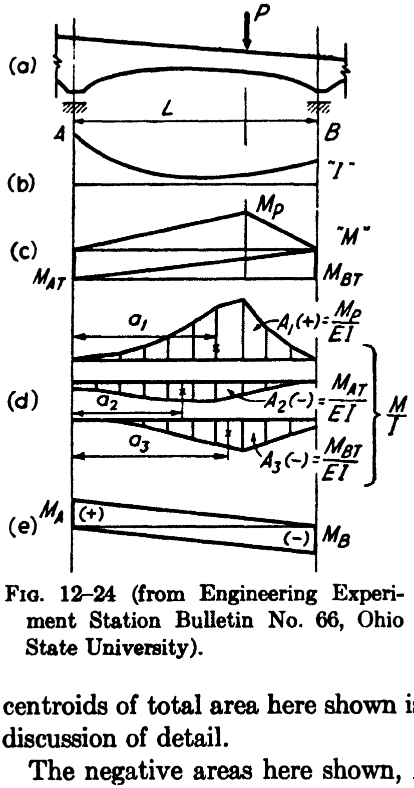


FIG. 12-24 (from Engineering Experiment Station Bulletin No. 66, Ohio State University).

The negative areas here shown, A_2 and A_3, are trial values and not the true values. The true areas may be expressed as $(A_2/M_{AT})M_A$ and $(A_3/M_{BT})M_B$, where M_{AT} and M_{BT} are the trial values and M_A and M_B the true values which are being sought.

The FEM for this loading are now found by solving these two equations:

$$A_1 - (A_2/M_{AT})M_A - (A_3/M_{BT})M_B = 0$$

$$a_1A_1 - a_2(A_2/M_{AT})M_A - a_3(A_3/M_{BT})M_B = 0$$

The determination of the carry-over factor from end A to end B, r_A, involves the determination of the moment set up at B, assumed to be fixed, when a given moment is applied at A, assumed to be hinged. It is evident that the M/I diagrams already constructed with areas A_2

and A_3 will serve again for this computation, noting that the A_2 area is now positive and of definite assumed value; A_3 and M_B are unknown. The second moment area theorem used above suffices here to give the necessary equation for finding the one unknown.

In order to find the carry-over factor from end B to end A, r_B, a similar procedure is in order with end A fixed and end B hinged and subjected to a given moment.

All the data have now been gathered for determining the stiffness factor, which will be expressed as a coefficient times the moment of inertia at some given section, usually the least I. From the general slope deflection equation we may write $M = CEK\theta$, which is the stiffness as here defined when $\theta = 1$, or $E\theta = M/CK$. The C factor for the A end may be determined by equating the two values of $E\theta$ which are now available: $(A_2 - r_A A_3)$ and M/CK, M being the arbitrarily chosen moment applied above at A for determining the carry-over factor, and K employing the desired value of I. The C factor for the B end is similarly computed.

Diagrams typical of those most used for problems involving variable I are shown in Fig. 12–25, chosen as representing the most commonly used types of non-prismatic beams. These two charts were also reproduced in the form here shown in the Portland Cement Association pamphlet, "Analysis of Rigid Frame Concrete Bridges." Another reference which will be helpful in office practice is "Concrete Beams and Columns with Variable Moments of Inertia," Portland Cement Association, pamphlet ST41.

The structural action of a beam with a curved soffit, as shown in Fig. 12–25a, differs from anything hitherto considered since its center line is curved and not straight and the overall length changes with load and reaction conditions. An approximate solution of this effect is given in the following example.

Example 12–10. Compute the maximum live load moment at the corner of the rigid frame in Fig. 12–26. The frame carries a uniform live load of 90 psf and a concentrated live load of 2500 plf of width, impact allowance included. *Note:* This is part of an analysis worked out completely in the Portland Cement Association booklet, "Analysis of Rigid Frame Concrete Bridges," drawings and computations are taken directly therefrom by permission. In this booklet are given the trial design rules which gave the trial section here analyzed. The student should obtain a copy and check through the whole problem.

Solution. As preparation for the moment determination the designer here chose to compute by moment distribution the corner moments resulting from the application of an arbitrarily chosen moment of +100 at the left corner. The ratios thus determined may be used directly for the several load conditions to be considered and thus avoid a separate distribution for each loading.

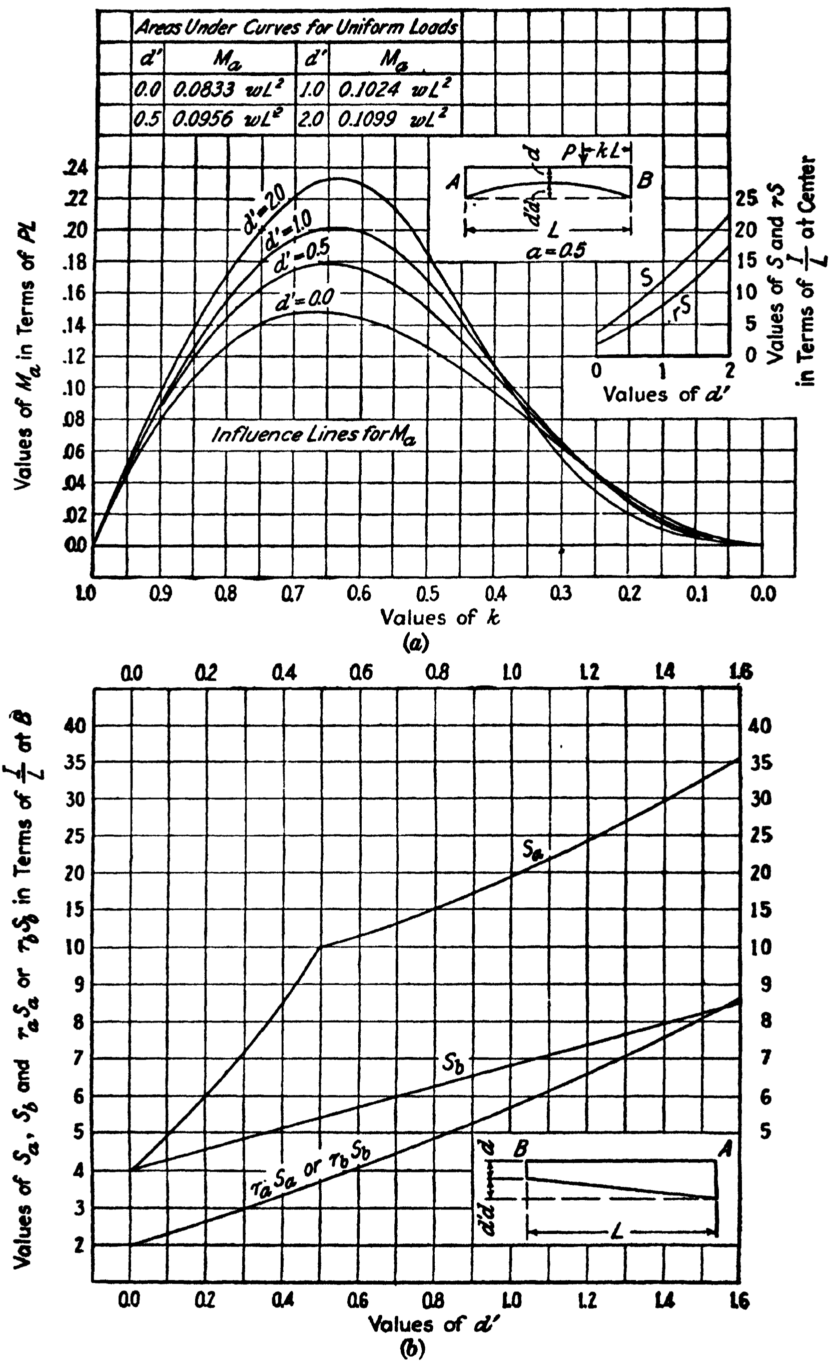


FIG. 12-25 (from "Continuous Frames of Reinforced Concrete," by Cross and Morgan).

Fig. 12-26*a* shows the elements into which the frame is assumed to be resolved for determining the analysis constants. In order to enter Fig. 12-25, the ratios d' are required: 1.43 for the deck and 0.70 for the end wall. From Fig. 12-25*a* S (the stiffness factor) is read as 16.25 (a precision actually impos-

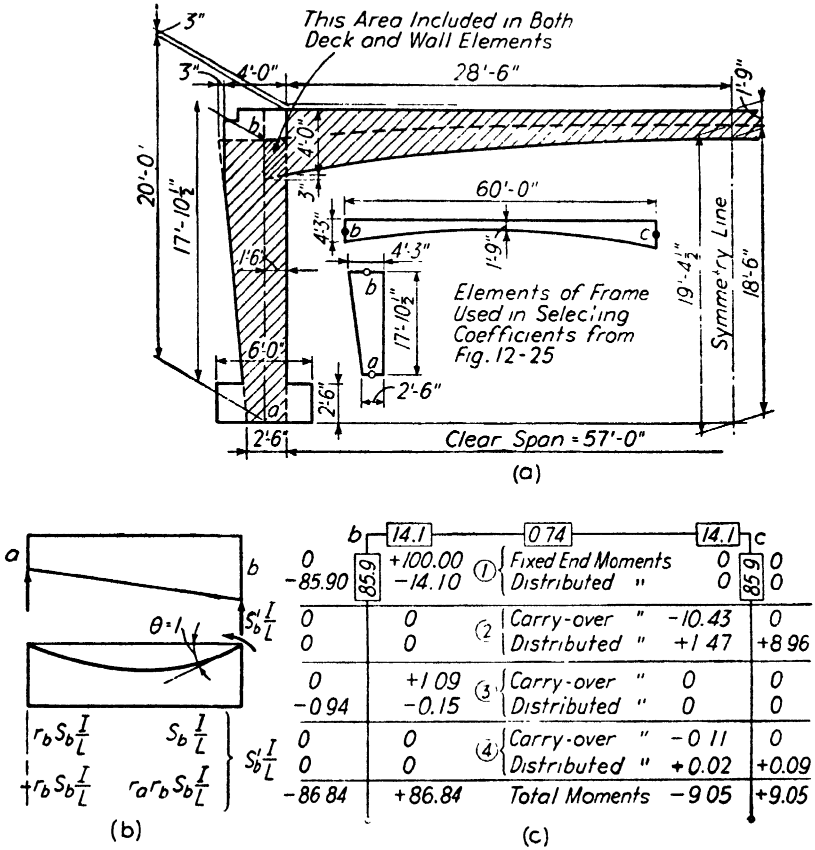


Fig. 12-26 (from "Analysis of Rigid Frame Concrete Bridges," P. C. A.).

sible from this particular edition of the chart); rS as 12 which gives r (the carry-over factor) as 0.74.

Foundation conditions for this structure were assumed to permit sufficient rotation at the lower end of the end wall to allow this support being taken as hinged. The stiffness factors read from the diagram, Fig. 12-25, assume the far end to be fixed. The modification in the stiffness factor for far end fixed made by freeing that end from restraint is shown in Fig. 12-26*b*: here assuming end *a* fixed and a moment applied at *b* to cause unit rotation, the end moments are as first recorded, S_bI/L at *b* and r_bS_bI/L at *a*; on unlocking joint *a* the moment there disappears and the carry-over to *b* is $-r_ar_bS_bI/L$, which combined with the initial moment equals the stiffness at end *b* when end *a* is hinged, $S_b'I/L = (S_bI/L)\ (1 - r_ar_b)$. By entering Fig. 12-25*b* with

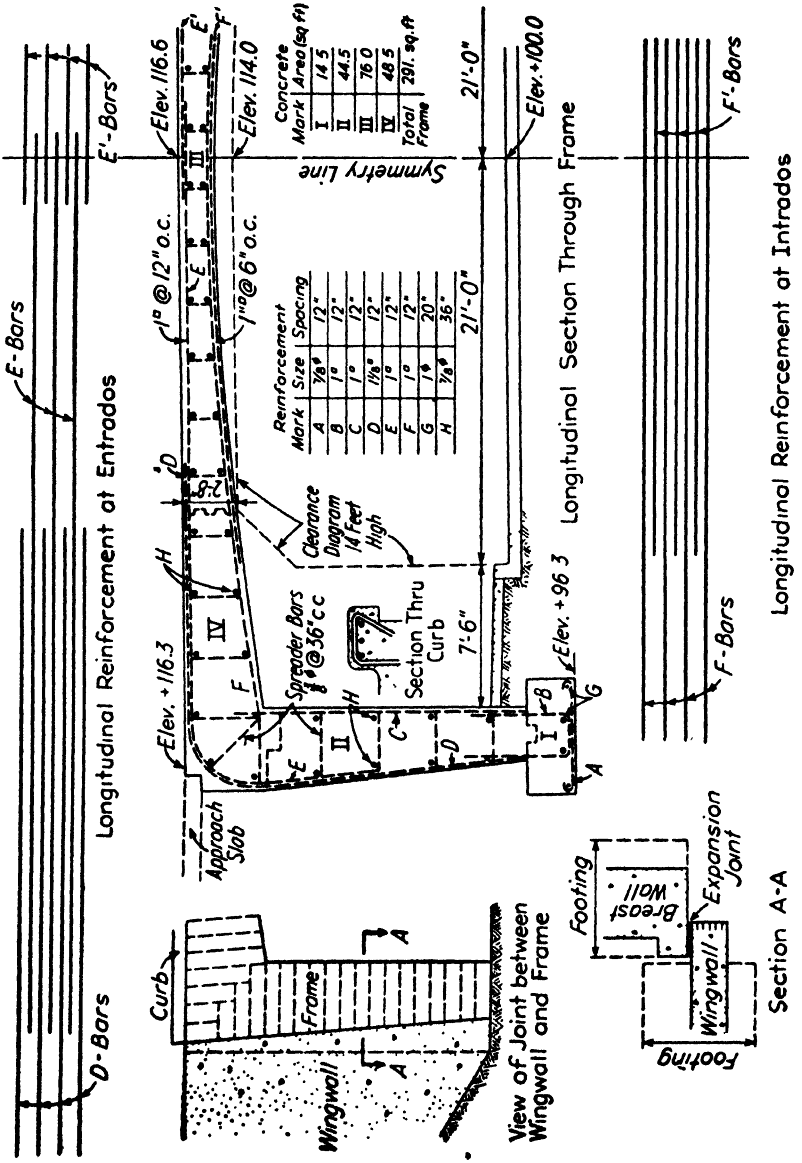


FIG. 12-27 (from "Analysis of Rigid Frame Concrete Bridges," P. C. A.).

$d' = 0.70$, we obtain $S_b = 13.3$; $S_a = 6.0$; $r_aS_a = r_bS_b = 4.4$; $r_a = 0.733$; $r_b = 0.331$.

For the deck member the stiffness factor is related to the center moment of inertia, here taken as proportional to the depth: accordingly, the stiffness of the deck is taken as $SI/L = 16.25 \times (1.75^3/60) = 1.45$. Similarly for the end wall the stiffness at the large end, small end hinged, is $[13.3 \times (2.50^3/17.88)](1 - 0.733 \times 0.331) = 8.83$. At the jointure of deck and end wall the relative stiffnesses in per cent are 14.1 for the deck, 85.9 for the end wall. These figures are recorded on the line diagram of the frame in Fig. 12–26c, together with the carry-over factor, 0.74, for the deck.

The distribution of the fixed-end moment of +100 assumed at b of the deck is shown in Fig. 12–26c and requires no comment.

For maximum live moment at joint b the live load must cover the entire deck and the concentrated load should be placed at about the 0.625 point, noting the indications of Fig. 12–25a. For $d' = 1.43$, the area under the influence line for corner moment with side sway prevented (Fig. 12–25a) is about (by interpolation) $0.106wl^2$, which makes the uniform load moment 34,300 lb-ft. For $k = 0.625$ (again by interpolation) the corner moment at b due to the concentrated load, $2500 \times 60 \times 0.216 = 32{,}400$ lb-ft: at c the moment is $2500 \times 60 \times 0.099 = 14{,}900$ lb-ft. These moments assume fixed joints; the true moments on the assumption of straight deck are obtained by multiplying by the factors found above (Fig. 12–26c) for an assumed corner moment of +100: $34{,}300(0.8684 + 0.0905) + 32{,}400 \times 0.8684 + 14{,}900 \times 0.0905 = 62{,}300$ lb-ft.

To correct for the effect of curvature we are advised to multiply by the ratio $(H + R/2)/(H + R)$, $(17.88 + 1.50/2)/(17.88 + 1.50)$, giving 59,900 lb-ft. the final moment.

CHAPTER XIII

BUILDING DESIGN — INDIVIDUAL MEMBERS

13–1. Many practical problems arise in the application of the theories which have been developed in the previous chapters for the design of slabs, beams, and columns. The present chapter deals with the practical application of theory to simple design problems involving individual members. It is intermediate between the theories of Chapters VII, VIII, and IX and the illustrative designs that follow. The members worked out here are taken from the designs that are developed completely hereafter.

13–2. One-Way Solid Slabs. A one-way solid slab is a rectangular beam of extended width. A strip 1 ft wide is designed as a rectangular beam with $b = 12$ in. The slab must be sufficiently strong to carry the live load and the dead weight of all construction without overstressing either the concrete or the steel in flexure, shear, or bond.

The dead load for a building consists of the weight of floor finishes, ceilings, partitions, walls, vaults, and all permanent fixtures that are built in as a part of the structure, as well as the weight of the construction itself. As soon as the layout of the job is determined the dead loads are computed from the known volumes of materials and their unit weights. Tables such as those in the Carnegie Pocket Companion, 1934, pp. 440 to 441, or American Institute of Steel Construction Manual, 1941, pages 339 to 340 are helpful. It is necessary to include in the load the weight of the member about to be designed. This can only be assumed from experience and arbitrary rules and corrected after a preliminary design is completed.

Live loads consist of all furniture, fixtures, safes, people, and movable contents of the structure. They must be assumed by the designer as representative of the anticipated occupancy. In some instances, such as churches, theaters and schools, the average amount of load is fairly definite and not likely to change. In others, such as commercial and loft buildings, the type of load is only roughly known and also is likely to vary considerably during the life of the structure. In general, live loads are established by the building codes and are expressed in pounds per square foot of floor area. Summaries such as those in Carnegie Pocket Companion, 1934, pages 356 to 357, and American Institute of

Steel Construction Manual, 1941, pages 335 and 344–345, show representative unit loads from different building codes. The designer can use these only as a guide and must select loads that are typical of the actual anticipated occupancy.

Building frames are frequently subjected to the stresses due to vibration, impact, earthquake, wind, centrifugal force, and the like.

Building codes establish stresses for given materials, but the designer selects the stresses he will use and so it becomes his responsibility to see that the materials specified and furnished are capable of carrying those stresses safely. Neither the loads alone nor the stresses alone determine the design. Insufficient loads with low working stresses or extra heavy loads with high stresses will produce comparable designs. Insufficient loads with extremely high working stresses will produce unsafe structures. For maximum combinations of loadings which will rarely if ever occur increased working stresses are permissible. The skilled designer strives to be safe under the condition of average loads at ordinary stresses, and also under the rare condition of the worst reasonable combination of loads, at stresses within the elastic limits by a fair margin.

Example 13–1. Design a one-way solid slab to carry a live load of 125 psf with a 1-in. granolithic finish for the situation shown in Fig. 13–1.* Specifica-

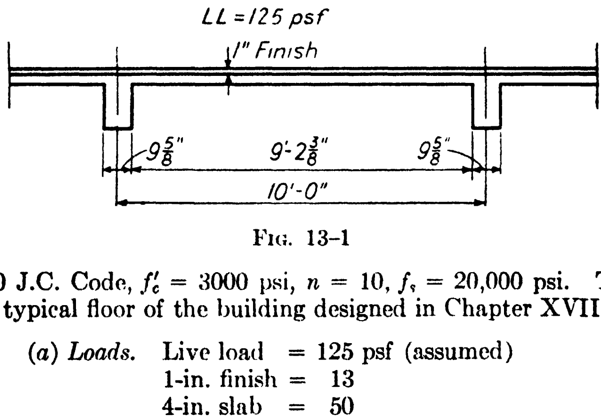


FIG. 13–1

tions: 1940 J.C. Code, $f'_c = 3000$ psi, $n = 10$, $f_s = 20{,}000$ psi. This is slab FS2 of the typical floor of the building designed in Chapter XVII.

Solution. (*a*) *Loads.*

Live load	=	125 psf (assumed)
1-in. finish	=	13
4-in. slab	=	50
w (total)	=	188 psf

In the above schedule the live load was set in the statement of the problem and represents a fair appraisal of the owner's needs. The weight of the 1-in. finish is on the basis of concrete at 150 pcf. The thickness of slab is unknown. For average load and span conditions it may be taken as between ⅜ in. and ½ in. per foot of span. The slab thickness is here assumed 4 in.

* The attention of the reader is again directed to the fact that designers invariably state problems in the form of sketches which in themselves are a long step toward a solution.

and the weight computed at 150 pcf. When the design is completed the assumption will be checked.

(*b*) *Span.* J.C. 804* recommends that the center-to-center span, L_m, be taken for moment computations of continuous beams and slabs. This is 10 feet 0 in. The reader should refer to Art. 17–5, page 362, for degree of precision in computations. For shear the clear span, L_v is used.

(*c*) *End Shear.* This is maximum on the face of the support and equals unit load times one-half the clear span (9.20 ft in this case).

$$V = \frac{188 \times 9.20}{2} = 865 \text{ lb per ft wide strip of slab}$$

(*d*) *Positive Moment.* If this were a simple span freely supported at each end, the bending moment curve would be a parabola of the form shown in Fig. 13–2*a*. If each end of the slab were held exactly horizontal while the beam deflected, the bending moment curve would be that shown in Fig. 13–2*b*. This case is intermediate between the two, being neither free ended nor completely fixed. Former codes recommended that for uniform loads on approximately equal spans the positive moment at mid-span might be taken as $+M = wL^2/12$, where L is the clearspan between faces of supports, and the bending moment would have been computed: $+M = 188 \times 9.20^2/12 = 1326$ lb-ft for $b = 12$ in. or lb-in. for $b = 1$ in.

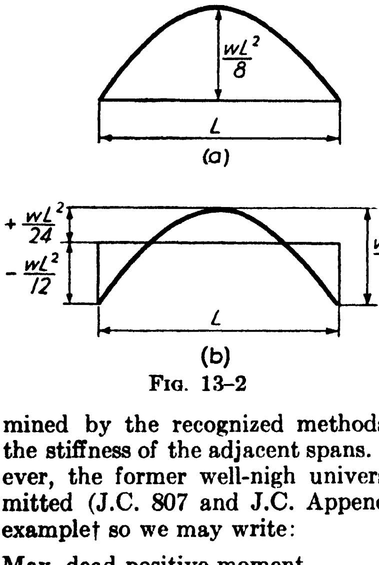


FIG. 13–2

The 1940 J.C. 801 recommends that in general bending moments be determined by the recognized methods which consider continuity, allowing for the stiffness of the adjacent spans. For equal spans and uniform loads, however, the former well-nigh universal method employing coefficients is permitted (J.C. 807 and J.C. Appendix 3). These conditions are met in this example† so we may write:

Max. dead positive moment
$= +0.046w_dL^2 = 0.046 \times 63 \times 10^2 \quad = +\ 290 \qquad +290$
Max. live positive moment
$= +0.085w_lL^2 = 0.085 \times 125 \times 10^2 \quad = \underline{+1060}$
max. (live + dead) positive moment $= +1350$ lb-in./in.‡
Max. live negative (mid-span) moment
$= -0.045w_lL^2 = -0.045 \times 125 \times 10^2 = \qquad \underline{-560}$
Max. dead + Max. live negative moment $= \qquad -270$ lb-in./in.

* The references to the J.C. Code are to the 1940 edition. The student is advised to obtain a copy of the 1940 J.C. Code and refer to it continuously while reading the following chapters.

† The reason for being able to apply the continuous beam (not frame) analysis to this slab is that, aside from some torsional rigidity of the beams, there is nothing but the stiffness of adjacent slab spans to affect the bending moments at supports.

‡ As computed the moments are either in pound-feet per 1 ft wide strip of slab or pound-inches per inch.

At mid-span the slab must be capable of taking a positive moment of 1350 lb-in. per in. width and a possible negative moment of -270 lb-in. per in. width; but on the basis of J.C. 806c the live negative moment could be reduced to -280 so that no negative moment need be provided for, as the reduced total moment is $+10$ lb-in. per in.

(*e*) *Stresses.* This discussion is purposely deferred to emphasize the fact that the external shear and moment are functions only of the load and span. Concrete testing 3000 psi in standard 6×12 in. cylinders at 28 days is readily obtainable and will be used. This gives for compression in flexure, (J.C. 878, Table 7) $f_c = 0.45f'_c = 0.45 \times 3000 = 1350$ psi. For shear as a measure of diagonal tension, $v_c = 0.02f_c = 0.02 \times 3000 = 60$ psi. Various values of v_c are set up in the Code to allow for special, i.e., end, anchorage of the tension steel with or without web reinforcement. For a simple slab shear is rarely a factor in the design and, if the actual stress is within the lowest allowable limit, the design is safe. The allowable bond on deformed bars is $u = 0.05f'_c = 0.05 \times 3000 = 150$ psi. Deformed bars of intermediate grade new billet or of rail steel will be used which will permit a stress of 20,000 psi in tension (J.C. 878). To recapitulate:

$$f_c = 1350 \text{ psi} \qquad f_s = 20{,}000 \text{ psi}$$

$$v_c = 60 \text{ psi} \qquad n = 10 \text{ (J.C. 878, Table 7)}$$

$$u = 150 \text{ psi}$$

(*f*) *Effective Depth.* The obvious procedure is to compute the theoretical depth necessary to keep the shear within the allowable limit [$v = V/bjd$; $d = V/bjv = 865/(12 \times 7/8 \times 60) = 1.4$ in.], and that required to keep the compression within its allowable limit ($M = Rbd^2$; $d = \sqrt{M/Rb} = \sqrt{1350/236} = 2.4$ in.). Many designers prefer this procedure. With a little practice, however, it is simple to assume a total slab thickness (4 in. was assumed in this case) and check the stresses, revising the depth if necessary. As slab thicknesses are ordinarily not dimensioned closer than ½ in., a second guess is seldom required. With a slab thickness of 4 in., allowing ¾ in. for fireproofing* and about ¼ in. for one-half the bar diameter, the effective depth becomes $4 - 1$ in., or 3 in.

(*g*) *Unit Shear.* Since shear rarely affects solid slab design it is sufficiently accurate to use the shear curve as for a simply supported beam. The variation in end shear produced by negative moments of different magnitudes on the ends of a span is discussed in the beam design, Art. 13–4*d*. Unit shear is computed:

$$v = \frac{V}{bjd} = \frac{865}{12 \times \frac{7}{8} \times 3} = 27 \text{ psi}$$

This is well within the 60 psi allowed, showing that the slab is amply deep. No attempt should be made to use a thinner slab until this is checked for flexure.

(*h*) *Flexure.* Next it is logical to compute:

$$R = \frac{M}{bd^2} = \frac{1350}{1 \times (3)^2} = 150 \text{ psi}$$

* J.C. 505 requires a minimum of ¾-in. protection; many fire protective codes specify 1 in. The former was used here. The latter would mean a very slight increase in steel requirements.

This value is of great significance. For working stresses of 20,000 – 1350, $n = 10$, $R = 236$ for balanced reinforcement, the condition where both steel and concrete are working at the maximum allowed stress Here R is less than 236 and the beam is underreinforced; that is, the steel is the determining factor and the concrete is not stressed to capacity. Had R been greater than 236 the reverse would have been true; the concrete would be the determining factor and the steel would not be stressed to capacity. The first condition is generally more economical. (See Fig. A–2 in the Appendix, and Chapter XXII.) If R is much greater than 236 a very expensive design results, as a good deal of steel is being used simply to lower the neutral axis a trifle to improve the stress in the concrete. In the case of a beam, if R exceeds 236 the beam should be deepened, a tee added, or double reinforcement used.

No thinner slab than 4 in. can be used here. Although for a 3½-in. slab d would be 2½ in. and R would be only 212, yet for negative moment (page 241) R would exceed 236. This shows the advantage of the method of assuming a design and checking back to see that it is within the allowable limits. Fig. A–2, shows that with $R = 150$ and $f_s = 20{,}000$, $f_c = 1000$ psi and $p = 0.0085$.

(*i*) *Steel Area.* From Fig. A–1 with $p = 0.0085$ and $n = 10$, $j = 0.89$ and, since $A_s = M/f_sjd$ we have:

$$A_s = \frac{1350}{20{,}000 \times 0.89 \times 3} = 0.0253 \text{ sq in. per in.}$$

Approximately equal results could be obtained from $A_s = pbd$. This is not particularly accurate as it is difficult to read p with much precision from the chart. Also it is neither necessary nor desirable to consult a chart at all. As long as R is less than 236 the concrete and steel stresses will be within the allowable and it is sufficiently accurate to take j at its approximate value of ⅞. As a practical point, since this computation is performed for every beam designed, time is saved by using a constant of $f_sj = 17{,}500$. This gives:

$$A_s = \frac{1350}{17{,}500 \times 3} = 0.0257 \text{ sq in. per in.}$$

This steel area is required in each inch width of slab.

(*j*) *Bars.* In selecting bars to make the required area per inch width of slab keep the following in mind:

(1) Bar sizes must be small enough to afford satisfactory bond.

(2) Bar spacing must not exceed about $2t$ to $3t$, or 12 in. in this case. This prevents a concentrated load from punching through between bars. The spacing must be small enough so that the zones of longitudinal shear around each bar overlap. The spacing must not exceed that for adequate shrinkage reinforcement.

(3) Bars under ¾ in. take increasingly large price extras, thereby adding to the cost (see Art. 2–9).

(4) Cost of placing is more nearly a function of the number of bars placed than of the weight.

For any bar size the bar area divided by the required spacing equals the area required per inch. A single setting of the slide rule makes all possible

combinations of bars and spacings available. Under the required area, A, sq in. per in. on the A scale, set the left index of the B scale. The required spacing for any bar size can then be read on the B scale directly beneath the bar area on the A scale. Thus we have:

A	0.0257	0.11 (3/8″ϕ)	0.20 (1/2″ϕ)	0.25 (1/2″ sq)	0.31 (5/8″ϕ)	A
B	1	4 + in.	7½ + in.	9½ + in.	12 + in.	B

At the lower limit discard 3/8 in. round at 4 in. c to c as bars that are too small and too close together, involving size and bending extras and additional expense for handling. Probably ½ in. round at 7½ in. c to c or ½ in. square 9½ in. c to c will meet the requirements for bond. At the upper limit discard the 5/8 in. round at 12 in. c to c. Although this combination is within the limits for spacing and meets the requirement of a few large bars to handle, it will have a high bond stress and is relatively heavy for this slab.

In checking negative moment in item (l) below it is found that bending alternate ½ in. square bars from each adjacent span provides insufficient top reinforcement over the beams. An arrangement of ½ in. round straight bars 18 in. c to c alternating with 5/8 in. round truss rods, also 18 in. c to c gives the same steel area for positive moment as ½ in. square bars 9 in. c to c but considerably more negative reinforcement and will be used.

(k) *Bond.* There are several points in a continuous beam or slab at which the bond stress should be investigated: (1) at the face of the support where the external shear is a maximum and Σo is the perimeter of the entire top bar combination; (2) where part of the top steel bends down and Σo is greatly reduced, the external shear being only moderately less; (3) at or near the point of inflection where alternate bars are bent up and Σo is the perimeter of the remaining bottom bars.

The point of inflection in a run of continuous beams of approximately equal spans is near the fifth point of the span. From Fig. 13-3a, assuming a full uniformly distributed live load, the external shear at the point of inflection is 60 per cent of the end shear. *Maximum* live shear at the point of inflection occurs with one portion of the beam loaded and the other not loaded. However, as is clear from Fig. 13-3b, which is drawn for a simply supported span, such partial loading affects the live shear relatively little and the dead shear not at all, so this refinement of partial loading is considered only in the case of unusually heavy girders. For ordinary cases of uniformly distributed live load, it is customary to take 60 per cent of the maximum end shear and compute the bond stress on those bars which are *not* bent up for negative moment. For concentrated loads the external shear at the point of inflection should be computed.

By J.C. 824, check the bond as follows:

At the support for 5/8 in. rounds, 9 in. c to c, $u = (vb/\Sigma o) = (27 \times 9/1.96) = 124$ psi.

At the bend-down point (approximately $L/4$) bond on the 5/8 in. rounds 18 in. c to c extended bars: $u = (13\frac{1}{2} \times 18/1.96) = 124$ psi.

At point of inflection for ½ in. rounds, 18 in. c to c, $u = (0.60 \times 27 \times 18/1.57) = 186$ psi. Since u exceeds the allowable value of 150 psi but is less than one and one-half times this value, i.e., 225 psi, special anchorage is required for the bottom bars (J.C. 878, Table 7). See Fig. 13-7 for the way this is obtained.

(*l*) *Negative Moment.* In a simple beam such as Fig. 13-2*a*, there is no moment at the end of the span. In a fully fixed beam such as Fig. 13-2*b*, the negative moment amounts to $wL^2/12$. Each of these occurs with full live load over the span. For a continuous slab such as we are considering,

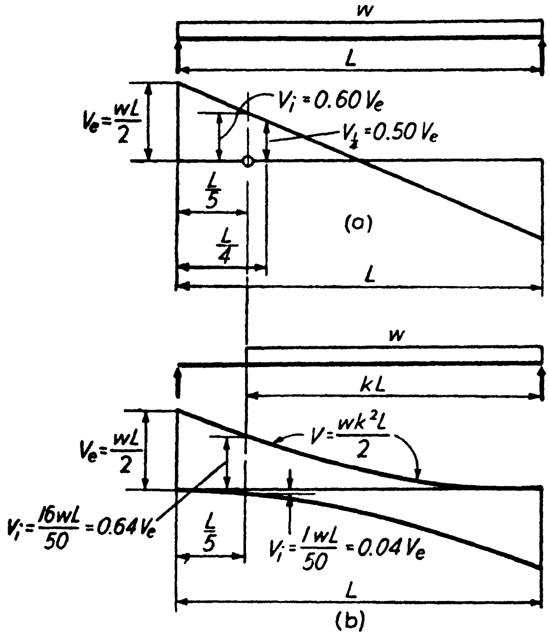


Fig. 13-3

the magnitude of the negative moment depends upon which slabs are loaded. For maximum negative moment over a support the two spans which meet at the support are loaded and alternate spans beyond these. Most codes until recently stated that for beams continuous over a series of approximately equal spans the negative moment over the support be assumed as $wL^2/12$. This occurs with a different loading from that producing maximum positive moment. Thus the moment diagrams would have been somewhat as shown in Fig. 13-4.

The 1940 J.C. Code gives coefficients in Appendix 3 for maximum moment at the center of the support:

Max. dead negative moment

$$= -0.080w_dL^2 = -0.080 \times 63 \times 10^2 = -\ 504 \text{ lb-in. per in.}$$

Max. live negative moment

$$= -0.115w_lL^2 = -0.115 \times 125 \times 10^2 = \underline{-1438}$$

Max. live + dead negative moment $= -1942$ lb-in. per in.

By J.C. 808*d*, the negative moment at the face of the support may be used in proportioning a member.* This moment is obtained by subtracting $Va/3$

* See "Continuity in Concrete Building Frames," Portland Cement Association, 1935, pp. 45–46, for a discussion of this and related problems in the design of continuous beams.

from the negative moment on the center of the support, where V is the end shear and a is the width of the support, assuming a triangular distribution of the reaction. Then, $-M = -1942 + (865/12) \times (9.63/3) = -1711$ lb-in. per in. width of slab. $R = M/bd^2 = 1711/(3)^2 = 190$, which is within the 236 allowed. No thinner slab can be used here, without compressive rein-

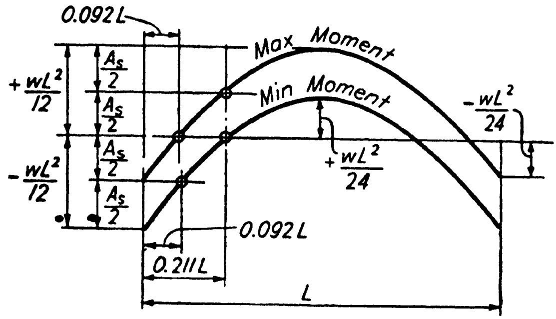


FIG. 13-4

forcement, because with $d = 2\frac{1}{2}$ in., $R = 266$ (allowing for the decrease in dead weight) which exceeds 236. As mentioned under item j above, this requires more reinforcement than the positive moment, so try bending up $\frac{5}{8}$ in. round rods at 18 in. c to c from each side, then $A_s = M/f_sjd = (1711/17500 \times 3) = 0.0326$ sq in. per in. and $\frac{5}{8}$ in. rounds 18 in. c to c from each side $= 0.0344$ sq in. per in.

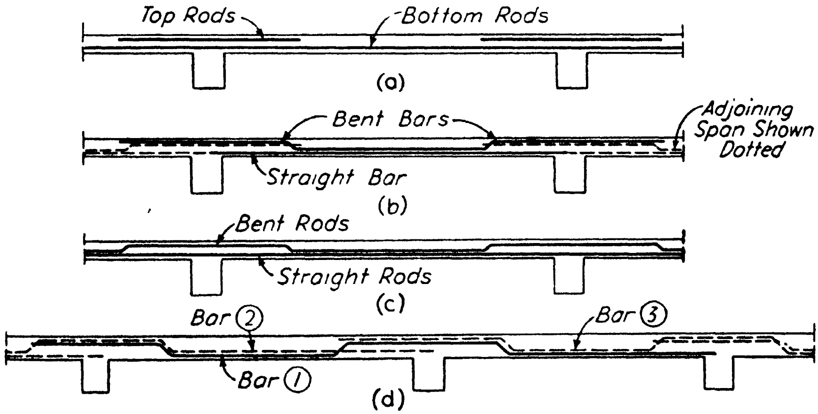


FIG. 13-5

(*m*) *Arrangement of Reinforcing.* With the positive and negative moments determined, it is possible to arrange the reinforcing steel to take care of the resulting tensions with a minimum of material. Several possible schemes are shown in Fig. 13-5. In (*a*) straight rods are provided in the bottom of the slab to furnish the required area for positive moment; separate straight bars over the supports take care of the negative moment. In (*b*) is shown an arrangement of alternate straight and bent bars. Both the straight and bent

bars are available at mid-span for positive moment. Over the support the upper ends of the bars from each of the adjoining spans are available for negative reinforcement. Scheme (*b*) has some advantage over scheme (*a*), mainly in assuring the designer that the top steel will be positioned as required. Loose top bars are easily misplaced or forgotten.

An arrangement sometimes used is shown in Fig. 13–5*c*. Since alternate bars only are bent the provision for negative reinforcement is only half that for positive reinforcement. As large negative moments will develop, the stress in this top steel will become very high, probably exceeding the yield point and causing tension cracks in the concrete parallel to and over the supporting beams.

An improvement over scheme (*c*) is shown in (*d*) which provides the required steel areas for both negative and positive moments. It gives somewhat longer bars to handle than scheme (*b*), cutting down on the number of pieces to place. Owing to the staggered arrangement of bars there is greater possibility of error in placing. Special short bars are required at the ends of runs to piece out the staggered arrangement.

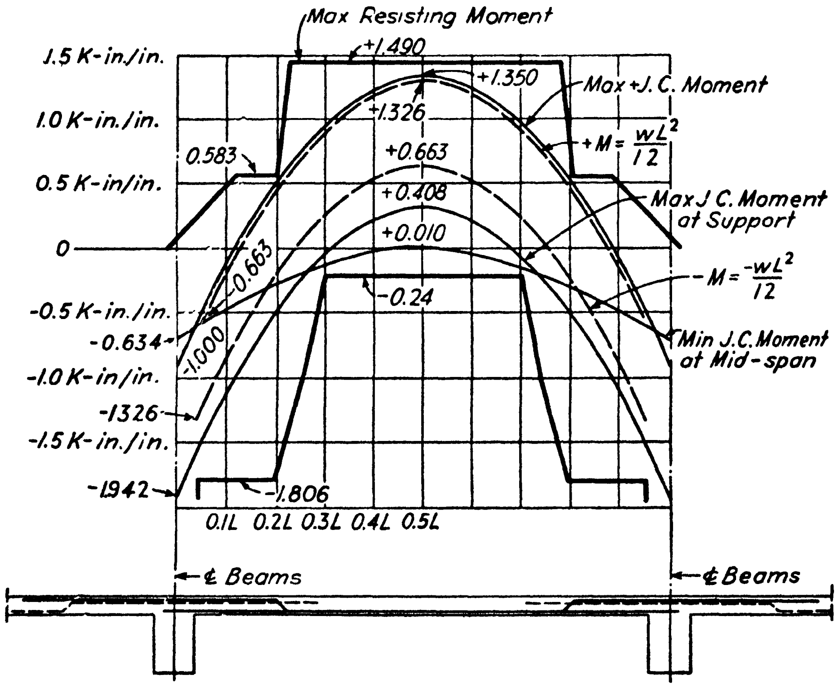


Fig. 13–6

Other combinations of staggered bent bars have been devised to serve the same purpose. Scheme (*c*) is inadequate. Any of the other schemes can be used, depending upon the designer's judgment as to relative costs and ease and certainty of placing.

(*n*) *Resisting Moment.* In order to show graphically the relation between bending moments and resisting moments, Fig. 13–6 has been prepared. The

dash lines of this diagram indicate the approximate moment curves from Fig. 13-4. The light solid lines indicate the values just computed using coefficients. The heavy lines show the moment of resistance of the reinforced slab at all points of the span. In a properly designed slab the resisting moment curves will always be above the maximum bending moment curve and below the minimum. The points of the resisting moment curve were obtained from the relation $M_s = A_s f_s jd$, using the arrangement of bars of scheme (*b*), Fig. 13-5. Resisting moments were computed for the full steel at mid-span and for alternate rods near the quarter-point. The curve drops from one value to the other as the steel is bent up. The small constant negative resisting moment at mid-span is for a concrete slab without reinforcement, f_r = 90 psi tension, as suggested in J.C. 867, for plain footings.

The reader's attention is particularly directed to this relationship between bending moment curves and resisting moment curves. Although fairly good approximations are possible from diagrams of standard moment conditions, it is an advantage for the designer to have this diagram well in mind when bending and arranging reinforcing.

(*o*) *Temperature Reinforcement.* Some reinforcement is needed crosswise of the panel at right angles to the main rods. This serves to space the steel during construction, and to resist stresses set up by shrinkage of the concrete slab in hardening or by expansion and contraction of the slab due to temperature variations. A concentrated load on a one-way slab (supported on two opposite sides) causes a bowl-like depression in the slab, thus setting up longitudinal extensions in the bottom of the slab as well as extension in the direction of the span. In building construction the amount of such steel is usually arbitrarily chosen as 0.002 to 0.0025 of the slab area.

$A_t = 0.002 \times 4 \times 12 = 0.096$ sq in. per ft   3/8 in. rounds 13 in. c to c $= 0.102$ sq in. per ft

$A_t = 0.0025 \times 4 \times 12 = 0.120$ sq in. per ft   3/8 in. rounds 11 in. c to c $= 0.120$ sq in. per ft

Use 3/8 in. rounds 12 in. c to c.

(*p*) *Office Practice.* The above explanation requires considerable space. It is customary to design a floor slab in a few lines of figures. The arrangement on page 244 is the result of many years of simplifying computations with a number of designers and detailers. It saves space in the computations, gives the designer a standardized form in which to set down his figures and gives the detailer at a glance the data he needs to prepare drawings. There is a considerable gain in precision by keeping the figures in the slide rule, saving resetting each value. In order to aid in reading this tabulation, there also is given a skeleton form with the numerical values replaced by the formulas and by the units (in parentheses) in which the results appear.

The weight of live load and floor finish are entered. Then the span length is filled in from the drawings. From this the slab thickness is estimated at 4 in. and entered in the computation of loads. Multiplying w by one-half the clear span gives the end shear. The maximum positive moment is computed from the 1940 J.C. coefficients given in Appendix 3.

The value of d is entered by deducting from the slab thickness the amount of fireproofing and half a bar diameter. From V/bjd, v is obtained; as this is a well-known relationship and the figures are immediately in front of the com-

LL (psf) Fin Slab $w = \overline{(psf)}$	$L = $(ft) $d = $(in.)	$V = wL/2$ (k per ft width) $v = \frac{V}{bjd}$ (psi) $u = \frac{V}{\Sigma ojd}$ or $\frac{vb}{\Sigma o}$ (psi) $u_1 = \frac{0.60V}{\Sigma ojd}$ or $\frac{0.60vb}{\Sigma o}$ (psi)	$M =$ moment factor $\times wL^2$ (k-in per ft or per in width) $R = \frac{M}{bd^2}$ (psi) $A_s = \frac{M}{f_sjd}$ (sq in per ft or per in width)

LL 125 psf Finish 13 } 63 4" Slab 50 } $\overline{188}$ psf Top Rods, $u = 124$ psi < 150	$L_v = 9.20$ ft $L_m = 10.0$ $d = 3$ in	$V = 0.865$ k $v = 27$ psi $u_1 = 186$ psi S.A.	$+M = 0.046 \times 63 \times 10^2 = 290$ lb in/in $0.085 \times 125 \times 10^2 = \underline{1060}$ " $+M = 1350$ " $R = 150 < 236$ $A_s = 0.0257$ sq in/in $\frac{1}{2}$"φ @ 18" Str } alt $= 0.0283$ $\frac{5}{8}$"φ @ 18" Bt } ——— $-M = -0.080 \times 63 \times 10^2 = -504$ lb in/in $-0.115 \times 125 \times 10^2 = \underline{-1438}$ " -1942 " $+\frac{Vo}{3} = \frac{865 \times 9\frac{1}{8}}{12 \times 3} = \underline{+231}$ " $-M = -1711$ " $R = 190$ $A_s = 0.0326$ sq in/in $\frac{5}{8}$"φ @ 9" c to c $= 0.0344$	FS2 4" Slab $\frac{1}{2}$"φ @ 18" c to c St $\frac{5}{8}$"φ @ 18" c to c Bt

puter it is customary to run this through on a slide rule, entering only the final result. R is obtained from M/bd^2 and entered without recording the computations. Since R is less than the 236 allowed, the concrete stress will be considerably below the maximum. Conversely, R is sufficiently close to 236 to suggest that an appreciably thinner slab would not be permissible, especially for negative moment.

A_s is obtained from $M/f_sjd = M/17{,}500d$ and is recorded as 0.0257 sq in. per inch of width. The steel selection of ½ in. round straight + ⅝ in. round bent alternate at 9 in. c to c provides 0.0283 sq in. per inch of width, which is also recorded. Negative moments are computed by the 1940 J.C. coefficients and reduced to give the moment at the face of the support, from which R and f_s are computed and found satisfactory. Bond stress is obtained from the relationship $u = vb/\Sigma 0$, and the designer simply records $u = 124$ psi in top steel and 186 psi in bottom rods. As the latter exceeds the 150 psi allowable he adds the note S.A. for special anchorage of bottom steel.

From this simple tabulation the complete computations can be quickly established. Little space is needed for the record. The detailer has available all the information he needs. The data which the designer will need for later computations of beams and columns are readily available.

(*q*) *Detail.* The best design is useless until complete instructions have been issued for construction. Chapter XXI gives some suggestions on detailing and should be read in this connection. Fig. 13–7 shows a detail of the slab just designed. The concrete outlines are first shown, usually diagrammatically, as a view of the forms just before concrete is poured. Chairs (page 495) are indicated to support and space the bars, located not over 5 ft apart. For easy handling the straight slab bars are made 20 ft 6 in. long and extending through two panels. This prevents staggering the truss rods over the supports, but they should be spaced far enough apart to insure concrete working down between. The truss rods are bent up at the quarter-points of each span at 45° and are extended to the quarter-point of the adjacent span, as

Fig. 13–6 shows that this adequately covers both maximum and minimum moment curves. The computations required special anchorage of the bottom rods. J.C. 828 mentions either hooking or embedment of the ends of the bar. Maximum stress allowed is 10,000 psi which would require ($L = f_s d/4u$)

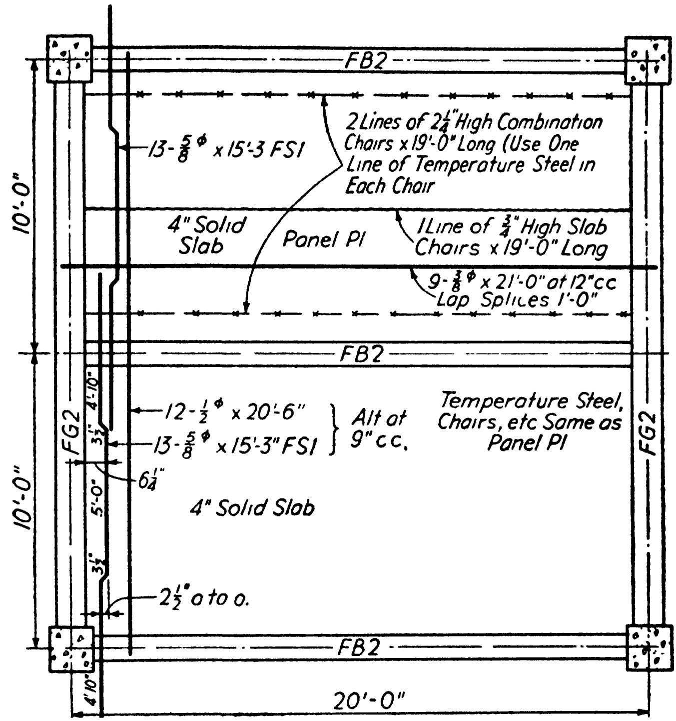


Fig. 13–7

16⅔ bar diameters (in this case 8 + in.) embedment past the face of the support or about 3 in. past the center line of the column. The number, size, length, bending diagram, and spacing of every bar, including temperature reinforcement, is clearly and compactly shown.

13–3. Rectangular Beams. The design of a simple rectangular beam parallels exactly that of a solid slab, the only difference being that the designer has freedom in choosing the breadth as well as the depth of section. This choice is discussed in Art. 13–4.

13–4. Tee-Beams. The load carried by a floor beam consists of the reaction from the tributary slab on either side, plus any load directly over the beam, including not only live load but possible partitions, plus the dead weight of the beam itself. When beams carry a relatively large area of floor it is unlikely that every square foot of tributary slab

will be fully loaded simultaneously. In a garage, for example, each strip of floor slab will at some time be called upon to support the weight of an entire axle load and so every portion of floor slab should be designed for such loading. The supporting beam, however, can only be called upon to carry the weight of the few vehicles that can simultaneously occupy the tributary slab area. Of course, in warehouse storage areas the entire floor, aside from aisle spaces, will often be entirely covered with live load. It is a problem of probabilities, but many codes allow certain live load reductions on main beams, permitting the live load to be reduced 10 per cent, 15 per cent, and often 20 per cent for floor beams that carry over 250 sq ft of floor area.

As the selection of live loads, stresses, etc., is made at the time the slab is designed little additional consideration is necessary when proportioning the supporting beam.

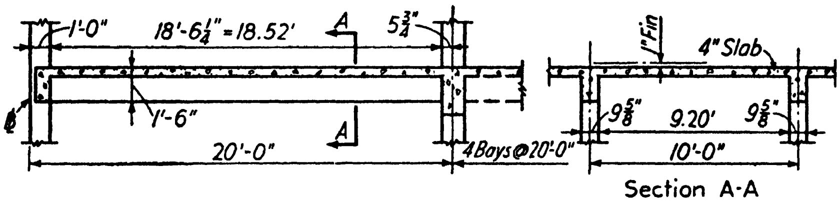


Fig. 13-8

Example 13-2. Design a reinforced concrete beam for a span of 20 ft from outside face of wall column to center of interior columns as indicated on Fig. 13-8, the remaining interior spans being 20-ft c to c columns. Specifications, 1940 J.C.: $f'_c = 3000$ psi, $f_s = 20{,}000$ psi, $n = 10$. This beam is one of the supporting beams for the slab of Ex. 13-1. It is Beam FB1 designed in Chapter XVII.

Solution: (*a*) *Unit Load.* Two methods are available for computing the load on the floor beam. One is to take the reactions of the slabs supported by the beam from the slab design. This load is computed on the clear span. It is necessary to add the weight of live load and floor finish directly over the beam to that of the beam itself. The second method is to take the weight of a panel of slab from center to center of beams and add the weight of beam stem below the underside of the slab.

	Scheme 1			*Scheme 2*	
From FS2	$2 \times 0.865 =$	1.73 klf	Slab	$10 \times 0.188 =$	1.88 klf
Beam		0.30	Stem		0.15
		2.03 klf			2.03 klf

For approximating the weight of the beam it is usually sufficiently accurate to assume between ¾ in. and 1 in. of beam depth per foot of span; the former

applies to light loadings, the latter to heavier. The breadth of beam should be about one-half the depth. Since concrete weighs 150 pcf a fairly close approximation is to take just over 1 lb weight per sq in. of cross-sectional area per lineal foot.

In this case the beam depth ranges between 15 in. and 20 in., or say 18 in. The width should be about 10 (or 9⅝) in. Beam widths are often taken as 7⅝, 9⅝, and 11⅝ in. to suit standard dressed plank soffits. Some designers record these as 7½, 9½, and 11½ in. to avoid the odd fraction.

(*b*) *Span.* For computing end shear and for determining bending moments as explained in item *e* below, the clear span from face to face of supports is used.*

(*c*) *Shear.* This follows from $wL/2 = 2.03 \times (18.52/2) = 18.8$ kips.

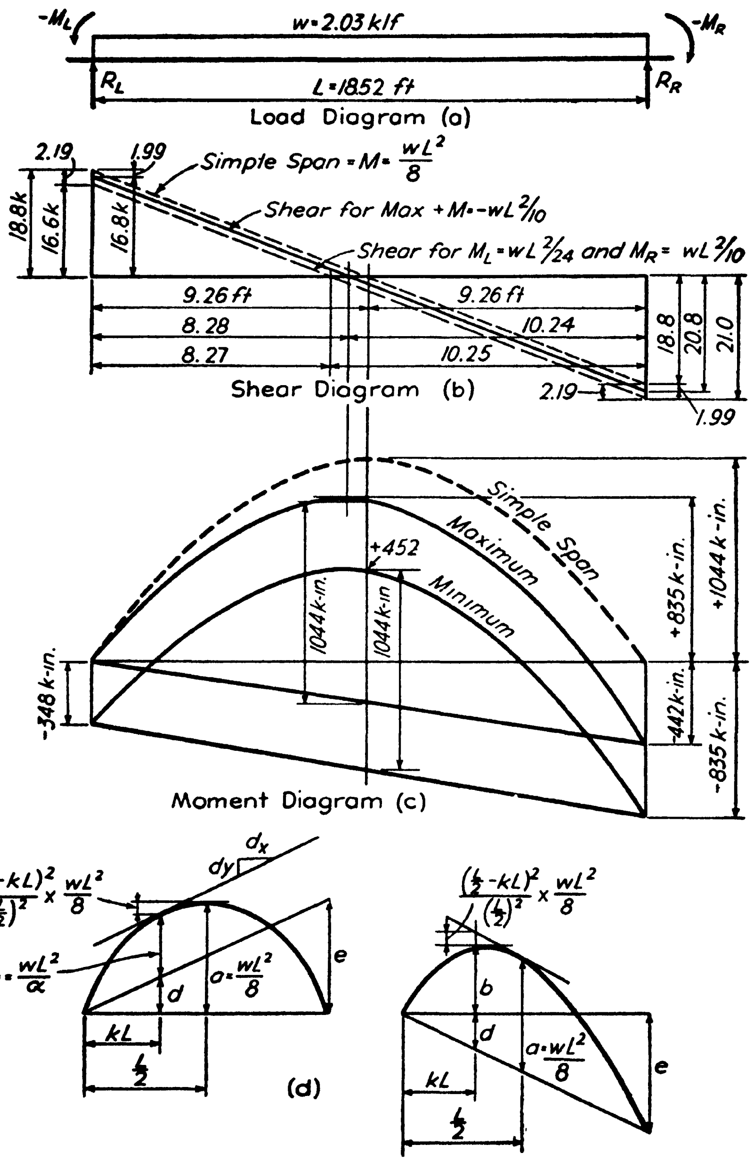
(*d*) *Shear Diagram.* Although it is not needed by the experienced designer the diagram of Fig. 13-9*b* will be helpful to the student. In dotted lines the shear diagram is drawn as for a simply supported span with $V = W/2$ at each end.

If the ends of a beam are subjected to unequal negative moments, M_L less than M_R (Fig. 13-9*a*), a couple is brought into action with forces acting down at R_L and up at R_R, each equal to $(M_R - M_L)/L$. From the curves of Fig. 13-9*c* (see discussion of moment below), if $M_L = 0$ and $M_R = -442$ k-in., then R_L is decreased and R_R increased by $442/(18.52 \times 12) = 1.99$ kips, as shown by full lines on the shear curve; if $M_L = -348$ and $M_R = -835$ k-in., the change is $487/(18.52 \times 12) = 2.19$ kips, as shown by dash lines. This illustrates the suggestion in many codes that at the inner end of semi-continuous spans the shear be taken as 15 or 20 per cent greater than the simple beam shear.

In many offices the dotted shear curve, as for a simply supported beam, is used in designing stirrups, but the curve, with maximum positive moment $wL^2/10$, shown in full lines, is more in keeping with the principles of continuous structures and is being used by careful designers.

(*e*) *Moment.* The choice of moment factors is a difficult matter. In the slab design on page 236 the effects of free ends, fully fixed ends, and partially restrained ends are illustrated. That slab was designed by arbitrary coefficients taken from Appendix 3 of the J.C. Code, which gives quite reliable results for equal spans and uniform loads on continuous *beams*. J.C., Art. 808, recommends that beams in building frames be designed by the principles of continuity, taking into account the stiffness of all the beams and columns in the *frame*. To apply these methods the relative stiffnesses of the members must be known. Except for the unusually skilled designer, it is likely that engineers will proportion flexural members by the use of approximate or arbitrary moment coefficients and check unusual conditions by more exact methods. Codes have long suggested $wL^2/8$ for simple spans, $wL^2/10$ for both positive and negative moments in semi-continuous spans, and $wL^2/12$ for positive and negative moments in fully continuous spans with a moment of $wL^2/24$ at the outer end of a semi-continuous span. In this article these coefficients will be used, and in Art. 13-4A a check by moment distribution will be made. There is a tendency of codes to reduce the positive moments

* In the more exact methods explained in Chapter XII and Art. 13-4A, the span is taken from center to center of supporting columns but corrections are made in the case of negative moments to obtain the value at the face of support.



-M_L
w=2.03 klf
-M_R
R_L
L=18.52 ft
R_R
Load Diagram (a)
2.19
1.99
Simple Span = M = wL²/8
18.8k
16.6k
16.8k
Shear for Max +M = -wL²/10
Shear for M_L = wL²/24 and M_R = wL²/10
9.26 ft
9.26 ft
8.28
10.24
8.27
10.25
18.8
20.8
21.0
Shear Diagram (b)
2.19
1.99
Simple Span
Maximum
Minimum
+452
1044 k-in.
1044 k-in
+835 k-in.
+1044 k-in.
-348 k-in.
-442 k-in.
-835 k-in.
Moment Diagram (c)
dx
dy
(L/2-kL)²/(L/2)² x wL²/8
b = wL²/α
d
a = wL²/8
e
kL
L/2
(d)
(L/2-kL)²/(L/2)² x wL²/8
b
d
a = wL²/8
e
kL
L/2

FIG. 13-9

and vary the negative moments to agree more closely with the results of more exact analyses.*

For the benefit of the student a set of approximate moment curves complying with these recommendations is shown in Fig. 13-9c. The dotted line indicates the moment curve for this load and span of a simply supported beam. The mid-ordinate, directly below the zero point of the dotted shear curve, is $wL^2/8 = 2.03 \times (18.52)^2 \times 1.5 = 1044$ k-in. From simple statics the ordinate of any moment curve representing any degrees whatever of end restraint measured vertically from a line connecting the two end moments must equal the simple beam moment. On this basis the maximum positive moment curve in full line is drawn through the free end and tangent to a horizontal line $wL^2/10 = 835$ k-in. above the axis, obtaining a negative moment of -442 k-in. on the continuous end.† The minimum positive moment curve is drawn, using $-wL^2/24 = -348$ k-in. on the left end and $-wL^2/10 = -835$ k-in. on the right end.

(*f*) *Stresses.* As the same quality of materials will be used in the beams as in the slab, we use the working stresses recorded on page 237. For the slab of the previous example the limiting stress in shear (J.C. 878, Table 7) was set at 60 psi, without special anchorage of longitudinal steel, 90 psi with special anchorage, no diagonal tension reinforcement being employed. It is economical and advisable to use diagonal tension reinforcement in beams and the Code limits the shear to $0.06f'_c$, or 180 psi in this case, without special anchorage of the main steel. This value could be increased with special anchorage but such construction would not be economical as a carefully detailed design must be made in these cases (J.C. 826, 827, 828, 829) and it is common to require particularly rigid inspection of such beams during construction.

(*g*) *Beam Size.* The selection of proper stem size is not only a matter of computation but of judgment and experience as well. It is a simple matter to solve the formula $v = V/bjd$ by turning it into the form $bd = V/jv$ and obtain a minimum area of stem that will carry the external shear within the specified diagonal tension allowances. In choosing b and d to make this area the designer must have in mind:

* The 1941 A.C.I. Code, Art. 701c, suggests for this case $+M = wL^2/14$ and $-M_R = wL^2/10$. The former is a reduction of 28 per cent from the $wL^2/10$ used here, whereas the latter is in exact agreement. Comparison should be made with Art. 13–4A, which gives a more precise analysis.

† A formula that is convenient in plotting positive moment curves may be derived from Fig. 13–9*d*. Determine kL, the abscissa to a point on any moment parabola whose ordinate is to be a predetermined amount, $b = wL^2/\alpha$, the mid-ordinate being $a = wL^2/8$, by equating the slope d/kL of the secant, to the slope of the tangent dy/dx at the point, which equals the shear at the point:

$$\frac{\dfrac{wL^2}{8}\left[1 - \dfrac{\left(\dfrac{L}{2} - kL\right)^2}{\left(\dfrac{L}{2}\right)^2}\right] - \dfrac{wL^2}{\alpha}}{kL} \qquad \left(\frac{wL}{2} - wkL\right)$$

and solving, $k = \sqrt{2/\alpha}$, and the negative moment $e = (0.5 - k)wL^2$. For $b = wL^2/10$, $k = \sqrt{0.20} = 0.447$. For this problem $e = 0.053 \times 2.03 \times 18.52^2 \times 12 = 442$ k-in.

(1) The depth d is often best made between two and three times b.

(2) The width b must be sufficient to cover the bars at proper spacings and afford fireproofing each side.

(3) The depth and width must be such as to keep $R = M/bd^2$ for negative moment within the allowable value for a double reinforced beam since, for economy and simplicity in form work, a prismatic beam is preferable to one with haunches.

(4) Frequently a better balanced and more economical design results if all the beams in the building are kept the same size even though in some cases more area is provided than the minimum bd computed.

(5) The width b should suit standard plank form sizes.

(6) The depth and the width must be selected to suit any clearances established by conditions of the building.

(7) Beams should preferably be shallower than the girders into which they frame so that the layers of reinforcing steel at the intersection will lie in different planes.

Instead of finding bd as above suggested it will be found simple to solve V/bj for vd as follows: Assume a width such as $7\frac{5}{8}$, $9\frac{5}{8}$, or $11\frac{5}{8}$ in.; divide V by bj and opposite various trial depths read off at one setting of the slide rule the corresponding intensities of shear. In this case $9\frac{5}{8}$ in. seems a satisfactory width; dividing 21.0 by (9.63×0.875) brings the right index of the C scale over vd on the D scale, then opposite various depths we can read shears as follows:

v	208	178	156	138 psi	on C1 scale
d	12	14	16	18 in.	on D scale

Obviously 12 in. is too shallow, because it brings an excessive value of v and 18 in. is rather deep. The effective depth should be between 14 and 16 in. With a view to keeping one beam size throughout the building take a total depth of 18 in.

(*h*) *Effective Depth.* J.C. 505 requires $1\frac{1}{2}$ in. fireproofing. The total depth is $1\frac{1}{2}$ in. plus one-half the bar size of about 1 in., or 2 in. more than the effective depth. For ordinary beams with one layer of steel it is sufficiently accurate to take the effective depth 2 in. less than the total depth. For double layers of steel increase the deduction to 3 in. Under item t below it appears that the steel must be placed in two layers, so take $d = (18 - 3) = 15$ in.

Although it is necessary to use this value of d in designing for positive bending moment it is still permissible to take d as 16 in. when computing web shear and bond because the top layer of steel will be bent up near the point of inflection. This refinement is often used when designing heavy girders with several layers of steel, but it is frequently not taken into account with ordinary light beams because of the added chances for errors and the very small saving in material.

(*i*) *Web Shear.* With a tentative stem size selected, compute the intensity of web shear from the relation $v = V/bjd$. This gives $v = 21{,}000/(9.63 \times \frac{7}{8} \times 15) = 166$ psi. This is higher than the 60 psi allowed by the Code (J.C. 878, Table 7) without stirrups or special anchorage, but less than the 180 psi allowed for beams with properly designed web reinforcement but without

special anchorage of the longitudinal steel. It appears that a small amount of web reinforcement will be all that is required and so we drop consideration of the stem temporarily and proceed with the moment computations.

(*j*) *Tee.* With an unlimited expanse of slab the maximum width of tee must not exceed (J.C. 804*d*):

> ¼ span, or $18.52 \times 12/4 = 55.6$ in.
> Stem plus $8t$ each side, or $9\frac{5}{8} + (2 \times 8 \times 4) = 73.6$ in.
> Half the distance toward adjacent beam each side, or 120 in.

The first value is the one to use.

(*k*) *Compression.* With t determined and t/d computed as 4 in./15 in. = 0.27, Table A-1 gives the maximum allowable R as 215. As t/d is considerably less than 0.403, R is correspondingly less than 236. The actual R is computed from M/bd^2 as $835{,}000/(55.6 \times 15 \times 15) = 66.7$ psi. This value is only a fraction of the allowable, so f_c is correspondingly low.

(*l*) *Steel Area.* $A_s = (M/f_s jd) = 835/(17.5 \times 15) = 3.18$ sq in.

The beam is underreinforced and j is greater than ⅞; (actually, the neutral axis lies within the flange, $p = 0.0038$ and $j = 0.917$) so this is on the safe side. The possible small saving in tension steel effected by using an accurate value of j is frequently not taken advantage of.

(*m*) *Bars.* In selecting bars several points must be in mind:

(1) The bars chosen should be divisible into two approximately equal groups to permit bending up one-half the steel, maximum negative and positive moments being the same.

(2) The size used should be small enough to keep the bond stress workable.

(3) The larger the bars the narrower the beam stem which covers them.

(4) The larger the bars the fewer pieces to detail, handle, and place.

(5) The combination of bars should be easy to detail, order, identify, and install.

The required area can be made with four bars, each having an area of about 0.80 sq in. suggesting accordingly two 1 in. round straight and two 1 in. round bent, or 3.16 sq in. Note the way this is recorded in the abbreviated computation, page 258. The first term shows the bars carried through in the bottom; the second, those trussed to resist negative moment and to help with web reinforcement.

(*n*) *Bond.* At the point of inflection (approximately $L/5$ from the right end) bond is checked on the straight bottom rods only: $u_s = (0.6V/\Sigma ojd) = (0.6 \times 21{,}000)/(6.28 \times 0.875 \times 15) = 152$ psi. This is just over the 150 psi allowed without special anchorage. Many offices allow an overrun of not more than perhaps 2% or 3% in stress. It is plain that the effectiveness of the bent rods as main tension reinforcement does not cease abruptly at the point of bending up. The Joint Committee recognizes this by providing (824*b*) that portions of bent bars within one-third of the depth of the beam ($d/3$) from the main steel may be counted on for bond resistance. Since the calculation immediately above shows excessive shear, a second computation is made at once a distance of $d/3 = 5$ in. nearer the support, the angle of bend being 45°, giving $u = 145$ psi.

At the left end $u = 16{,}800/(6.28 \times 0.875 \times 15) = 204$ psi. As this is greater than 150 psi and less than 225 psi, special anchorage is required.

Bond at the support must be checked in conjunction with the adjoining span after negative moment is investigated; see item *s* below.

(*o*) *Web Reinforcement.* The function of the stirrups is to carry at 16,000 psi (J.C. 878, Table 7), the excess shear above the capacity of the web concrete. Although the bent-up tension steel helps, only the center three-fourths of the sloping part is available (J.C. 819*c*); this might displace one, or at most two, stirrups. Except for large girders detailed completely on the design drawings, as on Fig. 14-3*b*, no allowance is made for bent-up bars.

Web concrete can be stressed $0.02f'_c$ without special anchorage of bottom bars or $0.03f'_c$ with special anchorage; the latter value will be used here and the bottom bars must then have special anchorage at each end.

To obtain A_v, compute a (Fig. 13-10), the distance from face of support to point at which the web concrete is capable of taking the entire external shear, respectively 32 and 57 in. in this case. This disregards the small shear near the center due to live load on part of the span only* but as no allowance is made for the bent-up longitudinal bars the extra computations were not deemed necessary. Compute the volume of the excess shear prisms, in this case triangular ones, and divide these volumes by 16,000 psi to obtain A_v. These required areas can be provided by ¼ in. round U-shaped stirrups, using 5 at the left and 14 at the right end, or by ⅜ in. round stirrups, 2 at the left and 6 at the right end.

(*p*) *Stirrups.* Stirrup diameters must be small enough to insure anchorage. J.C. 878, Table 7, permits maximum bond stress in stirrups of deformed bars of $0.05\,f'_c$ or 150 psi, on the area embedded in half the effective beam depth; for plain rods the bond stress is reduced to $0.04\,f'_c$, or 120 psi. J.C. 830 and 828 allow a standard hook to develop 10,000 psi of the stirrup stress, leaving 6000 psi of the pull to be developed by bond in the distance from mid-depth to the center of the hook.

If ¼ in. round stirrups are used at each end they will doubtless be of plain rod as ¼-in. deformed bars are somewhat difficult to obtain. It is not practicable to depend on embedment alone for anchorage, because, equating stress to bond, taking length of embedment as (7.5-1.25) in. (see Fig. 13-10*c*): $16{,}000 \times (\pi d^2/4) = 6.25 \times 120 \times \pi d$, giving maximum possible $d = 0.19$ in. or less than ¼ in. Good practice would require the stirrup to be hooked in any case. A ¼ in. round stirrup requires in addition to the hook an embedment of $(f_s A_v/u\Sigma o) = (6000 \times 0.05)/(120 \times 0.79) = 3.2$ in. This is less than the distance available above the center depth of the beam to the center point of the hook, about 5 in., as may be seen by sketching the beam cross section and the hook detail, as on Fig. 13-10c.

If ⅜ in. round *deformed* stirrups are considered, the bond above the center depth, omitting hooks for an embedment of $7\frac{1}{2} - 1\frac{1}{4} = 6\frac{1}{4}$ in. (see Fig. 13-10*c*) is $u = (f_v A/L\Sigma o) = (16{,}000 \times 0.11)/(6\frac{1}{4} \times 1.18) = 240$ psi as compared with 150 psi allowed. Thus the ⅜-in. stirrups must be hooked for anchorage and the additional embedment above center depth must be $[(6000 \times 0.11)/(150 \times 1.18)] = 3\frac{3}{4}$ in., which is less than the space available, as can be seen on the sketch suggested above. So ⅜ in. round stirrups

* Taking live shear at mid-span into account can be accomplished by basing the slope of the shear lines in Fig. 13-10 on $w_d + \frac{3}{4}w_l$ instead of $w_d + w_l$, having in mind that for a uniformly loaded simple beam live shear at the center is one-quarter of live end shear (Fig. 13-3). Fig. 13-10*b* was worked out for comparison only.

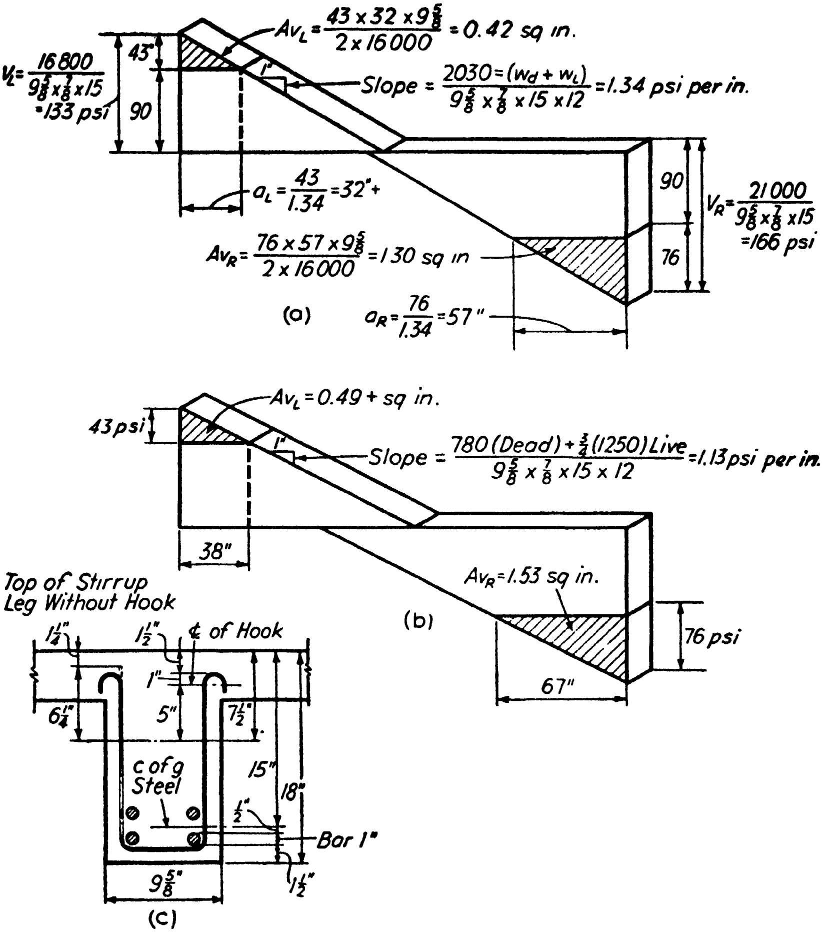


Fig. 13–10

could be properly developed, but it might be necessary to add stirrups to keep the spacing less than $d/2$ as required by J.C. 819*c*.

(*q*) *Stirrup Spacing.* In the computations for this beam, FB1 on page 379, the spacing for the stirrups is worked out with the use of the coefficients tabulated in Table A–2. This involved a single setting of the slide rule, as noted in the instructions attached to the table.

The slide-rule method, described in Art. 7–15, page 102, is illustrated for this case in Fig. 13–11, (*a*) being for ¼-in. and (*b*) for ⅜-in. stirrups. The problem is to divide the triangles representing the total pull on the stirrups into equal areas as indicated. On the base lines of the triangles are given the slide-rule readings, $L/\sqrt{N}$ multiplied successively by $\sqrt{N}$, $\sqrt{N-1}$, etc.

The differences of these readings are the bases of the several equal areas and on line A are shown the stirrup spacings with a stirrup at approximately the centroid of each area. Since stirrups are not to be placed farther apart than the half-depth (J.C., 819C), additional stirrups should be used as indicated on line B. This also offsets the neglect of possible live shear at the center of the span. It is evident that the spacing of lines A and B can be entered directly by

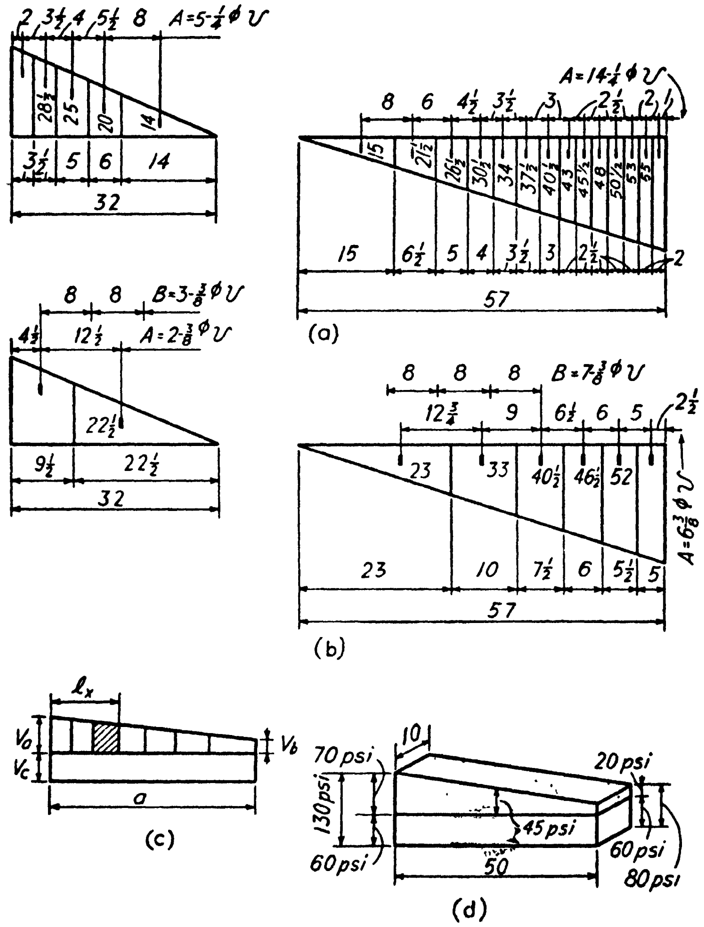


FIG. 13–11

the designer from inspection of his slide rule with a single setting, only the runner being moved. It is sufficiently accurate to read directly the differences between the values, $L - L\sqrt{(N - \frac{1}{2})/N} - L\sqrt{(N - 1\frac{1}{2})/N} - L\sqrt{(N - 2\frac{1}{2})/N}$, etc., and use these for stirrup spacings, giving for the left end of Fig. 13–11a, for

example: $32 - 32\sqrt{4.5/5} - 32\sqrt{3.5/5} - 32\sqrt{2.5/5} - 32\sqrt{1.5/5} - 32\sqrt{0.5/5}$ = 2, 3½, 4, 5, 7½ in.*

(*r*) *Negative Moment.* At the free end the negative moment of $wL^2/24$ is easily taken care of by bending up two of the 1 in. rounds and hooking them into the column, since required $A_s = (M/f_sjd) = 348/(17.5 \times 15) = 1.33$ sq in., and two 1 in. rounds furnish 1.58 sq in. At the junction of FB1 with

* This method of spacing is applicable whenever the excess shear prism is triangular, as in the majority of cases. With a concentrated load the prism is frequently trapezoidal, and requires the division of a trapezoid into N equal areas; this can be done in several ways. First, charts are available as in A. R. Lord's Handbook of Reinforced Concrete Design, of the A.C.I., 1928, pp. 109–111.

A second method is to compute the value of a stirrup, f_vA_v, and divide by the excess shear per lineal inch of beam at a point assumed as the centroid of the volume carried by the xth stirrup, $(v_x - v_c)b$, to obtain a spacing x. As these mean excess values are approximated, some variation results and an adjustment is made so that the sum of the x dimensions equals the base of the prism.

Third, a direct algebraic solution is possible by the use of the formula (see Fig. 13–11*c*): $l_x = \frac{a}{v_a - v_b}\left[v_a - \sqrt{\frac{(n - x)v_a^2 - xv_b^2}{n}}\right]$, but owing to the combination of figures under the radical, the computations are rather lengthy and owing to the taking of a difference in the bracket several significant figures must be carried. It is well to understand all three methods so that the one most suitable for the problem on hand may be used.

To compare these, let it be required to determine stirrup spacings for the condition shown in Fig. 13–11*d*. $N = (10 \times 45 \times 50)/(16{,}000 \times 0.22)$ = seven ⅜ in. round U's. By the first method, $v_b/v_a = 20/70 = 0.28$, and $a = 50$ in.; so read from the chart 2, 7, 12, 18, 24, 32, 42, and, by taking differences, obtain 2, 5, 5, 6, 6, 8, 10. By the second method compute the value of a stirrup as $0.22 \times 16{,}000 = 3520$ and divide by the shears at points approximated as follows: $s_2 = 3520/680 = 5.2$; $s_7 = 3520/630 = 5.6$; $s_{12} = 3520/580 = 6.1$; $s_{18} = 3520/520 = 6.8$; $s_{24} = 3520/460 = 7.7$; $s_{32} = 3520/380 = 9.3$ and $s_{42} = 3520/280 = 12.5$. Placing a stirrup at the center of each of these spaces gives 2½, 5, 6, 6½, 7, 8, 11. By the third method, compute $a/(v_a - v_b) = 50/(70 - 20) = 1$ for this case. The remainder of the computations are best arranged in a schedule using $l_x = 1\left[70 - \sqrt{\frac{(7 - x)70^2 - x20^2}{7}}\right]$:

x	$(7 - x)70^2$	$x20^2$	$\sqrt{\frac{(7 - x)70^2 - x20^2}{7}}$	l_x	Stirrup Spacing
1	29,400	400	65.4	4.6	2.3
2	24,500	800	60.1	9.9	5.3
3	19,600	1,200	54.6	15.4	5.5
4	14,700	1,600	48.3	21.7	6.3
5	9,800	2,000	41.0	29.0	7.3
6	4,900	2,400	32.3	37.7	7.8
7				50.0	12.3

It should be noted that meticulous spacing of stirrups in the fashion suggested by this footnote is insisted on only by those unfamiliar with the approximate nature of the theory of stirrup action.

its companion beam FB2 the negative moment of $wL^2/10 = 835$ k-in. gives $R = M/bd^2 = (835{,}000/9\frac{5}{8} \times 15^2) = 386$, which exceeds the 236 allowed, indicating the need of compressive reinforcement (or a haunch). Refer to Fig. A–10 and read for $R = 386$ ($d'/d = \frac{2}{15} = 0.133$); $p = 0.222$; and $p' = 0.019$; so $A_s = 3.21$ and $p'bd = 2.74$ sq in., the latter value reducing* to $\frac{9}{16}$ of $2.74 = 1.54$ sq in., by J.C. 804c. The two 1 in. round rods bent up from this beam leave $3.21 - 1.58 = 1.63$ sq in. minimum to be supplied by bars bent up from FB2. The truss rods in FB2 will probably be two 1 in. rounds = 1.58 sq in., which is close enough. The bottom area of 1.54 sq in. is more than adequately supplied by two 1 in. round straight bars in this beam and the two $\frac{7}{8}$ in. rounds of FB2 = 2.78 sq in.

(*s*) *Bond at Support.* (Read item *k*, page 239.) Assuming two 1 in. round truss rods in FB1 and two 1 in. rounds from FB2, the bond at the face of the support is computed $u = V/\Sigma ojd = 21{,}000/(4 \times 3.14 \times \frac{7}{8} \times 15) = 127$ psi, which is less than the 150 psi allowed. The truss rods in FB1 bend down about $3\frac{1}{2}$ ft from the face of the support (Fig. 13–13) where the shear is about $21{,}000 - 2030 \times 3\frac{1}{2} = 13{,}900$ lb. Here the bond on the two rods from FB2 is $13{,}900/(2 \times 3.14 \times \frac{7}{8} \times 15) = 168$ psi, which is greater than the 150 psi allowed. Although J.C. 824 permits figuring bond at $d/3 = 5$ in. further from the support, the bond stress at that point would still exceed 150 psi and so require special anchorage of the rods in the top of the beam.

The bottom bars are in compression and must be developed. In designing stirrups special anchorage was assumed. J.C. 828 may be taken as requiring an embedment of $L = fd/4u = 10{,}000/(4 \times 150) = 16\frac{2}{3}$ bar diameters (17 in. for this case) beyond the near face of the support. The bars need only extend into FB2 as additional compressive reinforcement for that beam until its two $\frac{7}{8}$ in. round bottom bars afford sufficient compressive reinforcement. Taking the point of inflection as $L/5 = 44\frac{1}{2}$ in. from the face of the support, the two $\frac{7}{8}$ in. rounds are approximately sufficient at $(1.54 - 1.20/1.54) \times 44\frac{1}{2} = 10$ in. from the face of support, and, as the two 1 in. rounds are working at an extremely low fiber stress, little or no additional embedment is necessary to develop them. Let the two 1 in. rounds extend 13 in. past the far face of the support to give a bar length in 3-in. multiples.

(*t*) *Arrangement of Reinforcing.* The amount of positive and negative steel having been determined, the placing and bending of bars remain to be decided upon. J.C. 504 requires that bars be spaced $2\frac{1}{2}$ diameters on centers for round rods and 3 for square. As previously explained, a $9\frac{5}{8}$-in. stem width will not accommodate four 1 in. round rods in one layer, as 3 spaces of $2\frac{1}{2}$ in., plus 2 half bars of 1 in., plus 2 stirrup legs of $\frac{3}{8}$ in. each and 2 fireproofings of 1 in. each, give a minimum stem width of $11\frac{1}{4}$ in. Hence the bars will be placed in two layers, the straight bars in the bottom and the trussed bars in the upper layer. Some increase of effective depth could be made by placing 3 bars in the bottom layer and 1 above, giving $d = 18 - 2 - (\frac{1}{4} \times 2) = 15.5$ in., but no change in the main reinforcement could be made.

For bending up the truss rods Fig. 13–12 shows in dashed lines the maximum and minimum bending moment curves, assuming the arbitrary coefficients of

* The value $\frac{9}{16}$ is here used instead of $\frac{1}{2}$ because otherwise f_s' would exceed 16,000 psi. If k is taken as 0.4 then $f_s' = [(0.4 \times 15 - 2)/(0.4 \times 15)]\ 1350 \times 10 = 9000$ psi, so $A_s' = \frac{9}{16}$ of $p'bd$. For this case $k = 0.405$ (Fig. A–9), but the above approximation is amply close.

$-WL/24$, $+WL/10$, and $-WL/10$. In dotted lines it indicates the values obtained from the study in Art. 13-4A. In heavy lines are shown the resisting moments based on the reinforcing steel just selected. The steel is bent up as soon as it can be spared from the positive moment zone, to afford negative

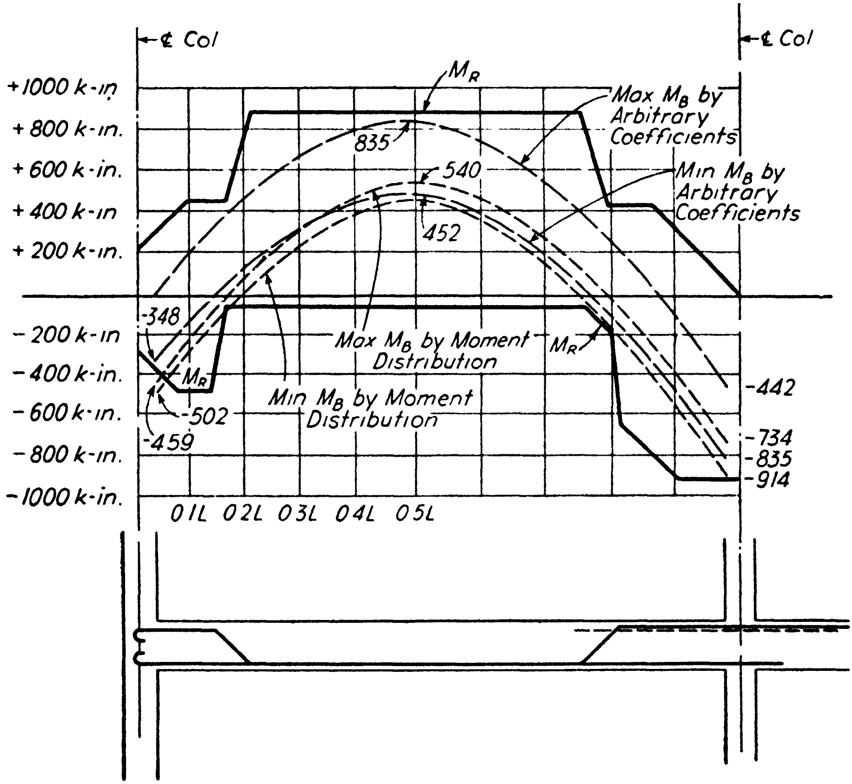


FIG. 13-12

reinforcement. The small negative resisting moment at mid-span is that for a rectangular unreinforced section with $f_t = 90$ psi. A reading of Ex. 13-1(n) should make the balance of this diagram clear.

(u) *Office Practice.* The form shown below records every step necessary in the design of this beam with numerical values replaced by the formulas and, in parentheses, the units in which the results appear.

Load (w-psf) x S=(plf)	L = (ft)	$W = wL$ (k)	M = Moment Factor x WL (k-in)	Beam Number
Beam = ___	bd = (in. x in)	$v = \frac{V}{bjd}$ (psi)	$R_{actual} = \frac{M}{bd^2}$ (psi)	
Total (plf)	T = (in)	$a = \frac{v-v_c}{v}\,\frac{L}{2}$ or $\frac{v-v_c}{\Delta v}$ (in.)	$A_s = \frac{M}{f_s jd}$ (sq in)	
$R_{allowable}$ = (psi)		$A_v = \frac{ba(v-v_c)}{2 \times 16000}$ (sq in.)		
			Rods =	
	Stirrups =		$u = \frac{V}{\Sigma o jd}$ or $\frac{vb}{\Sigma o}$ (psi)	
			$u_1 = \frac{0.60 vb}{\Sigma o}$	

The computations for this example can be filled in as shown below, the detailed operations being explained in the text following.

10 ft @ 188 = 1 88 klf Stem = 0 15 w = 2 03 klf	L = 18 52 ft bd = 9⅝ x 18 in. T = 55 6 in d = 15 $\frac{t}{d}$ = 0 27 R_{allow} = 215 psi	W = 37 6	$\pm \frac{Wl}{10}$ = 835 k-in $-\frac{Wl}{24}$ = - 348 k-in. R_{actual} = 66 7 < 215 A_s = 3 18 sq in 2-1 in rd (s), 2-1 in rd (b) } = 3 16	FB1 
V_L = 16 8 k v_L = 133 psi a_L = 32 in A_{vL} = 0 42 sq in $Stirrups_L$ = 3 2-⅜ rd @ 4½, 12, 8, 8	V_R = 21 0 k v_R = 166 psi a_R = 57 in A_{vR} = 1 30 sq in $Stirrups_R$ = 7 6-⅜ rd @ 2½, 5, 6 6½, 9, 12¾, 8, 8, 8		u_L = 204 S A u_1 = 152 psi $u_{(1-d/3)}$ = 148 psi u_R = 119 psi Bend Down u_R = 148 psi	2-1"ϕ s (S A) 2-1"ϕ b 3-⅜"ϕ U 4½, 8, 8 left 7-⅜"ϕ U 2½, 5, 6, 6½, 8, 8, 8 right

The load on the beam was obtained as 10 ft of slab at 188 psf plus the weight of the beam stem. The span is obtained by deducting from the 20-ft panel the assumed thickness of exterior column and one-half the interior column. Total load W is obtained by multiplying the unit load by the span. Holding this value in the slide rule, the bending moment $WL/10$ is obtained by multiplying again by the span and by 1.2. The factor 1.2 results from multiplying by 12 to change kip-feet to kip-inches and multiplying by the moment factor $1/10$. For fully continuous spans with $M = WL/12$ this factor is 1.0, and for simple spans with $M = WL/8$ the factor becomes $12/8 = 1.5$.

The beam size is assumed and recorded. The external shears are here obtained from $W/2 \pm (M_R - M_L)/L$. Many engineers design stirrups and check bond by using the shear curve of a simple span, and that practice is illustrated in the design of this same beam on page 379. The unit shears are obtained from $v = V/bjd$. R results from M/bd^2; as this is well within the allowable value the concrete compression is low. The steel area, A_s, comes from $M/f_s jd$. The values "a" come from $(v - v_c)/\Delta v$. The total steel area required by the stirrups at either end of the beam, A_v, is obtained as the volume of the excess shear prism divided by 16,000.

Stirrups are chosen as indicated to provide at least this area. The spacing of stirrups is obtained from Table A-2. The tension rods are selected to provide area at least equivalent to A_s. Note that s indicates straight rods and b bent rods.

Since all the computations are from simple standard formulas it is unnecessary to record detailed computations, as they can be reestablished immediately from the data shown. It is advisable that the beginner record units meticulously, as above. This is not needed as one gains in proficiency.

The computation of negative moment cannot be made at the same time as the positive moment because it involves the junction of two beams. Consequently negative moment computations would be set up as follows:

$\frac{-WL}{10}$ = 835 k-in. d = 15"	R = 386 psi	$\frac{d'}{d}$ = 0.133 u = 127 psi. u_1 = 168 psi S.A	p' = 0.019 p = 0.222	A'_s = ⅝ x 2 74 = 1 54 sq in. A_s = 3.21 sq in.	2-1 in. rd (s) + 2-⅞ in. rd (s) = 2 76 sq in. 2-1 in. rd (b) + 2-1 in. rd. (b) = 3 16 sq in.	FB1 – FB2

(*v*) ***Detail.*** As in the case of the slab, the design is valueless until complete information is given the field, preferably in the form of a drawing. Fig. 13–13 gives a detail for the construction of this beam. It must show the outlines of the concrete for building forms and computing the yardage of concrete; the number, size, length, and spacing of all reinforcing bars and their bending

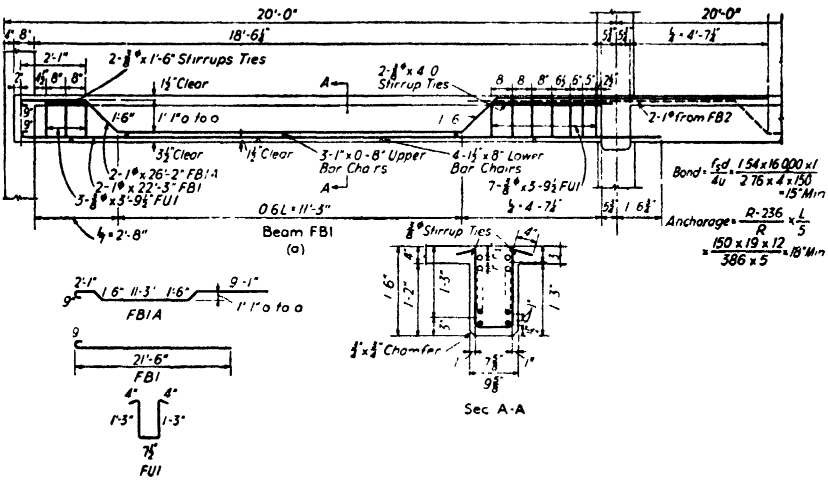
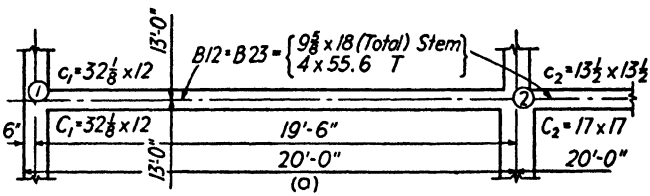


Fig. 13–13

diagrams; chairs for supporting and spacing the bars in the forms, and tie rods for holding the stirrups in place. The bending points and ends of truss bars are taken arbitrarily at $L/7$ and $L/4$ as noted. Fig. 13–12 shows how this covers the maximum and minimum moment curves for ordinary beams. Detailing in schedules instead of by individual drawings is illustrated in Chapter XXI.

13–4A. Check Tee-Beam by Moment Distribution. The 1940 J.C. Code, in Arts. 801 and 803, recommends that the bending moments be determined by principles of continuity, allowing for the stiffness of the columns and beams of the adjacent spans, and in Appendix 2 an abbreviated procedure for estimating these moments is given. It is based on suggestions in Continuous Frames for Reinforced Concrete, by Hardy Cross and N. D. Morgan (John Wiley & Sons, Inc., 1932). An exact analysis becomes very involved and for ordinary design purposes it is not required. (See Chapter XVIII.) Before the moments in any member of a frame can be computed it is necessary to know the stiffness (I/L) of all connecting members for at least one panel in every direction from each end of the span. The size of these members must first be approximated, probably by computations such as those in Art. 13–4. Fortunately considerable variation in the relative stiffnesses does not cause corresponding variations in the final moments. The following example illustrates the computation of moments and

shears in one beam of a building frame by the abbreviated moment distribution recommended in J.C. Appendix 2, the size of columns, etc., being adopted from the design in Chapter XVII.



Member	Size	Moment of Inertia = I	L	$K = I/L$
$C_1 = C_1$	$32\frac{1}{8} \times 12$	$32\frac{1}{8} \times 12^3/12 = 4630$	13'-0"	1.67
c_2	$13\frac{1}{2} \times 13\frac{1}{2}$	$13\frac{1}{2}^4/12 = 2770$	13'-0"	1.0
C_2	17×17	$17^4/12 = 6960$	13'-0"	2.52
$B12$	$55.6 \times 4 = 222$ $14.0 \times 9.63 = 135$ 357	$222\left[\frac{(4)^2}{12} + (3.4)^2\right] = 2860$	19'-6"	2.24
$B23$	$kd = 2 + \frac{135}{357} \times 9 = 5.4$	$135\left[\frac{(14)^2}{12} + (5.6)^2\right] = 6440$ 9300	20'-0"	2.18

(b)

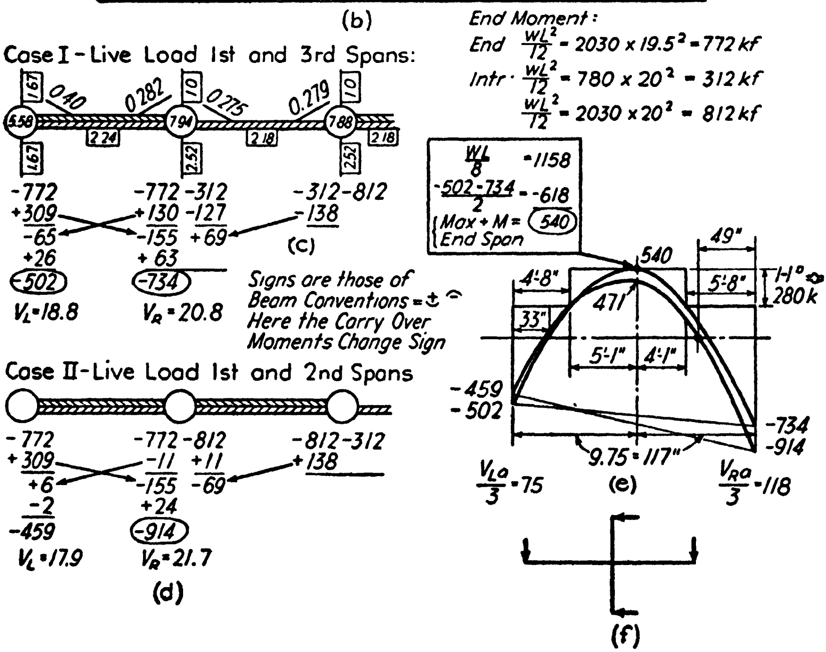


Fig. 13-14

Example 13-2a. Check by moment distribution the shears and moments in the beam designed in Ex. 13-2, page 246, the sizes and stiffnesses of adjacent beams and columns being as shown on Fig. 13-14*a*.*

* Comparable results are obtainable by the use of tables of coefficients, notably those on pp. 128–131 of Reinforced Concrete Design Handbook of the A.C.I.,

Solution. As suggested elsewhere, there are many methods of attacking this problem. For the present purpose we follow the suggestion of J.C. 808*b* that, where no great irregularities of span, story height, or loading exist, it is sufficiently exact to follow the general method of abbreviated moment distribution given in J.C. Appendix 2. This is limited to two cycles of operations, as no higher degree of accuracy is obtainable by further cycles, because the unknown slopes of the far ends of the connecting members (tacitly assumed fixed by this method) then affect the results.

Since only relative values of $K = I/L$ are required, the smallest K value is taken as unity. Two cases are considered: (1) full load on this beam and dead load on the adjacent interior beam, giving maximum values for M_L and positive moment, and (2) full load on both spans for maximum M_R. The FEM's are tabulated; one distribution is made into the beams only at each joint; the carry-over moments are computed; and a final distribution is made. Although such an analysis is only approximate, it is reasonably accurate and can be prepared quite rapidly. It is ordinarily sufficiently accurate to obtain the positive moment by deducting the mean of the two end moments, but this may not suffice if the two end moments differ materially.* Finally the shears are computed as explained on page 247. The results of this analysis have already been plotted on Fig. 13–12.† For the design moments at the faces of the columns deduct $Va/3$; this gives $M_L = -427$ and $M_R = -796$.

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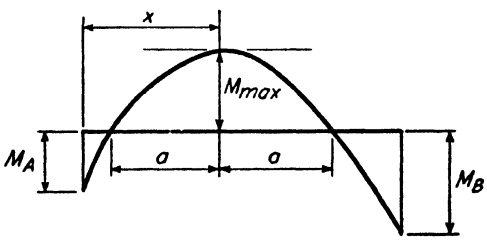
adapted from "Design by Coefficients for Building Frames," by A. J. Boase and J. T. Howell, Proceedings, A.C.I., Sept., 1939. To apply these to Ex. 13–2*a*: $W_L/W_D = 1250/780 = 1.60$; ΣK Col/K Beam $= 3.34/2.24 = 1.49$ left end and $3.52/2.24 = 1.57$ right end; then from Table 1A, p. 129, by interpolation $-M_L = -0.056 \times 2030 \times 19.5^2 \times 12 = -521$ k-in.; $+M = +0.057wL^2 = 531$ k-in.; $-M_R = -0.098wL^2 = -912$ k-in., which agree closely enough with the approximate moment distribution for practical purposes.

* If occasion arises to use the more precise value the student should have no difficulty in deriving the relationships indicated on the figure and below:



$$x = \frac{L}{2} + \frac{M_B - M_A}{wL}$$

$$+M_{\max} = \frac{wL^2}{8} + \frac{M_B + M_A}{2} + \frac{(M_A - M_B)^2}{2wL^2}$$

$$a = \sqrt{\frac{L^2}{4} + \frac{M_B + M_A}{w} + \frac{(M_A - M_B)^2}{w^2L^2}}$$

† This is the floor beam framing between columns. The intermediate floor beam between spandrel and interior girders has on its outer end only the torsional resistance of the spandrel. A means of evaluating this is shown in Ex. 14-7, p. 291.

As the beam stem is determined by shear and the shear variations are slight, no change in concrete sizes would result from the frame analysis. The decrease in positive moment reduces A_s to $M/f_sjd = 540/(17.5 \times 16) = 1.93$ sq in.; one ¾ in. round and one ⅞ in. round straight and one 1 in. square bent furnish 2.04 sq in. At the left end $R' = 174$ and $A_s = 427/(17.5 \times 16) = 1.53$ sq in.; one 1 in. square and two ⅝ in. round top bars give 1.60 sq in. At the right end $R' = 323$; $d'/d = 0.125$; $p = 0.019$; $p' = 0.0105$; $A_s = 2.93$ sq in.; and $A'_s = 9.3/16$, $p'bd = 0.94$ sq in.* The one 1 in. square bar

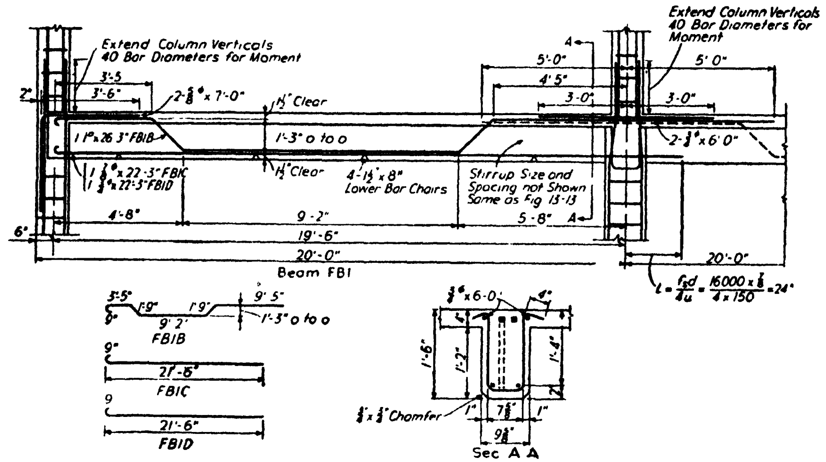


Fig. 13-15

from FB1 and the one 1 in. square from FB2 must be supplemented by two ¾ in. round top bars at the right end, and the one ¾ and one ⅞ in. round bottom bars must be carried through for compressive reinforcement. *Bond, bending up of bars, etc., should be checked as in Ex.* 13-2. *This is especially important in frame analysis because arbitrary rules may be considerably in error.* See Fig. 13-14*e* for the bending up of bars. For comparison with Fig. 13-13, a detail of the reinforcement by this analysis is shown in Fig. 13-15. In particular note the two ⅝ in. rounds bent into the top of the beam at the left end, and the fact that column verticals must overlap sufficiently and be of proper size to take care of the moments in the columns. The detailer must think of each joint as being a group of cantilever beams as shown in Fig. 13-14*f*.

13-5. Beams Reinforced for Compression. Compression reinforcement is not economical. As shown in Art. 7-10, a tee-beam is the satisfactory means of obtaining compressive strength. Sometimes, however, owing to limitations of head room and of width at the same time, it becomes necessary to obtain as high flexural resistance as possible in a minimum of space. In such cases compressive reinforcement is the solution.

* See explanation, p. 256; based on assuming $k = 0.40$. Here k is about 0.402.

The typical designs so far considered in this chapter have dealt with uniform loads. There are certain simplifications in the handling of concentrated loads which will be shown in the following example of the design of a doubly reinforced concrete beam.

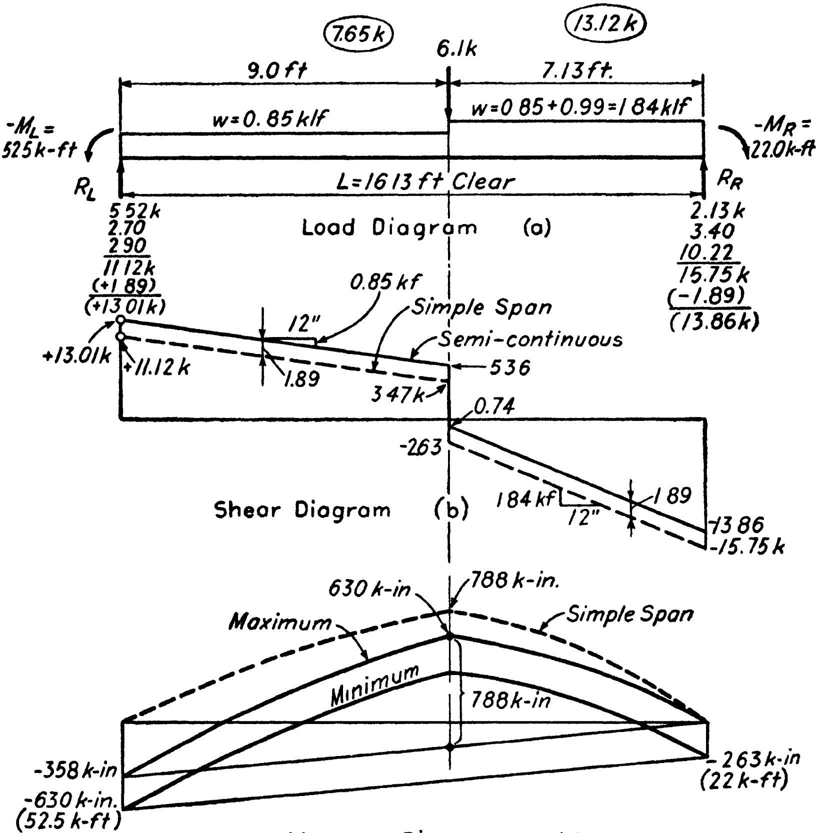


FIG. 13–16

Example 13-3. Design a reinforced concrete beam for a clear span of 16.13 ft loaded as shown in Fig. 13-16*a*, keeping the width 8 in. and depth 18 in. if at all possible. Specifications: 1940 J.C., $f'_c = 3000$ psi, $f_s = 20{,}000$ psi. This is beam FB8, designed in Chapter XVII.

Solution: (*a*) *Load.* The concentrated load was obtained from the computations of FB4 on page 379, and is the reaction of beam FB3 which carries 12 ft of brick wall at 80 psf plus 100 plf dead weight of beam. Its reaction is 5.75 ft @ $(12 \times 80 + 100)$ plf = 6.1 k.

The uniform load on the left portion of FB8 was obtained as follows:

Weight of sash	0.04 klf
Spandrel 7.5 × 80	0.60
Sill	0.06
Beam	0.15
	0.85 klf

The load on the right-hand portion is equal to the above plus the weight of the concrete stairs which was approximated at 0.99 klf. This would be checked after the stair computations are completed. (See Ex. 16–6.)

Attention is directed to this method of computing loads as it is frequently more difficult to establish the proper load diagram than it is to complete the computations. The diagram records the total clear span, the distance from left reaction to concentration and from concentration to right reaction; also the unit uniform loads and the concentrated load.

(*b*) *Load Computations.* Note in Fig. 13–16*a* that the uniform load for the left portion is multiplied by the length and recorded in a circle as 7.65 k, the total resultant of the left-hand uniform load. Similarly the total load on the right-hand portion is recorded as 13.12 k.

(*c*) *Reactions.* Instead of computing the reactions by taking the sum of the moments of the various loads it is quicker, as well as (later) more easily checked, to obtain the reactions of each individual load independently as follows:

For the left-hand uniform load:

$$R_r = \frac{7.65 \times 4.5}{16.13} = 2.13 \text{ k}$$

$$R_l = \frac{7.65(7.13 + 4.5)}{16.13} = 5.52$$

$$\text{Check by } \Sigma V = 0: \quad \overline{7.65} \text{ k}$$

It is unnecessary to record so elementary a computation. The reaction components are recorded directly from a slide rule, and each time the distribution is checked by adding the two terms. The total reactions result immediately from the addition of the components. This is a simple and direct method for handling concentrations.

(*d*) *Shear Diagram.* A shear diagram as for a simply supported beam is shown by dotted lines in Fig. 13–16*b*. The presence of a negative moment of 630 k-in. = 52.5 kip-ft acting on the left end of the beam and 263 k-in. = 22.0 k-ft on the right end induces a couple of 30.5/16.13 = 1.89 kips acting upward at R_l and downward at R_r, as discussed in Ex. 13–2, item *d*, changing the shear diagram to that shown by full lines.

The dotted shear diagram of a simple beam will be used in this problem for designing stirrups and checking bond stress, but not without emphasizing the approximate nature of arbitrary moment coefficients and the desirability of using methods such as those of Chapter XVIII.

(*e*) *Positive Moment.* Maximum positive moment occurs at the point of zero shear. On the basis of a simple span, starting at the left reaction the left

end uniform load is not enough to balance the reaction. Adding the concentrated load more than balances: zero shear is directly beneath the concentrated load. This is obvious from a glance at Fig. 13-16*b* (dotted) but the designer usually reasons this out as just explained and does not bother drawing the shear diagram.

Taking moments about the point of zero shear gives:

$$\begin{aligned} 11.12 \times 9.0 &= 100.1 \text{ k-ft} \\ -7.65 \times 4.5 &= \underline{-34.4} \\ & 65.7 \text{ k-ft} = 788 \text{ k-in.} \end{aligned}$$

In this example we shall make use of the arbitrary coefficients suggested in earlier codes,* viz., $WL/24$, or ⅓ of the simple beam moment at the outer end of the span, $+WL/10$, or 80 per cent of the simple beam moment at zero shear near mid-span, and $-WL/10$, or the same percentage of the simple beam moment as negative moment at the inner end. The positive moment then becomes $+WL/10 = (788 \times 8/10) = 630$ k-in. The end moments become respectively $(-788 \times ⅓) = -263$ k-in. and $(-788 \times 8/10) = -630$ k-in., respectively. Although this method of arbitrary coefficients leaves the designer some latitude in drawing moment curves, the fact that the overall depth of diagram equals the simple beam moment of 788 k-in. suggests an arrangement of maximum and minimum moment curves about as shown on Fig. 13-16*c*.

(*f*) *Stresses.* Follow the stresses recorded in Ex. 13-1*e*.

(*g*) *Beam Size.* The width of beam will be made 8 in. so that the beam will line up with the 8-in. brick spandrel wall. The depth will be maintained at 18 in. if possible to line up with the other spandrel beams.

(*h*) *Effective Depth.* It is customary to provide a slot in the soffit (bottom) of the spandrel beams to receive the steel sash. Sash manufacturers recommend a slot approximately 1½ in. high. The stirrups can rest directly on top of the form for this slot. The effective depth is 18 in. − (1½ in. slot) −(⅜ in. stirrup) − (½ in. for half the bar diameter) or 15⅝ in.; assume $d = 15.6$ in.

(*i*) *Shear.* The maximum unit shear equals $V/bjd = 15{,}750/(8 \times ⅞ \times 15.6) = 144$ psi, which is less than the 180 psi permitted with diagonal tension reinforcement.

(*j*) *Compression.* Determine R from $M/bd^2 = 630{,}000/(8 \times 15.6^2) = 324$. This exceeds the 236 allowable. Concrete compression would considerably exceed 1350 psi. The beam must be increased in size or compression reinforcement must be used.

* A brief check will indicate that the columns above and below this floor (12 by 32½ in.) each have a relative stiffness about 12½ times that of this beam; this means that the beam's participation in any unbalanced joint moment will be of the order of magnitude of some 3 or 4 per cent. Hence to carry out the J.C. suggestion of design by the principles of continuity, it would be unnecessary to use even an approximate moment distribution, taking about 95 per cent of the fixed-end moment at each end. Also, with such stiff columns the effect of loads on the adjacent spans would be negligible and various positions of live loading need not be investigated.

(*k*) *Steel Areas.* Determine $d'/d = 2/15.6 = 0.128$. Using Fig. A-10*:

$$p = 0.019 \quad A_s = 2.37 \quad 2\text{—}1\tfrac{1}{8} \text{ in. squares} = 2.52$$

$$p' = 0.011 \quad A'_s = 1.37 \qquad \frac{9.2}{16} \text{ of } 1.37 = 0.79 \text{ sq in.}$$

Two ¾ in. rounds = 0.88 sq in., which is apparently sufficient. Such rods must be heavy enough not to buckle. As a general rule probably ¾ in. rounds are as small as would ordinarily be used for compressive reinforcement.

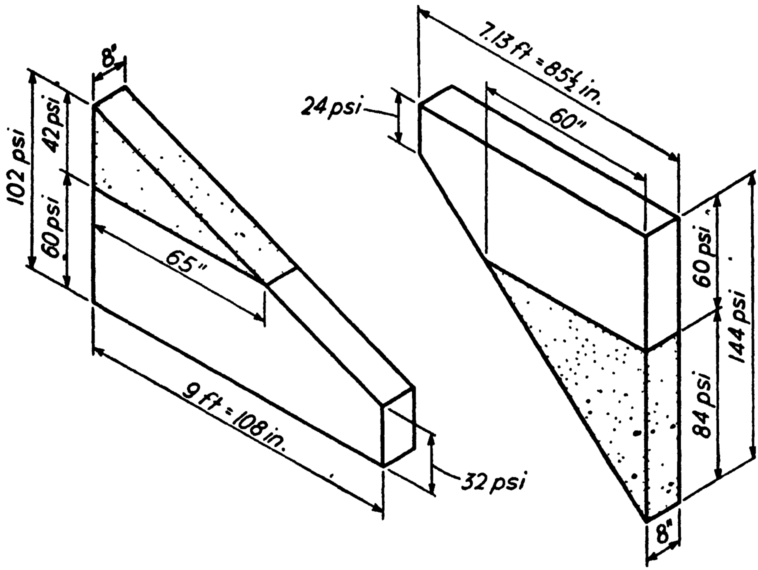


Fig. 13-17

Again, if these rods are carried across the supports to act as negative steel as well, then their size will be determined by negative moment requirements as explained under item *n* below, where two ⅞ in. rounds are found necessary.

J.C. 804*c* requires ¼ in. round ties not over 16 bar diameters centers to prevent buckling of the compressive reinforcement regardless of web shear requirements. Try ¼ in. round ties, 12-in. centers.

(*l*) *Web Reinforcement.* For the determination of stirrups under concentrated loading conditions a sketch is helpful. Fig. 13-17 gives the necessary information, based on the shear diagram as for a simple span.

For computing the values shown, which are obtained from $v = V/bjd$, the

* This plate is drawn on the basis of the elastic theory assuming the stress in the compression steel to be *n* times the stress in the adjoining concrete. J.C. 804*c* takes account of plastic flow by doubling the effectiveness of the compressive reinforcement but f'_s is not to exceed 16,000 psi. Here the elastic stress (assuming $k = 0.40$) is about $[(0.4 \times 15.6 - 2)/(0.4 \times 15.6)]\ 10 \times 1350 = 9200$ psi, giving $A'_s = 9.2/16 \times p'bd$, as explained on p. 256.

simplest procedure is to solve $1/bjd = 1/(8 \times \frac{7}{8} \times 15.6) = 0.0092$, which may be left in the slide rule without recording. Multiplying by the left reaction, 11.12, gives 102 psi; multiplying by $(11.12 - 7.65 = 3.47)$ gives 32; multiplying by the right reaction, 15.75, gives 144; and by $(15.75 - 13.12 = 2.63)$ gives 24.

It may be that the ties for the compressive steel will provide sufficient web reinforcement without additional stirrups. To be on the safe side, not having determined whether or not special anchorage will be required, v_c will be taken as 60 psi (J.C. 878, Table 7).

The point at which no stirrups are needed from the left end is obtained from $[(102 - 60)/(102 - 32)]\, 9 \times 12 = 65$ in. At the right-hand end, $[(144 - 60)/(144 - 24)]\, 7.13 \times 12 = 60$ in.

The greater number of stirrups is required on the right-hand end and equals $(60 \times 8 \times 84/2 \times 16{,}000) = 1.26$ sq in. The compressive reinforcement requires only $\frac{1}{4}$ in. round ties, 12-in. centers. In 60 in. of length there would be at least 5 ties each with a sectional area of 0.1 sq in., providing 0.50 sq in. of web reinforcement where 1.26 is required. This is insufficient, so on the right end use $\frac{3}{8}$ in. round ties spaced 3, 6, 6, 6, 8, 8, and 12 in. to center of span and on the left end space similar ties 3, 6, and 12 in. to center of span, the spacing being obtained as shown on page 253.

(*m*) *Cutting Compressive Reinforcement.* The compressive reinforcement could stop as soon as the concrete alone is able to carry the compression, except that it is necessary to extend the bars past this theoretical point to develop them* (J.C. 829*c*). In this case a better detail results from permitting the two $1\frac{1}{8}$ in. square bars to extend into the supports at each end without bending either of them and then permitting the two top bars to extend through the support and to the point of inflection of the adjoining span, thus saving not only extra compressive reinforcement at mid-span but extra tension reinforcement for negative moment as well.

(*n*) *Negative Moment.* The negative moment at the first interior support is taken as $-WL/10 = -630$ k-in. Complete computations could be made for negative moment, as was done before, but since this is a rectangular beam the same amounts of tension and compression steel are needed over the sup-

* The distance y from the right support to the point at which the concrete will just carry the bending moment is obtained by equating the bending moment expressed in terms of y to the maximum resisting moment, Rbd^2, of the concrete section without compressive reinforcement as follows (see Fig. 13–16*b* — dotted):

$$\left(15.75 - \frac{1.84}{2}y\right)y = \frac{236 \times 8 \times \overline{15.6}^2}{12} = 38.3$$

$$y = 2.95 \text{ ft}$$

The distance x, because of the negative moment on the left end, is best computed from the concentrated load to the point where the concrete alone can just carry the compression. At the concentration $M = 630$ k-in. or 52.5 k-ft. Then:

$$52.5 - \left(3.47 + \frac{0.85}{2}\right)x = 38.3$$

$$0.43x^2 - 3.47x = +14.2$$

$$x = 3.02 \text{ ft from concentration or } 9.0 - 3.02 = \text{(say) } 6 \text{ ft from left reaction}$$

port, as was provided at mid-span but interchanged from top to bottom of beam. Thus for top reinforcement 2.37 sq in. are required. Assuming that one 1 in. round will be supplied by the bent-up bar from the adjoining beam FB7 (page 381), there remain 1.58 sq in. to be furnished. This can be made with the two ⅞ in. rounds for negative reinforcement at the outer end plus a ¾ in. round top bar = 1.64 sq in. The length of this added top bar can be computed from the moment curve as in Fig. 13–12. If the bottom bars of FB8 overlap those of FB7 there will be ample compressive reinforcement.

At the outer end only 10/24 as much top tension steel is needed as in the bottom at mid-span or about 0.99 sq in. The two ⅞ in. round bars chosen above = 1.20 sq in. and are more than sufficient.

(*o*) *Bond.* At the right end check the bond on the bottom bars from Fig. 13–17 as $u = vb/\Sigma o = (144 \times 8/9) = 128$ psi which is within the 150 psi allowed without special anchorage. At the point of inflection near the left end the external shear is much lower and the bars are the same, so the bond there is well on the safe side.

It is unnecessary to compute the bond on the top bars at the interior support. The shear is less than three-fourths as much as at the right end, the perimeter of the two ⅞ in. top bars in this beam alone is nearly two-thirds of the perimeter of the bottom bars; in addition bars will be brought through from the adjoining span and the bond stress already computed is only 128 psi whereas 225 psi is allowable with the special anchorage that will result with the bars extended for negative moment, so we can simply note " Bond O.K." The design of an ordinary reinforced concrete structure is a lengthy job and wherever it is obvious from previous computations that a given detail is adequate and safe it is not customary to repeat figures. However, to prevent overlooking any considerations, it is desirable that the computer record some indication that he has considered the point, such, for example, as the notation suggested.

At the outer end $u = 128 \times (9/5.5) = 209$ psi; this requires special anchorage of the top bars at the outer support.

(*p*) *Office Practice.* The tabulated form shown in blank schedule on page 257 fits the design of this beam with one or two adaptations to take care of concentrated loads and compressive reinforcement. The computations are clearly shown in Chapter XVII. The student should study and check these carefully in conjunction with the foregoing explanation.

13–6. Tee-Beams with Compressive Reinforcement. On rare occasions it is necessary to use compressive reinforcement in a tee-beam as, for example, when the compressive area required exceeds the amount of flange width available within the limitations of J.C. 804*d*. These cases, however, occur but rarely and ordinarily in very heavy girders which require care in analysis and detailing. The authors prefer to solve such conditions by use of transformed areas, making an approximate solution by guess and checking it by computations.

13–7. Tied Columns. The design of a tied column for a given load is a simpler matter than the determination of the load to be carried. For columns which carry square panels of uniform floor load it is easy

to obtain part of the loading from the floor area and the unit weight. In many cases the loads are not uniform and in all cases allowance must be made for the weight of the beams, girders, and columns. It is recommended that the accumulation of column loads be obtained from the beam reactions and, if desired, roughly checked on the basis of tributary areas.

In most building construction it is permissible to " reduce " the live load carried by columns, that is, to design the column for less than full live load over the area it supports. The percentage of reduction depends upon the local building code whose use should be tempered by consideration of the probability of how nearly all the tributary areas on all the floors could be loaded with live load simultaneously. The chances of the top floor being fully loaded are relatively high but, except in warehouses, the chances of the two upper floors being loaded at the same time are less and full load on three consecutive upper floors is still less likely.

Example 13-4. Design a column stack to support three floors and a roof for a panel 20 ft square carrying a roof load of 40 psf and a floor load of 125 psf; use live load reductions of 10 per cent on the third floor, 20 per cent on the second, and 30 per cent on the first floor. Specification, 1940 J.C.: $f'_c = 3000$ psi, $f'_s = 40{,}000$ psi. This is column B3, page 360, the tabulated design of which is on pages 377 and 385.

Solution. Since the computation of column loads involves the dead weight of the structure, refer to the design computations on pages 371 to 385. Little comment on the tabulations of loads and sections there shown is required.

(*a*) *Roof Loads.* From page 367, column B3, supporting the roof carries two beams B2 and girders G1 and G2. The reaction for each beam or girder is taken directly from its design computations, being one-half the total load. In cases of concentrated loads the appropriate end reaction is used. These reactions are all computed on the clear span between faces of columns. In assuming the dead weight of column it is necessary to include some allowance for any strip of floor slab that may not be included in the clear span lengths of the beams. (See Fig. 17-2.)

b. Third-Story Columns Supporting Roof. (See Computation Sheet BG6, page 377.) Although J.C. 852 limits the height of column to ten times the side, and limits the size of main columns to 10 in., because of the light load to be carried, the roof columns will be made 9⅝ by 9⅝ in. (to suit standard 2 by 10 plank forms), using a reduced stress for long columns (J.C. 858). The weight of this column is about 1100 lb, or closely the 1.3 kips assumed.

The capacity of this column is recorded and found to be in excess of requirements, 65.4 kips if of normal length or 58.2 kips for excessive length, as against a load of 46.1 kips. The reinforcing steel is chosen from J.C. 855*a*, where four ⅝ in. round rods are a minimum. The rods must represent at least 1 per cent of the area of the column; here four ⅝ in. rounds are 1.29 per cent.

(*c*) *Third-Floor Loads.* (See Computation Sheet BG10, page 385.) The third-floor column carries not only the third-floor system but the weight of

the roof column as well. The live load reduction scarcely requires explanation; the 10 per cent value has already been established. Since the panel is 20 ft square there are 400 sq ft of tributary area each of which carries a live load of 125 psf.

(*d*) *Second-Story Columns Supporting Third Floor.* In designing this column it is simpler to estimate its size and check back than to solve by mathematical formulas. As a guide in selecting sizes the concrete alone has an assumed capacity of 540 psi. One per cent of vertical steel would increase the capacity of each square inch of gross area to 700 psi. Dividing 128.0 by 700 and extracting the square root indicates a column approximately 13.5 in. square. Assume a 13½ in. square column with a weight of 2100 lb, whose capacity is computed as 129.2 kips. Note that four ⅞ in. bars represent 1.32 per cent of the column area, which is in excess of the 1 per cent minimum limitation of J.C. 855*a*.

(*e*) *Second-Floor Loads.* (See Computation Sheet BG10, page 385.) Since the floor loads and areas are the same as for the third floor it is unnecessary to itemize each beam reaction. The dead weight of the column shaft is slightly greater than for the story above, say 3.3 kips total weight.

On the larger columns no attempt was made to fit commercial sizes of lumber, as the column forms would be made up of more than one piece and probably with reduction strips (page 55) to suit the columns above.

(*f*) *Office Practice.* The abbreviated form of recording computations is shown in the schedule on page 385. This should be clear from the foregoing explanation. The capacities of tied columns may also be obtained from Table A-4, in the Appendix.

13–8. Spiral Columns. The computation of loads for a spiral column is the same as for a tied column. The capacity of a spiral column can be obtained from the formulas on page 137. These formulas are designed rather for determining the capacity of an assumed column than for making a design. There are at least three variables, the column diameter, the amount of vertical steel, and the amount of spiral hooping.

A limited number of solutions are practicable because: (*a*) the shaft diameters should be in multiples of 2 in. to accommodate standard forms; (*b*) the vertical reinforcement should be not less than six ⅝ in. round rods (J.C. 854*b*); (*c*) the vertical reinforcement will ordinarily consist of 6, 8, or 10 rods all of the same size; (*d*) the percentage of reinforcement and of hooping will ordinarily be greater than in the column immediately above and less than in the column below, so as to produce a gradually increasing capacity in the lower stories.

In selecting spiral columns it is possible to assume a design and check back by the formulas to determine its capacity or it is possible to pick a design directly from tables such as Table A-6. Both methods are illustrated for comparison in the following example.

Example 13–5. Determine an alternative design, using a spirally reinforced circular concrete column in place of the square tied column supporting the

first floor in Ex. 13-4. Specifications, J.C. 1940: $f'_c = 3000$ psi; $f'_s = 40{,}000$ psi.

Solution. (*a*) *Load.* From Computation Sheet BG10, page 385, $P =$ 280.0 k.

(*b*) *Capacity.* Using J.C. 854, Formula 9,

$$P = A_g(0.225f'_c + f_s p_g)$$
$$= A_g(675 + 16{,}000p_g)$$

The steel ratio p_g must be between 1 per cent and 8 per cent (J.C. 854*b*). If p_g is taken as 4 per cent, $P = 1315A_g$ and $A_g = 213$ sq in. This suggests a column about 16 or 18 in. outside diameter with p_g to suit. Assuming an 18-in. outside diameter column:

$$P = 254(675 + 16{,}000p_g) = 280.0 \qquad (675 + 16{,}000p_g) = 1105$$

$$p_g = 2.69 \text{ per cent} \qquad A_v = 0.0269 \times 254 = 6.83 \text{ sq in.}$$

The spiral, according to J.C. 854*d*, must be not less than:

$$p' = 0.45(R - 1)\frac{f'_c}{f'_s} = 0.45\left(\frac{18^2}{15^2} - 1\right)\frac{3000}{40{,}000} = 1.49 \text{ per cent}$$

The required pitch of ⅜ in. round hooping to supply this percentage is easily computed as 0.0149 = (volume of spiral/volume of core) = ($\pi \times 14.69 \times 12 \times 0.11$/pitch $\times$ 177 sq in.), from which the pitch is 1.8 or, practically, 1¾ in. Spiral pitch is usually set as a multiple of the quarter-inch.

(*c*) *By Tables.* The above computation can be avoided by referring to a safe load table such as Table A-6, which gives the total capacity for spirally reinforced concrete columns according to the formula $P = A_g(0.225f'_c + f_s p_g)$, based on 3000 psi concrete.

Pick from the table a column capable of carrying 280.4 kips. Several combinations might be used, such as an 18-in. diameter with six 1⅛ in. square rods or a 20-in. with six 1 in. squares; the required amounts of spiral are indicated in the table as ⅜-in. wire, 1¾-in pitch, and ⅜-in wire, 2-in pitch, respectively. The maximum and minimum amounts of vertical steel are shown by the heavy broken lines.

(*d*) *Conclusion.* For the relatively light load involved with the recommended formulas, a tied column is about as economical as a spirally reinforced column. The spiral column is tougher and more satisfactory but not much more economical in this case. If a few more stories of building were involved the spiral column would be required.

13-9. Bending and Direct Stress. Since the beams and columns of a building are poured monolithically and the reinforcing steel ties from one into the other, the beams cannot deflect and pick up their bending moment without at the same time bending the columns. This interaction is described in Art. 12-1 and is clearly shown in Fig. 12-1 (page 193). Many specifications for design state that uniformly loaded beams of approximately equal spans, built into columns, should be designed to carry at their exterior ends a negative moment of $wL^2/24$. This moment could be resisted only by the columns framing into the end of the

beam from above and below. These columns apportion the moment between them according to their relative stiffnesses. In the following example the design of columns for bending and direct stress is illustrated by the use of moments rather roughly estimated and apportioned. When reading Chapter XVIII the student can investigate the correctness of the moments used.

Example 13–6. Design the exterior stack of columns for a three-story and basement building with 20 ft square panels carrying a roof load of 40 psf and a floor load of 125 psf; use live load reductions of 10 per cent on the third, 20 per cent on the second, and 30 per cent on the first floor. Specification, 1940 J.C.: $f'_c = 3000$ psi, $f'_s = 40{,}000$ psi. This is column D3, designed in Chapter XVII. Upon completion of the design check the stack of columns for combined direct stress and the bending due to negative moment in the exterior end of girder G1 (Fig. 17–2) and see if any change in concrete size or amount of reinforcing steel is required to take care of this moment at the higher unit stresses permitted for such combination.

Solution. (*a*) *Direct Loads.* The direct vertical loads are accumulated in a schedule form as already explained in Ex. 13–5:

Roof				
2 beams RB7	13.4		$32\frac{1}{8} \times 12 =$	385.5 sq in.
RG1	12.9		-2 slots @ $2 \times 2 =$	-8.0
Col	7.3	33.6		377.5
Third Floor			9×6—$\frac{7}{8}$ in. rd. @ $0.60 =$	32.4
2 beams 3B7	22.5		Transformed area $=$	409.9 sq in.
3G1	23.8		For 6 — $\frac{7}{8}$ in. rounds:	
Col	6.6		$p = \frac{3.60}{377.5} = 0.95\%$ or sufficiently close to the 1% minimum of J.C. 855*a*	
	52.9			
-180 sf @ $10\% \times 125$ −	2.3			
	50.6	84.2	pn (both faces) $= 0.095$	
Second Floor	52.9		pn (one face) $= 0.048$	
-180 sf @ $20\% \times 125$ −	4.5		Capacity of column by elastic theory:	
	48.4	132.6	$P = 409.9 \times 540$	
First Floor	52.9		$= 221$	
-180 sf @ $30\% \times 125$ −	6.8			
	46.1	178.7	J.C. 855*a* and 854*a*	

(For the capacity by the yield-point theory see p. 385.)

(*b*) *Moments.* From the positive moment computations for RG1 on page 373, $WL/10 = 999$ k-in., from which the negative moment $-WL/24$ is found to be 416 k-in. Since only the roof column frames into the end of this girder the entire 416 k-in. must be transmitted to the roof column.

The computations for FG1 on page 381 show a positive $WL/10$ moment of 1888 k-in. from which the negative $-WL/24$ moment is found to be 786 k-in. It so happens that the columns above and below each floor are the same size, determined by the limitations of steel sash, minimum thickness, etc. Also, the story heights are substantially the same, so that for our present

purpose it will be assumed that one-half of the moment is transmitted into the column above the floor and the other half into the column below.

(*c*) *Column Size.* As explained on page 361, the column spacing having been determined and the width of steel sash selected, there is left between adjoining sash a distance of 32⅛ in. The whole of this will be taken as the column width. Although wider sash are obtainable, the appearance of the building would not be good with narrower piers The thickness of the column perpendicular to the exterior wall is determined by the fact that the exterior spandrel wall is 8 in. thick and there is a 4-in. pilaster projection for appearance; this limits the column size to a minimum of 12 in.

(*d*) *Reinforcement.* The minimum amount of reinforcement for a tied column is 1 per cent, or 3.78 sq in. in this case. In such a wide pier it is desirable to use at least six rods; six ⅞ in. rounds were arbitrarily chosen, being 0.95 per cent of the column area.

(*e*) *Computations.* Since the minimum size of column established appears ample for the loads carried, it seems worthwhile to check the stresses without attempting preliminary readjustment. Accordingly, a schedule is set up showing the loads and moments at each floor. Necessary values are computed to permit entering Figs. A–12, A–18, and A–19 to obtain the coefficients for figuring steel and concrete stresses in the selected section. If any section shows stresses higher than those permitted it must be increased.

In the following table the first column gives the floor supported; the second, the vertical load; and the third, the moment from the girder. The eccentricity $e = M \div P$ of the normal thrust to produce this moment is recorded. The depth h of column is next given and then the ratio of depth to eccentricity h/e. For this column stack $d'/h = 2/12 = 0.167$ and reference to Fig. A–12 shows that these are all Case II with tension over part of the section. Fig. A–17 being for $d'/h = 0.10$ and Fig. A–18 for $d'/h = 0.15$, values of C for $d'/h = 0.167$ can be obtained by extrapolation, using pn (one face) = 0.048 and h/e as computed, so record k and the coefficient C. Calculate $f_c = C(M/bh^2)$. Entering Fig. A–19 with the value of k just found and $d'/h = 0.167$, read A and B. Calculate $f_s = nf_cA$ and $f_s' = nf_cB$.

Floor Supported	P (kips)	$M = Pe$ (k-in.)	e (in.)	h (in.)	h/e	k	C	f_c Actual	A	f_s	B	f_s'	e/t	f_c Allowable
Roof	34	416	12.2	12	0.985	0.30	10.3	**930**	1.75	16,300	0.45	**4200**	1.015	**1140**
3rd	85	393	4.62	12	2.60	0.55	9.6	**820**	0.50	4,600	0.70	**5800**	0.385	**970**
2nd	133	393	2.96	12	4.06	0.79	9.4	**800**			0.79	**6300**	0.246	**885**
1st[1]	179	393	2.19	12	5.49	0.97	10.3	**880**			0.83	**7300**	0.182	**832**

[1] The column supporting the first floor is here treated as free-standing, although on Fig. 17–2 it is shown as an integral part of the foundation wall. Since these computations do not require any increase in section, advantage is not taken of the adjoining wall.

In the last column is recorded the maximum allowable combined stress from equation 8–10 (page 119), $f_c = f_a(1 + 6\cdot e/t)/(1 + C6\cdot e/t)$ where $C = f_a/0.45f_c'$. f_a = (409.9 sq in. × 540 psi)/377.5 sq in. = 586 psi. C = 586/1350 = 0.434. The recorded values e/t are the reciprocals of h/e, as both t and h in this case represent the depth of column. In each case the actual concrete stress is within the maximum allowed. No increase in section is required.

13–10. Summary. This chapter bridges a gap between the mathematical theory and actual design procedure and indicates the numerous practical considerations that must be taken into account as actual office design proceeds. These computations are simple and condensed but difficulties arise, since many decisions must involve consideration of other details in advance of their being developed. Assumptions must be made and the validity of the guess checked after the other details have been designed. Some rough rules have been suggested as an aid to guessing. Frequently with unusual structures it is good practice to run through a very rough, crude design to get the feel of the problem and to have some idea of the approximate sizes required. Often such rough computations can be made mentally, rounding out all results and carrying only one or two significant figures, thus saving time that would be lost in slide-rule computations. For final computations refer again to Art. 17–5 for the degree of precision required. In spite of the questionable points in theories and the impossibility of knowing exactly the loads to be carried, it is desirable to adhere to " slide-rule precision " if only for the purpose of preventing gross errors.

As a measure of war economy the War Production Board requires the use of a lower concrete stress than that employed in this chapter, $f_c = 0.35f'_c$, and a higher steel stress, 24,000 psi on intermediate and hard grade and on rail steel. This results in the use of deeper members and a smaller amount of the critical material, steel. It is also required that the designer employ all the refinements of continuous frame analysis and in every possible way reduce the amount of steel required. The performance of structures designed under these often radical rulings will determine the extent to which they will modify engineering practice after the emergency has passed. Further discussion of this topic appears in Art. 22–5.

CHAPTER XIV

PROBLEMS IN TEE-BEAM DESIGN

14–1. In addition to the details of choosing loads, spans, and moment factors, several other practical problems arise in the design of tee-beams that yield to mathematical treatment. Since these items are intermediate between the application of the simple mathematical theory and the practical design problems involving judgment and experience, they will be developed here. Some of the details to be considered include: width of flange and distribution of stress across the flange of tee-beams in monolithic structures; holes through the webs of beams; variable width of tee; holes through the flanges of tee-beams; unsymmetrical tee-beams with flange on one side only; longitudinal shear at the junction of the flange and stem; torsion on beams and deflection of beams.

These problems are very common in practice but are very little considered in textbooks. No standardized analyses are available for the guidance of the engineer, and there are few reported tests by which theoretical studies might be checked. The computations and analyses here presented are typical of the solutions attempted in engineering practice in situations where adequate information is not to be had.

14–2. Width of Flange and Distribution of Compression across the Flange. An analysis of this problem was made by Theodore von Kármán and is discussed in Timoshenko's Theory of Elasticity (1934), page 156. The elementary theory of bending assumes a constant compressive stress across the entire top plane for the full width of a tee-beam, but when this width is very large, the flanges at a distance from the web do not take their full share of the compression, the stress variation being as pictured in Fig. 14–1*a*.

For an infinitely long continuous tee-beam on equidistant supports, having all spans equally loaded with loads symmetrical about the centers of the spans and so arranged that the bending moment curve is a cosine line, the width of flange infinitely large, and the thickness very small in comparison with the depth of the beam, the effective width of tee over which an extreme fiber stress of f_c uniformly distributed will produce the same total compression is obtained as:

$$T - b' = \frac{4L}{\pi(3 + 2m - m^2)}$$

where L is the span and m is Poisson's ratio. Taking m as 0.2 gives:

$$T - b' = 0.378L \text{ or } \frac{T - b'}{2} = 0.189L$$

For the same continuous beam with equal concentrated loads at mid-spans only, the flange width is 85 per cent of that obtained with a cosine moment curve.

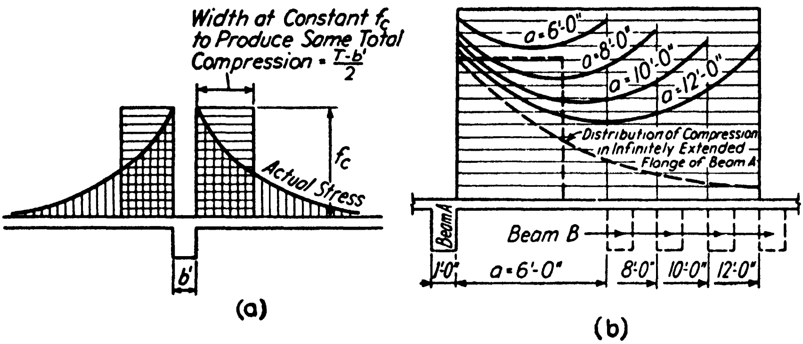


FIG. 14-1

For these cases the effective flange width either side of the stem varies from 16 to 19 per cent of the span. Actual structures differ by having a flange thickness that is not negligible as well as finite limits to the length and width of beam. The study in Fig. 14–1*b* shows a section of reinforced concrete floor and indicates in dotted lines the compressive distribution curve for an infinitely extended slab and the width of tee which at uniform distribution at maximum value would produce the same total compression. With customary beam spacings the distribution curves for adjacent beams overlap so that the compressive stress which is the summation of the effect of adjoining beams would be approximately as shown by the full lines in Fig. 14–1*b* for beam spacings, respectively, of 6, 8, 10, and 12 ft in the clear.

The A.C.I. and J.C. codes arbitrarily fix the overhanging width of flange as eight times the slab thickness or 12½ per cent of the span less one-half the stem width, or half the clear span of slab to the adjacent beam. The second of these requirements is in fairly good agreement with this theory.

14–3. Holes through Beam Stems. In buildings it is often necessary to provide holes through beams to permit the later installation of piping or other mechanical equipment. These are preferably made through the web of the beam in the central half of the span, and at or below the neutral axis. Some special reinforcement is required around

such openings to prevent their becoming the origin of diagonal tension cracks. As stress intensities at reentrant corners become very high it is well to make such holes circular where possible or else to fillet the corners to avoid a sharp, reentrant cut.

The stress situation about a round hole through a beam web below the neutral axis is shown in Fig. 14-2. The free body first taken for study consists of a square block, *abcd*, extending through the web and centering on the hole with sides sloping at 45° as shown in (*a*). On the

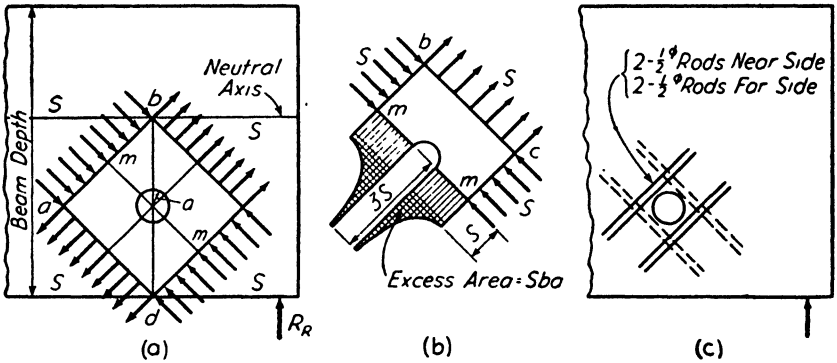


FIG. 14-2

assumption that the concrete below the neutral surface carries no tension and that this is accordingly a region of pure shear on horizontal and vertical planes, the stresses on the inclined faces of this block are tension and compression without shear, of intensity equal to that of the shear on a vertical plane. The unit tensile stress on the inclined plane *mm* through the center of the hole, Fig. 14-2*b*, is far from uniform, as is explained in all standard texts on strength of materials and mathematically demonstrated in texts on theory of elasticity, reaching a maximum at the edge of the hole of perhaps three times the average intensity. On a central section at right angles to *mn* the unit compression varies similarly to the tension shown.

The concrete of a beam web is usually sufficiently strong to carry the extra compression induced by the presence of a hole but reinforcement is required to carry the excess tension, the amount of which evidently equals the projected area through the hole multiplied by the normal stress intensity *s*. For this reason inclined reinforcing steel on two sides of the opening is needed and for practical reasons it is often better to reinforce all four sides with like amounts to avoid the possibility of the steel being incorrectly located on the compression sides of the hole only. In extreme cases this added steel may be helpful as compressive reinforcement.

Example 14–1. Design web reinforcement around a 24-in. diameter hole located below the neutral axis of a 22 by 96 in. balcony girder at a point where the effects of flexure need not be considered, and where the unit intensity of shear is 45 psi.

Solution. If $v_c = 60$ psi, no stirrups would normally be required. To make up for the hole, add

$$A_v = \frac{12 \times 22 \times 45}{16{,}000} = 0.75 \text{ sq in.}$$

on each side of the opening.

Four ½ in. round diagonal rods on each side of the hole (arranged in pairs in each face of the beam) provide 0.80 sq in. As previously pointed out, it might be well to put such groups of rods on all four sides of the opening to avoid placing errors. This is the minimum amount of steel possible, as no account is here taken of flexural stresses.

14–4. Variable Width of Flange. In the case of heavy girders, particularly in ribbed slab floor construction (see Art. 16–2), it is economical to vary the resisting moment along the length of the girder, keeping just outside of the bending moment curve at all points. This permits narrowing the tee and cutting off some of the tension steel away from the maximum moment point. Since the variation in tee width from section to section is very slight, the transverse stresses due to deflecting the total compression in each flange of the tee are relatively small; these can be considered in connection with Ex. 14–6.

Example 14–2. Design the bending up of tension reinforcement and variable width of flange for a girder carrying a live load of 200 psf on 12 in. plus 4 in. ribbed slab floor construction weighing 90 psf. Girders are 30 ft c to c and span 70 ft. Specification, 1940 J.C.: $f'_c = 3000$ psi; $f_s = 20{,}000$ psi.

Solution. As shown on Fig. 14–3*a*, the total load on the girder is computed as 805 kips, and the maximum bending moment as 84,530 k-in. A girder size of 36 by 75 in. is assumed and the shear is computed to see that the stem size is reasonable. The parabolic moment curve is plotted and the span is arbitrarily divided into 10 equal parts. At each section the moment, flange width, steel area, and number of 1¼ in. square bars is computed. On the sketches the steel pan forms that make the void spaces between the ribs are stopped just clear of the required flange width at any point. The tension steel was bent up in such a manner as to leave at any section the necessary number of bottom bars in the beam. Note that tension steel not needed should be bent up and anchored into the compression flange. It should not be extended past the theoretical stopping point and left straight in the bottom of the beam, as the end of the bar will be the starting point for a tension crack in the concrete.

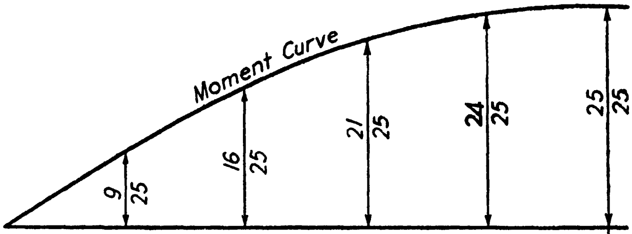
Although not a part of the moment computations, the provisions for web reinforcement are shown in Fig. 14–3*b* to illustrate heavy girder designs. Note that absolute maximum live shear is combined with normal dead shear. The actual effective depths are used in computing diagonal

Solution:

$LL = 200$ psf
$12+4 = \underline{90}$
$290 \times 30 = 8700$ plf
$DL = \underline{2800}$
11500 plf

$L = 70'\text{-}0''$
$W = 70 \times 11500 = 805$ k
$\frac{WL}{8} M = \frac{805 \times 70 \times 12}{8} = 84\,350$ k-in.
Try $36'' \times 75''$; $d = 69''$
$kd = 27.6''$

$v = \frac{402\,500}{36 \times \frac{7}{8} \times 69} = 186$ psi
$\frac{t}{d} = \frac{16}{69} = 0.232$
$R_{Allowable} = 200$
$k = 0.40$, $j = 0.90$



Moment k-in	30 500	54 200	71 100	81 300	84 530
Flange $= \frac{M}{Rd}$ in	32	57	75	86	89
$A_s = \frac{M}{f_s jd}$ sq in	24 6	43.7	57 5	65 5	68.2
No of $1\frac{1}{4}''$	16	28	37	42	44

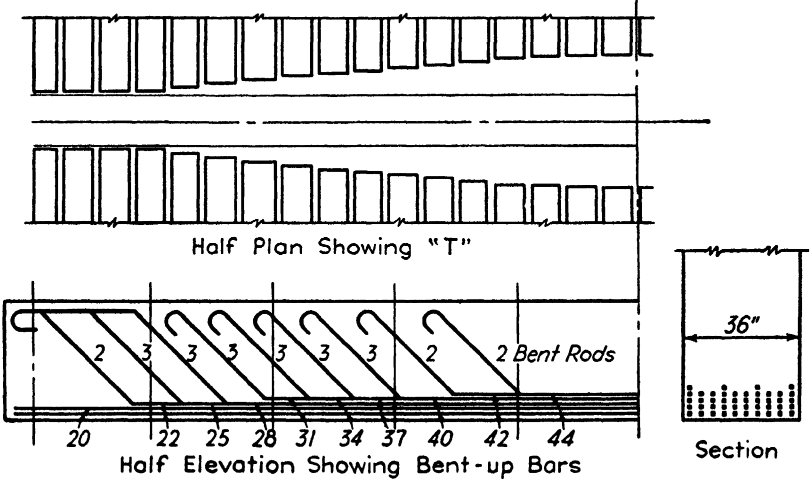


FIG. 14–3*a*

tension intensities. The concrete carries 90 psi with special anchorage. The value of the bent-up bars was next determined and, as their zones of influence overlap, they take care of the middle trapezoid. Stirrups are then added to care for the balance of the shear prism.

14–5. Holes through the Flanges of Tee-Beams. Wherever possible holes through the flanges of tee-beams should be avoided, especially holes close to the junction of stem and flange. The top fibers of a tee-

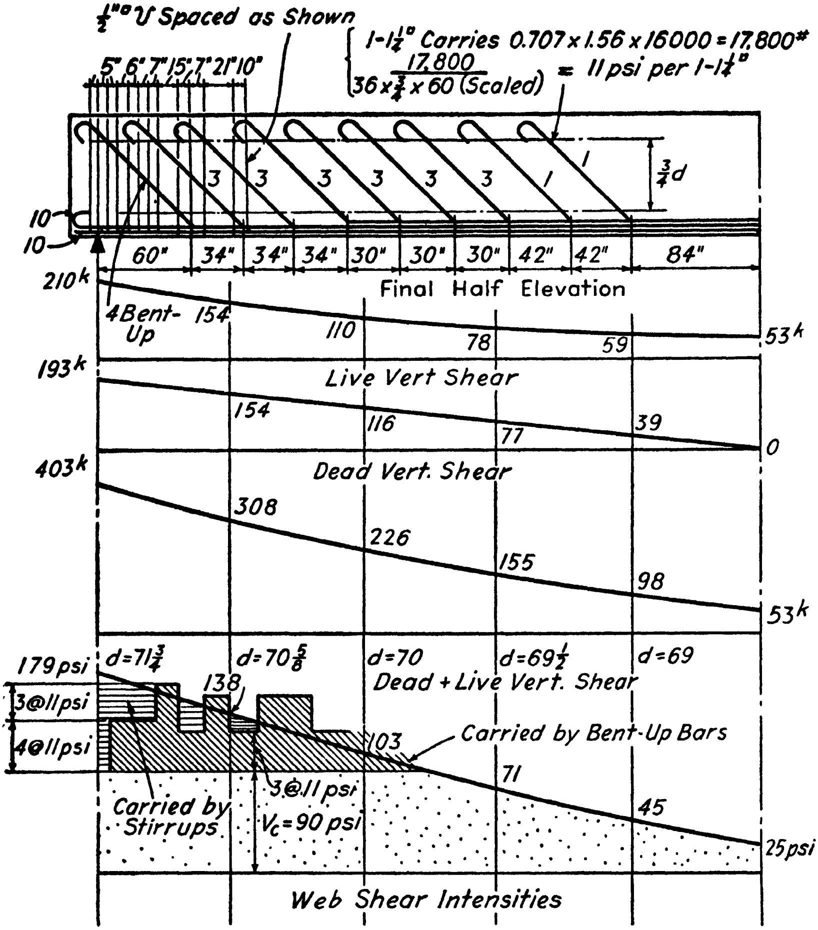


Fig. 14–3b

beam are often undergoing a compressive stress approximately equal to the safe working intensity. The presence of a hole develops high stress concentrations. Such high intensities are especially marked at reentrant corners and for that reason, if openings are unavoidable, their corners should be rounded or chamfered.

Fig. 14–4a shows in plan and elevation a tee-beam as part of a concrete floor system with a circular hole at or near mid-span. A block of concrete with sides parallel and perpendicular to the longitudinal axis of the beam is isolated in this figure and is shown with a longitudinal compression of varying intensity on two opposite faces, designated as

f_c on the top fibers. Then the mean intensity at mid-slab depth will be $\dfrac{f_c(kd - t/2)}{kd}$, which is called s in the figure.

Fig. 14–4*b* pictures the distribution of normal stress on a section *nn* through the center of the hole at right angles to the axis of the beam, and shows that the compression decreases rapidly from its maximum intensity of 3*s* at the edge of the hole asymptotic to the applied intensity *s*.

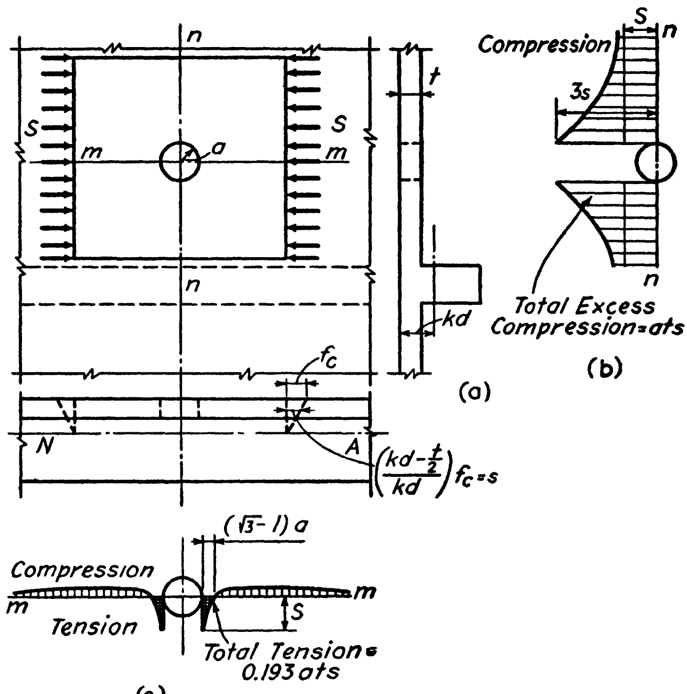


FIG. 14–4

Fig. 14–4*c* pictures the distribution of the normal stress on a section *mm* through the center of the hole parallel to the axis of the beam and shows that at the edge of the hole there is a *tension* of intensity *s*, which rapidly fades away to nothing and then passes to a compression whose maximum value is negligible. The total tension is equal to about 20 per cent of the projected value of the hole at the applied stress intensity *s*.

The total excess compression on a section at right angles to the axis is equal to the projected value of the hole at the average stress intensity at mid-slab depth. Some compressive reinforcement is desirable, but the exact amount is largely a matter of judgment. The designer must

decide what maximum amount of overstressing of the concrete in compression is permissible. In the corners of rigid frames, for example, experiments have shown that, although the theoretical compression is infinite, a redistribution of stress occurs and the neutral axis moves back into the member. For negative moments in beams it was long customary to allow higher compressive values for the very short peak of the moment curve than for positive moment at mid-span. For tied columns undergoing combined bending and flexure higher stresses are permitted than for direct compression alone. Since here this high intensity exists only at one point (the fiber stress decreases from the top down to the neutral axis, and decreases away from the hole laterally as shown by the rapid falling away of the curve, Fig. 14–4*b*), and since this condition will be relieved by plastic flow of the concrete, it is possible to use a maximum f_c higher than the $0.45 f_c'$ allowed for ordinary bending. The authors' practice is illustrated in the example below where the allowed value is chosen from experience in the light of the above facts. As more data are collected on this subject modifications may be made.

Considering the tension normal to the section parallel to the axis, reinforcement must be provided close to each side of the hole and at right angles to the beam axis, sufficient to carry a total tension equal to 20 per cent of the projected value of the hole at the mean applied stress intensity.

Both sets of reinforcement must extend far enough each side of the opening to be developed in bond and the compressive reinforcement must be adequately tied to prevent buckling. Usually fastening securely to the tension steel on either side of the hole, with a U-shaped anchor back into the slab on the center of the hole, will prove sufficient.

Example 14–3. In the beam of Ex. 14–2 it is desired to form a 12-in. diameter hole through the flange at mid-span, the edge of the hole to be 6 in. from the side of the web. Detail the necessary provisions for such an opening.

Solution. The best solution of this problem is to move the hole to some other location. Assuming that this is impossible, a combination of analysis and judgment suggests the following: The results of the study just completed indicate that the effect of a hole in the elastic slab might be very crudely pictured by substituting a trussed frame, as in Fig. 14–5*a*, to divert the total compression represented by the projected value of the hole at the average stress intensity at mid-slab depth. The slope of the inclined members is here assumed as 1 : 5 by reason of the foregoing analysis.

First, the tee should be widened 12 in. at mid-span to make up for the concrete displaced by the hole. This additional width should be accumulated gradually, say at a maximum rate of 1 : 5. At the extreme fiber we know from Ex. 14–2, $f_c = 1350$ psi; at the bottom of the flange $f_c = \dfrac{0.403 \times 69 - 16}{0.403 \times 69} \times 1350 = 574$ psi; at mid-slab depth average $f_c = 962$ psi. The projected

value of the half-hole is $6 \times 16 \times 962 = 92{,}500$ lb, the amount of compression to be taken care of on each side of the hole, corresponding to the stress in members *AC* and *BD* of the analogous frame of Fig. 14–5*a*.

For the reasons discussed in the preceding text, assume that a considerably higher average stress intensity, f_c, is here permissible to the extent that the compressive reinforcement is called upon to carry only one-half the projected value of the hole. Then $A'_s = (\frac{1}{2} \times 92{,}500/16{,}000) = 2.89$ sq in.; try three 1 in. squares = 3.00 sq in.

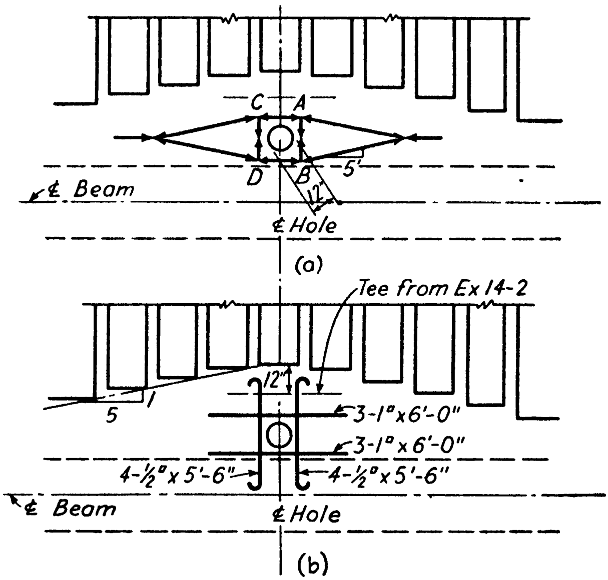


Fig. 14–5

The tension on the other two sides of the hole, corresponding to the stress in members *AB* and *CD* of the analogous frame in Fig. 14–5*a*, equals $0.20 \times 92{,}500 = 18{,}500$ lb. $A_s = 18{,}500/20{,}000 = 0.925$ sq in., which can be supplied by four ½ in. squares.

Fig. 14–5*b* shows the detailed arrangement of pan forms and reinforcing steel around the hole.

14–6. Beams with Unsymmetrical Flanges. It is not always possible to have symmetrical flanges on either side of a beam stem. Spandrels along the outside walls of a building can have a flange on the inside face but must be flush on the outside (Fig. 14–6*a*). Balcony girders are usually one-sided because of the seat steps (Fig. 14–6*b*). The ordinary theory of reinforced concrete beams is based on the limitation that the beam is symmetrical about the plane of loading. Then the beam deflects in the plane of loading and the neutral axis is perpendicular to that plane. In an unsymmetrical beam the total compressive force *C*

would not lie in the same vertical plane as the total tension force, T* (Fig. 14–6c), if the compressive stress were uniformly distributed across the flange according to the usual theory. In considering the unsymmetrical beam two cases must be distinguished: (1) a free-standing

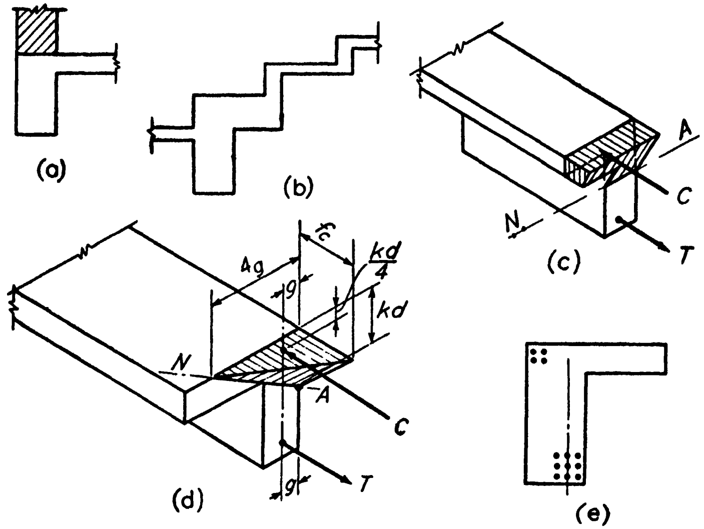


FIG. 14–6

beam, resting freely on its supports and having no cross-bracing of any kind, which deflects in a plane inclined to the plane of loading, in which case the neutral axis is not normal to the plane of loading; and (2) a braced beam constrained by connecting bracing to deflect vertically.

An unsymmetrical free-standing beam resting freely on its supports deflects in an inclined plane and so the compression must vary across the section as shown in Fig. 14–6*d*.

In " Der Eisenbetonbau " (Stuttgart, 1922, Vol. 1, p. 377) Professor E. Mörsch outlines the mathematical relationships involved for this stress distribution, but the case of a free-standing, unbraced unsym-

* The twist due to shear forces can be approximated by the method described for torsional effects on p. 289. The twisting of rolled steel channels under transverse loads is discussed by F. B. Seely, W. J. Putnam, and W. L. Schwalbe, in " The Torsional Effect of Transverse Bending Loads on Channel Beams," Univ. of Ill. Bul. 211, in which it is shown that for vertical deflection without twisting the load plane should pass through the " shear center " which, for a channel iron loaded parallel to the web, is on the axis of symmetry and distant from the back of the channel on the side away from the flanges an amount that ranges from ½ to ⅓ the flange width. For a steel angle iron this shear center is approximately at the intersection of the long center lines of the rectangles that form the two legs.

metrical tee-beam is so rare that the above suggestion should be sufficient for the unusual case.

If the beam is braced so that it can deflect vertically only, the neutral axis will be approximately horizontal and the compression will be approximately uniform on any horizontal line across the section. This bracing may be accomplished by connecting the beam to a stiff floor slab, by tying across to a symmetrical arrangement on the opposite side with cross beams or diaphragms, or by such diagonal bracing beams as may occur in a theater balcony.

Another method for eliminating twisting is to use compressive reinforcement in the upper corner near the flush side of the beam, located as far as possible from the unsymmetrical tee (Fig. 14–6*e*). Rarely can sufficient steel be used to balance the tee completely, even at the 16,000 psi suggested in Art. 7–9, but by coupling with the bracing effect of cross beams a good deal can be accomplished.

The following example illustrates the use of compressive reinforcement and careful detailing to neutralize as much as possible an unsymmetrical tee condition.

Example 14–4. A balcony girder, required by available space to have the outline shown in Fig. 14–7, carries a bending moment of 45,000 k-in. Determine (*a*) the tension steel area, (*b*) the desirable compressive steel area, and (*c*) the horizontal offset between the resultant compression and tension as an indication of the torque transmitted to the cross beams to maintain a constant f_c across the top row of fibers. Specifications, 1940 J.C.: $f'_c = 3000$ psi; $f_s = 20{,}000$ psi; but use $f'_s = 16{,}000$ psi for compressive reinforcement.

Solution. To illustrate the principles involved an approximate computation will be made, assuming that the arm of the internal couple is $\frac{7}{8}$ of 69 in. or 60 in.; that the neutral axis is $\frac{3}{8}$ of 69 or 26 in. below the extreme fiber; and neglecting any compression in the beam stem below the underside of the flange.

$$C = T = \frac{45{,}000}{60} = 750 \text{ k}$$

(*a*) $A_s = \dfrac{750}{20} = 37.5$ sq in. Twenty-four $1\frac{1}{4}$ in. squares = 37.44 sq in.

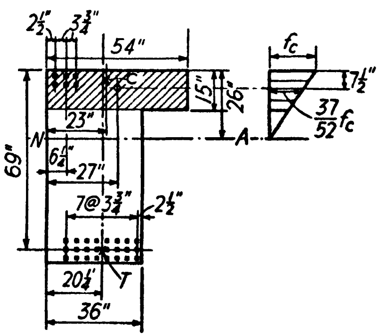


FIG. 14–7

(*b*) For compressive steel try six $1\frac{1}{4}$ in. squares as shown.

$$\begin{aligned} C_{total} &= & 750 \text{ k} \\ C_s &= 6 \times 1.56 \times 16{,}000 = & \underline{150 \text{ k}} \\ C_c &= & 600 \text{ k} \end{aligned}$$

If depth to neutral axis is 26 in., then mean compression at mid-slab depth $= 37f_c/52$, so

$$C_c = 54 \times 15 \times (37f_c/52) = 600 \text{ k};\ f_c = 1040 \text{ psi}$$

This shows that the concrete is not stressed to the limit of 1350 psi, and might suggest omitting the compression reinforcement, but note the following.

(*c*) The center of compression can be calculated since the compressive steel represents 150 kips centered 6.25 in. from the flush side, and the concrete represents 600 kips centered 27 in. from that side, resulting in 750 kips centered 23 in. from the flush side. The tension steel can be grouped as shown, centered 20.25 in. from the flush side. There is a horizontal projection of 2.7 in. between the center of compression and of tension as a measure of the inclination of the neutral axis. Without the compression steel this horizontal projection would increase to 6.75 in.

14–7. Flange Reinforcement. The flange of a girder (a beam supporting other beams) may consist of a portion of slab with span parallel to the stem, or of a block of concrete supplied for the sole purpose of resisting compression; in both cases, therefore, without reinforcement across the stem. Such cross-reinforcement must be added to resist the diagonal tension stresses at the junction of the flange and stem. The forces involved* may be understood by study of Fig. 14–8. There is a longitudinal shear on the vertical planes *ae* and *a'e'* (Fig. 14–8*a*), because the total compression C_f in one flange must pass through this surface in shear. It is shown in standard texts in strength of materials that the longitudinal shear per inch of length along plane *ae* for a homogeneous beam is $z = VQ/I$, V being the external shear at the section in question, Q the statical moment about the neutral axis of the flange area outside the plane *ae*, and I the moment of inertia of the cross section of the girder about its neutral axis. For a rectangular tee it can be shown† that the distribution of this longitudinal shearing force over section *ae* varies from 0 at the neutral axis to a maximum at the extreme fiber, and that at any height y above the neutral axis the intensity of longitudinal shear v_f on plane *ae* is $(V/I)\,[(T - b')/2]y$.

Fig. 14–8*b* shows an element from the top of the flange at junction of flange and stem, the element being taken so small that the obvious inequalities of stress intensities on opposite faces are ignored (for ex-

* For a discussion of this problem with application to the flanges of steel I-beams see Strength of Materials, by Professor George F. Swain, 1924, pp. 118–203.

† A demonstration of this can be made as follows, using Fig. 14–8*d*. On top of shaded element the shear per lineal inch of beam is: $z = VQ/I = (V/I)[(T - b')/2][(kd - y)][(kd + y)/2] = (V/I)\,[(T - b')/4][k^2d^2 - y^2]$. On bottom of shaded element: $z_1 = (V/I)[(T - b')/4]\,[k^2d^2 - y_1^2]$. The difference must go into the web or stem through the end of the shaded element, so $v_f dy = z - z_1 = (V/I)[(T - b')/4][y^2 - y_1^2] = (VQ_{\text{shaded element}}/I) = (V/I)[(T - b')/2]y\,dy$ and $v_f = (V/I)[(T - b')/2]y$. This demonstration applies only when T is constant.

ample, f_c on the near face really differs from f_c on the far face). The solution for the maximum diagonal tension or compression for the combination of the above is approximately $s = (f/2) + \sqrt{(f^2/4) + v_f^2}$.

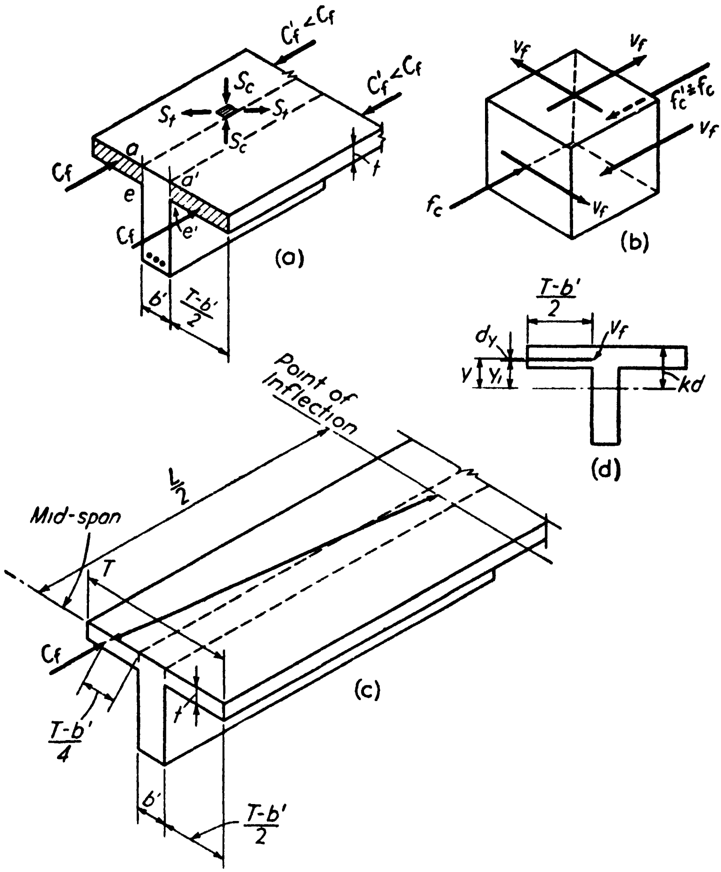


Fig. 14–8

It should, therefore, be a relatively simple matter to approximate this diagonal tension, and provide any steel required.

The problem is complicated by the fact that C_f acts through the center of gravity of the flange area, and so has a moment arm about a vertical axis in plane ae. Since there is no compression in the flange at the point of inflection, the flange acts as a beam cantilevered out from the vertical stem with its maximum compressive fiber at the point of inflection and its extreme tension fiber at mid-span. Thus, wide thin flanges on short spans will develop the most cross tension of this character. By re-

ferring to Fig. 14-8c and using the familiar relationship $f = M/S$, we can estimate this tension: $M = C_f(T - b)/4$ and $S = t(L/2)^2/6$; and so $f_t = 6C_f(T - b)/tL^2$. Fig. 14-8c shows the maximum compression in a girder flange diverted into the stem at the inflection point, and it is this diverting of the flange compression that requires cross-reinforcement. A thorough and exact analysis of this problem is difficult and unnecessary. The following examples will show the importance of the designer's recognizing the problem and will give some idea of a method of attack.

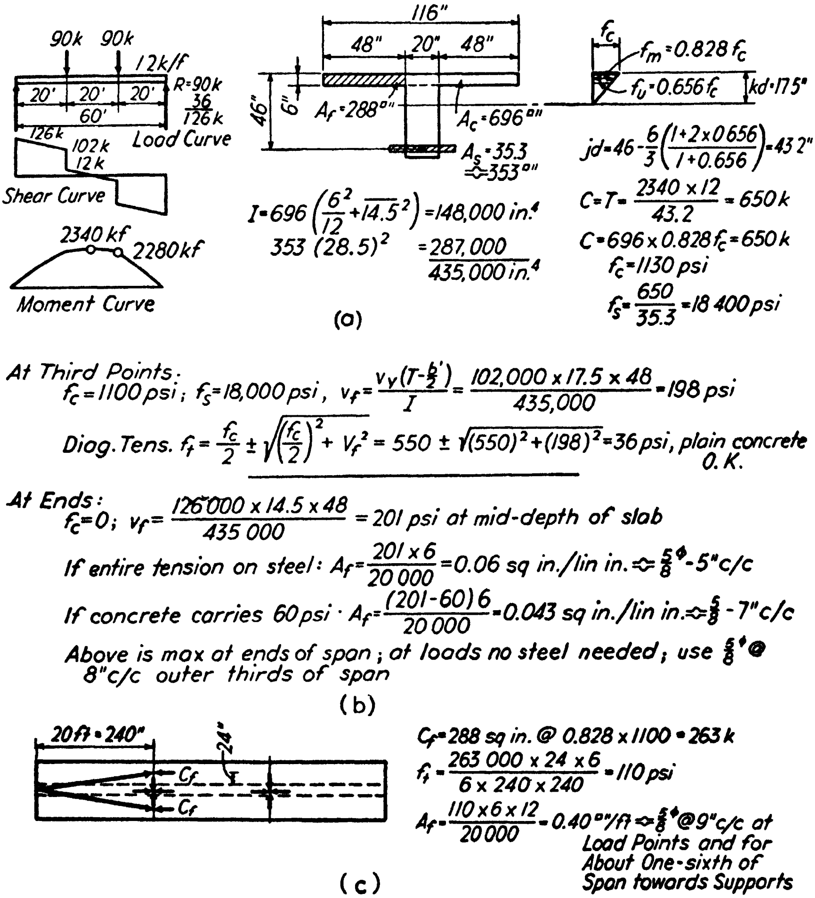


FIG. 14-9

Example 14-5. The beam of Fig. 14-9a carries concentrated loads of 90 kips at each third point of a 60-ft span. The beam weighs 1.2 klf. The size of beam, amount of tee, tension reinforcement, and properties are shown in

Fig. 14-9a. Determine the amount of cross-reinforcement required in the flanges. ($n = 10$.)

Solution. Using the formula $v_f = \dfrac{Vy\dfrac{(T - b')}{2}}{I}$ compute the intensity of longitudinal shear at the third point of the span = 198 psi. The extreme fiber stress in flexure is 1100 psi. The combination gives a diagonal tension of only 36 psi, which is no more than the concrete can safely stand. No cross-reinforcement is required at this point.

At the supports the intensity of longitudinal shear is 201 psi and, since there is no direct stress, the diagonal tension is of like amount, requiring 0.06 sq in. per inch of span if the steel is assumed to take all the tension, or 0.043 if the concrete is assumed to take 60 psi. The required steel is ⅝ in. round 5 in. c to c in the first case, or 7 in. c to c in the second. The cross-reinforcement varies from none at the third point to a maximum of ⅝ in. round 5 in. c to c at the ends. For ease in placing use an average of about ⅝ in. rounds 8 in. c to c.

A top view of the beam in Fig. 14-9c shows how the flange compression must be diverted to the stem in the outer thirds of the span. The total flange stress being 263 k, the tension in the bracket or cantilever beam formed by the third of the span is 110 psi, requiring ⅝ in. rounds 9 in. c to c.

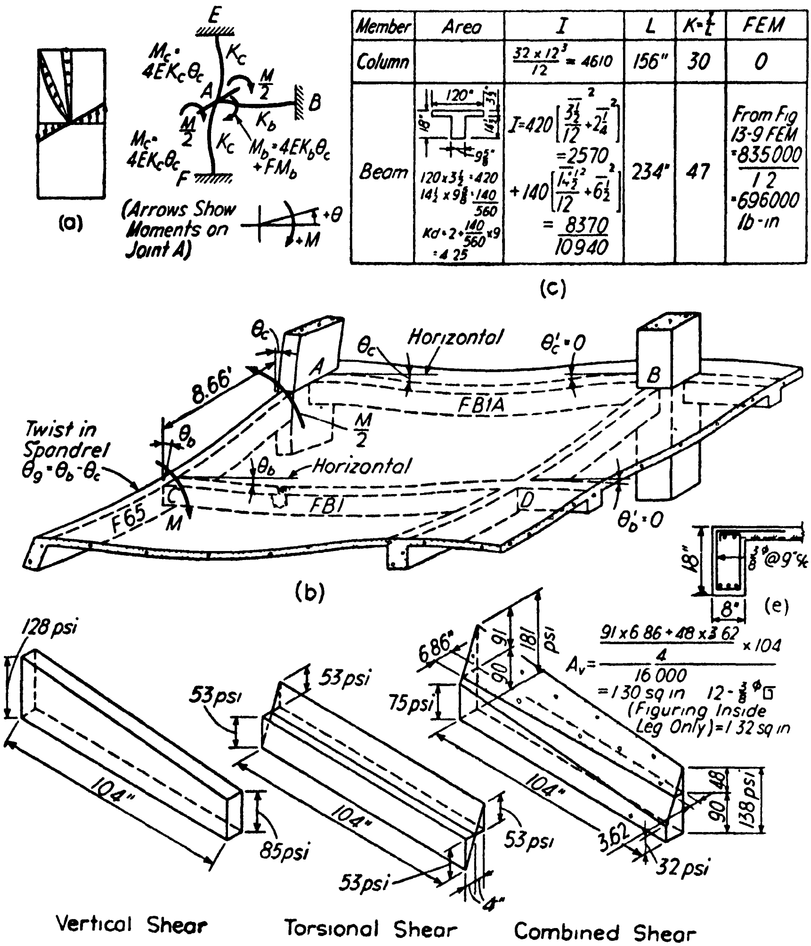
From this it appears that unless there is slab reinforcement of at least this amount crossing the beam stem at right angles, special steel must be added for the purpose of developing the tee. For convenience, in placing these rods they should be spaced uniformly across at least the outer thirds of the span.

Example 14-6. Refer to Ex. 14-2; determine the amount of cross-reinforcement required between flange and stem for the girder there shown.

Solution. This example is somewhat complicated by the variable width of tee. Consider the necessity of diverting the total flange compression C_f within the distance from mid-span to the point where the tee stops at $L/10$ from the support, a distance of 336 in. $f_c = 1350$ psi and $f_m = 19.6 \times (1350/27.6) = 958$ psi. $C_f = 26.5 \times 16 \times 958 = 407{,}000$ lb. Arm $= 26.5/2 = 13.3$ in. $M = 407{,}000 \times 13.3 = 5440$ k-in. $f_t = [(5{,}440{,}000 \times 6)/(16 \times 336^2)] = 18$ psi. $A_s = [(18 \times 12 \times 16)/16{,}000] = 0.22$ sq in. per foot of girder (½ in. square rods at 12 in. c to c = 0.25 sq in.).

14-8. Torsion on Marginal Beams. Marginal beams in monolithic construction, such as spandrel beams of buildings and beams around stair and elevator openings, receive their load from one side only and so, if restrained against rotation, are subjected to considerable torsion, the magnitude of which can be estimated from the fact that the end slope (θ_b, Fig. 14-10b) of the supported beam or slab equals the sum of the twist of the column supporting the marginal beam (θ_c), and the twist (θ_g) of the marginal beam between the restrained section, and that at the supported beam. Because the applications of this suggestion have not as yet been completely formulated approximate methods of analysis will be suggested in the following examples.

In a homogeneous beam of circular cross section the torsional shear varies from 0 at the centroid to a maximum of $v_t = 2M/\pi r^3$ at the circumference. In a rectangular homogeneous beam a plane cross section



Member	Area	I	L	$K = \frac{I}{L}$	FEM
Column		$\frac{32 \times 12^3}{12} = 4610$	156"	30	0
Beam	$120 \times 3\frac{1}{2} = 420$; $14\frac{1}{2} \times 9\frac{5}{8} = 140$; 560; $Kd = 2 + \frac{140}{560} \times 9 = 4.25$	$I = 420\left[\frac{3\frac{1}{2}^2}{12} + 2\frac{1}{4}^2\right] = 2570$; $+140\left[\frac{14\frac{1}{2}^2}{12} + 6\frac{1}{2}^2\right] = 8370$; 10940	234"	47	From Fig 13-9 FEM $= \frac{835000}{12} = 696000$ lb-in

Fig. 14–10

is warped somewhat on twisting and the torsional shear is distributed as shown in Fig. 14–10a, being a maximum at the center of the long side and approximately equal* to $v_t = [M(15h + 9b)]/5b^2h^2$. The

* See Fuller and Johnston, Applied Mechanics, John Wiley & Sons, Inc., 1919, Vol. II, p. 382, or Fred B. Seely, Resistance of Materials, John Wiley & Sons, Inc., 1935, p. 370.

values at other points are as indicated on the figure. Marginal beams in concrete construction are not simple rectangular sections but are L-shaped, the slab forming a flange on one side, the effect of the flange being to increase the torsional rigidity; but, since the deflection of the floor slab is an active agency of rotation, the problem becomes involved and in the following analyses the marginal beam will be treated as a simple rectangular section.

The torsional shearing stress on a section approximately normal to the longitudinal axis of the beam is equivalent to a tension and a compression at 45° with this section each equal in intensity to the shearing stress. In a brittle material such as concrete, fracture will begin at the middle of the long side and develop along a helicoid at 45° with the longitudinal axis progressively as the diagonal tension exceeds the ultimate tensile strength. Longitudinal corner rods are of no value in resisting these stresses and are useful only for spacing and supporting the ties. The most effective reinforcement would be a 45° spiral; the most practicable reinforcement for a horizontal beam would be to increase the number of vertical stirrups, or preferably closed ties, that are being used to resist the diagonal tension from vertical shear.

An approximate expression for the angle of twist between two cross sections,* taken from the above sources, letting the shearing modulus equal $0.45E$, is $\theta = [-8TL(b^2 + h^2)]/Eb^3h^3$. Values of the deflection angles in the columns and beams are adapted from the slope deflection derivations in Chapter XII.

Example 14–7. What provisions should be made for the effect of torsional shear on spandrel girder FG5 shown on Fig. 17–2, caused by the end slope of floor beam FB1, neglecting any assistance from the floor slab?

Solution. The deformation of the floor panel will be somewhat as illustrated in Fig. 14–10*b*. The slope of the exterior column will equal that of the exterior end of floor beam FB1A that frames into it, both being designated as θ_c. The slope of the exterior end of FB1 at the middle of FG5 will be greater, as the restraint of the column is not present. This slope is called θ_b. The twist of the girder within a length of 8 66 ft will be the numerical difference between θ_b and θ_c and is here called θ_g. For simplicity assume that FB1 is fixed at its interior end (slope = 0), and the columns are fixed at their far ends.

Formula 12–4 reads "$M_{\text{near}} = 2EK(2\theta_{\text{near}} + \theta_{\text{far}} - 3R) + FM_{\text{near}}$." For sign conventions see page 208, and also Fig. 14–10*c*. Applying this to FB1, the end moment at C is $M = 4EK_b\theta_b + FM$, so $E\theta_b = (M - FM)/4K_b$. This moment twists the spandrel FG5 and develops a resisting torque $M/2$ at the column face. Applying the same formula to the junction of FB1A and the column: $M_{\text{upper column}} = 2EK_c(2\theta_c)$, $M_{\text{lower column}} = 2EK_c(2\theta_c)$; $M_{\text{beam}} = 2EK_b(2\theta_c) + FM$. The sum of all the moments on joint A equals zero,

* See Paul Andersen, "Rectangular Concrete Sections under Torsion," *Journal, A.C.I.*, Sept.-Oct., 1937, an experimental study.

$2M_c + M_b + M = 0$ or $8EK_c\theta_c + 4EK_b\theta_c + FM + M = 0$; so $E\theta_c = (-FM - M)/(4K_b + 8K_c)$. From Art. 14-8, $E\theta_g = -8M_tL[(b^2 + h^2)/b^3h^3]$ where L is the distance between the planes of the couples, 8.66 ft in this case. Since the end moment in FB1 twists two symmetrical halves of FG5, $M_t = M/2$ and $E\theta_g = (-8ML/2)[(b^2 + h^2)/b^3h^3]$. Note that a negative sign indicates a clockwise rotation.

The values for moments of inertia and stiffness coefficients based on the gross concrete section, neglecting the reinforcing steel,* are evaluated on Fig. 14-10c, so we may write:

$$\theta_g = \theta_b - \theta_c - \frac{8ML(b^2 + h^2)}{2b^3h^3} = \frac{M - FM}{4K_b} - \frac{-FM - M}{4K_b + 8K_c}$$

from which

$$\frac{-(8 \times M \times 8.66 \times 12)(64 + 324)}{2 \times 512 \times 5830} = \frac{M - FM}{4 \times 47} + \frac{FM + M}{4 \times 47 + 8 \times 30}$$

or

$$-0.054M = 0.0053M - 0.0053FM + 0.0023FM + 0.0023M$$

$$M = 0.047FM$$

The fixed end moment is evaluated on Fig. 14-10c as +696,000 lb-in., so $M = +32{,}000$ lb-in.† and $M_t = 16{,}000$ lb-in.

The maximum torsional shear is $v_t = [(15 \times 18 + 9 \times 8)M_t/(5 \times 64 \times 324)] = 0.0033\, M_t = 53$ psi. This value is constant from the face of the column to the face of FB1. It causes diagonal tension on the inside face of the girder that must be combined with that resulting from the direct vertical shear; on the outside face the two are opposed and it is only necessary to consider the difference. Fig. 14-10d shows the vertical shear about as on page 383, the torsional shear from the preceding analysis, and the combination for which web reinforcement must be provided. Assuming special anchorage of the longitudinal steel and considering that no allowance has been made for the torsional rigidity of the girder flange nor the torque developed by the exterior brick veneer, it is felt that $v_c = 90$ psi may be safely used.

Thus twelve ⅜ in. stirrups are required on the inside face of the girder spaced nearly uniformly the full length. On the outside face none at all are required for the combined vertical and torsional shear but in no case should the number of stirrups in the outside face be less than that required for vertical shear only. The computer provides for the torsional effect by using ⅜ in. round closed ties securely fastened to longitudinal rods in the four corners of the girder, spacing them, say, 9 in. c to c.

Example 14-8. What provisions should be made for the effect of torsional shear on spandrel beam FB7, shown on Fig. 17-2, caused by the end slope of the floor slab FS1?

Solution This example is not so clean-cut as Ex 14-7, because the torque from the floor slab is applied to the spandrel in varying amount along its

* This practice is recommended for the design of rigid frames of reinforced concrete. See p. 406.

† If it is desired to use these results in a moment distribution study of FB1 it is only necessary to assume a column or member at the outer end with $K_c = (0.047/0.953)K_b = 0.05K_b$. Also the angle changes may be evaluated for $E_c = 3{,}000{,}000$ as $\theta_c = -0.00055$, $\theta_b = -0.0011$ and $\theta_g = -0.00055$, all in radians.

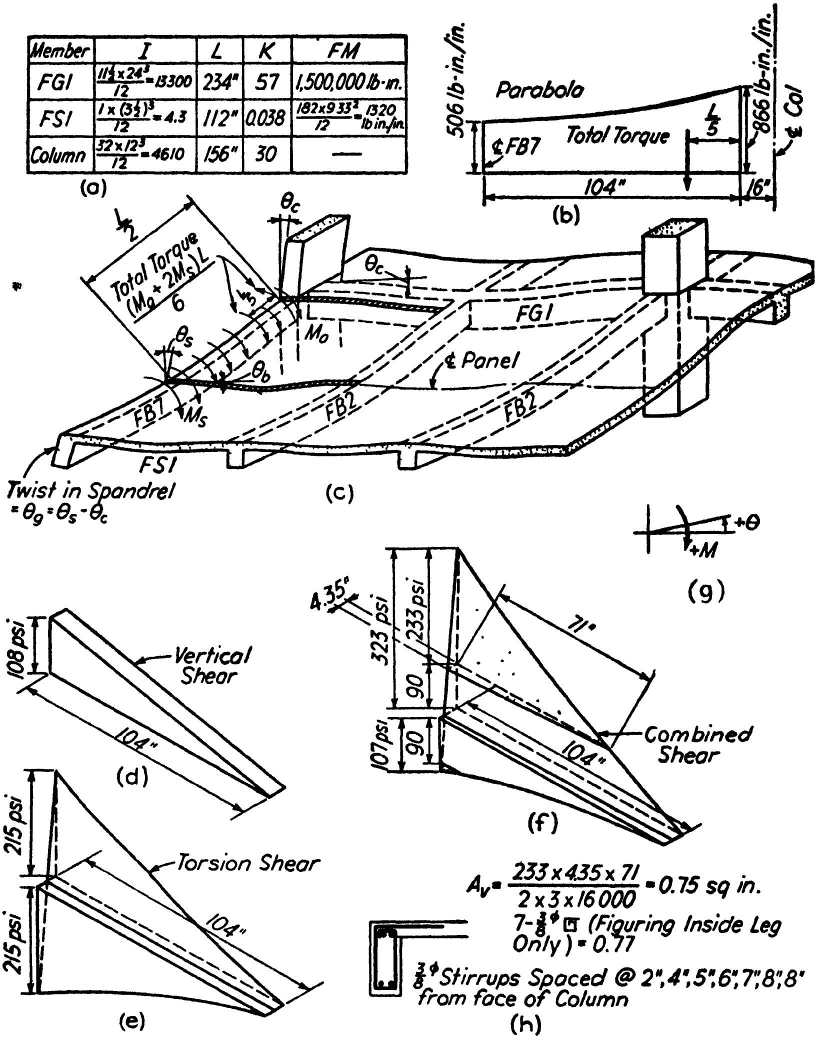
length, a portion of the slab near each column even adding to the joint restraint, and because the load spreads through the slab, some going directly to the columns instead of through the spandrels. The condition is roughly pictured in Fig. 14–11c. By neglecting the deflection of floor beam FB2, by assuming that the tangent to the deflected slab is a horizontal plane along the top of FB2, by overlooking the fact that the presence of the floor slab prevents free rotation of the spandrel because the top can move inward only the slight change in length of the deflected slab and all rotation must be outward at the bottom, and by making the approximation that the torque varies as a parabola from its value at the column to that at mid-span of the spandrel, some indication of the magnitude of the torsion can be arrived at by the method already used in Ex. 14–7.

If M_s is the end moment in that strip of slab 1 in. wide that frames on the center of the span of FB7 then, by applying formula 12–4 to this strip: $E\theta_s = (M_s - FM_s)/4K_s$, using the sign convention of page 208 and Fig. 14–11g. If M_a is the end moment in a similar 1 in. wide strip of slab immediately adjoining the column, $E\theta_c = (M_a - FM_s)/4K_s$. At the junction of FG1 and the column: $M_{\text{upper column}} = M_{\text{lower column}} = 4EK_c\theta_c$; $M_{\text{girder}} = 4EK_g\theta_c + FM_g$. The torque from two spandrels FB7 $= 2(M_a + 2M_s) \times L/6$, where L is the clear span of spandrel FB7. Placing the sum of the moments at the joint equal to zero; $8EK_c\theta_c + 4EK_g\theta_c + FM_g + (M_a + 2M_s) \times L/3 = 0$; so that $E\theta_c = [-FM_g - (M_a + 2M_s) \times L/3]/(4K_g + 8K_c)$. From Art. 14–8, if θ_g is the twist in the spandrel under a torque of mean intensity $(M_a + 2M_s)/3$ applied along a length $L/2 = 104$ in. (which is roughly equivalent to two couples concentrated about $L/5$ apart), then $E\theta_g = -8\left(\frac{M_a + 2M_s}{3}\right)\frac{L}{2} \times \frac{L}{5}\left(\frac{b^2 + h^2}{b^3h^3}\right)$.

We now have two unknown moments, M_s and M_a, but the latter is easily expressed in terms of the former because we have two expressions for $E\theta_c$ which can be equated: $(M_a - FM_s)/4K_s = [-FM_g - (M_a + 2M_s) \times L/3]/(4K_g + 8K_c)$. The values for the moments of inertia and stiffness coefficients are given on Fig. 14–11a so we may write: $(M_a - 1320)/(4 \times 0.032) = [-1{,}500{,}000 - (M_a + 2M_s) \times 208/3]/[(4 \times 57) + (8 \times 30)]$ or $7.8M_a - 10{,}300 = -3260 - 0.15M_a - 0.30M_s$, so that $M_a = 885 - 0.038M_s$. Since $\theta_g = \theta_s - \theta_c$ we have: $-\frac{4}{15}(M_a + 2M_s)L^2\frac{(b^2 + h^2)}{b^3h^3} = \left(\frac{M_s - FM_s}{4K_s}\right) - \left(\frac{-FM_g - (M_a + 2M_s)\frac{L}{3}}{4K_g + 8K_c}\right)$; from which $\frac{-4}{15}(988 + 1.962M_s) \times 208^2 \times$

$$\frac{(8^2+18^2)}{8^3\times 18^3} = (M_s - 1320)/(4\times 0.032) - \frac{-1{,}500{,}000 - (885 + 1.962M_s)\frac{208}{3}}{4 \times 57 + 8 \times 30}$$

or $-1330 - 2.98M_s = 7.8M_s - 10{,}300 + 3260 + 133 + 0.296M_s$, so $M_s = 506$ lb-in. per in. and $M_a = 885 - 19 = 866$ lb-in. per in. The torques applied to the spandrel are approximately as shown on Fig. 14–11b. Using $V_t = 0.0033M_t$ from Ex. 14–7, the torsional shearing intensities are as on Fig. 14–11e, the vertical shearing intensities as on Fig. 14–11d, and the combination of the two as on Fig. 14–11f, requiring stirrups as shown on Fig. 14–11h. For comparison with Ex. 14–7 the slopes have been evaluated for $E_c = 3{,}000{,}000$ psi, as $\theta_c \equiv 0.0012$, $\theta_s \equiv 0.0021$, $\theta_g \equiv 0.0009$, all in radians.



Member	I	L	K	FM
FGI	$\frac{11\frac{1}{2}\times 24^3}{12} = 13300$	234"	57	1,500,000 lb-in.
FSI	$\frac{1\times(3\frac{1}{2})^3}{12} = 4.3$	112"	0.038	$\frac{182\times 9.33^2}{12} = 1320$ lb in./in.
Column	$\frac{32\times 12^3}{12} = 4610$	156"	30	—

FIG. 14-11

14-9. Deflection of Beams. The deflection of a homogeneous beam is obtained from the well-known relation, $D = WL^3/kEI$, where k is a constant dependent on the type of loading and supports, values for which may be obtained from any text on the strength of materials.*

* A convenient summary is given in Formulas for Stess and Strain, by Raymond J. Roark, McGraw-Hill Book Co., 1938, pp. 92–101, and in the American Institute of Steel Construction Manual, 1941, pp. 358–369.

An accurate determination of deflection for a reinforced concrete beam is a difficult matter, even for a simple rectangular reinforced concrete beam resting freely on two knife-edge supports. Although it is safe, in computing stresses, to neglect tension in the concrete since some tension cracks will develop when the steel stress approaches its yield point, this simplification is not applicable in figuring deflections, because these are ordinarily desired at relatively low steel stresses before tension cracks have developed and, particularly, because the presence of small tension cracks would affect the deflections but little. This leads to the suggestion that it might be possible to use the deflection formulas for a homogeneous beam, taking for the moment of inertia that of the gross concrete section and neglecting the reinforcing steel entirely.* It so happens that the moment of inertia of the transformed section might not differ greatly numerically in many cases, but reasons for its use here are lacking.

As discussed on pages 36 and 72 the stress-strain curve for concrete is not rectilinear but more nearly parabolic.† The value for E varies with different stress intensities and we must have a fair idea of the stress range within which we are working before selecting a value for E. Another factor is the shrinkage of the concrete in setting. It has been shown, on page 109, that in a symmetrically reinforced section this shrinkage causes initial tension in the concrete and initial compression in the steel. In a beam reinforced near one face only, these shrinkage

* Such a procedure is obviously an approximation, as tests show that a heavily reinforced beam deflects less than a beam of the same size and same concrete but with less reinforcement. The practice is justifiable only when applied to actual designs carefully prepared, in which cases the beam is designed to carry the load at ordinary working stresses and with p varying only within a limited range. The subject of beam deflections is so complicated by the factors mentioned in this article and measured tests of actual structures are so limited that extreme refinement is hardly possible. Turneaure and Maurer, in Principles of Reinforced Concrete Construction, John Wiley & Sons, Inc., 4th ed., 1935, pp. 150–168, develop formulas for the various cases of reinforced beams in which the effect of the tension concrete below the neutral axis down to the level of the reinforcement is included. The effect of the fireproofing below the level of the reinforcement is omitted, probably merely to simplify the resulting formulas. Its effect can be compensated for by decreasing the assumed value of n a certain amount, say about 2.

George A. Maney, in Proc., A.S.T.M., Vol. XIV, 1914, p. 310, proposed the formula $D = (CL^2/d)(e_c + e_s)$ based on the geometry of the strain relations, and in Parcel and Maney, Statically Indeterminate Structures, pp. 48–51, John Wiley & Sons, Inc., varied it to read $D = (CL^2/d)(nf_c + f_s)/E_s$.

These studies should be read for the deflection of reinforced concrete members in flexure.

† The reason for this has been discussed by Glanville and by Shank who feel that the true elastic stress-strain curve under practically instantaneous loading would be a right line and that the falling away to a curve is a time effect due to plastic flow even during the short time interval of an ordinary compression test with deformation readings.

stresses are in the nature of an eccentric load, the unreinforced side shrinking naturally and decreasing in length whereas on the reinforced side shrinkage is largely prevented by the heavy reinforcement. Thus the member bends under the action of shrinkage with the reinforcement on the convex side, i.e., the curvature from shrinkage adds to that due to load. Precast concrete joists often show this curving due to shrinkage. A decreased value for E could be chosen that would allow for this and give the initial elastic deflection under load.

Another factor to consider is plastic flow, which has been discussed on pages 38, 83, 110, and which goes on for years under applied compression. As this effect is more or less proportional to the applied compressive stress intensity it is greatest on the extreme fibers of the compression side of the beam in the region of maximum bending moment and decreases gradually to the neutral axis, thus causing decrease of length on the compression side — an effect that adds to the deflection caused by load and by shrinkage. Hence the length of time the load remains in place enters the problem and the decreased value of E that is chosen should have a time factor, E decreasing with the length of time the load is continuously applied. This decrease is somewhat offset by the fact that the value of E taken from test cylinders shows a tendency to increase with the age of the specimen.

When the beam is part of a building other factors concern us. The beam is no longer free on the ends but part of a continuous structure. If the degree of end restraint can be approximated the value of k can be adjusted to suit, because constants for fully free or fully restrained ends are tabulated, and for partial fixity k can be expressed in terms of the slopes of the end tangents, or may be interpolated between the ideal conditions. Also, when a beam is part of a monolithic floor system the surrounding slab has an effect. The width of flange allowed on a tee-beam for stress computations (pages 77 and 275), is quite arbitrary and may not apply very well to deflection computations. Again, as the shoring is usually left in place for a considerable time, at least under main girders, the full effect of shrinkage may not develop as freely as in a casting yard. The plastic effects will be mainly those due to dead load, including any superimposed walls or partitions, as the live load is ordinarily of too short duration to develop plastic deformations.

Various reasons exist for wanting deflections. They may be desired as a guide to cambering the forms to compensate for probable deflections, as in the case of proscenium or other exposed girders where an appearance of sag is harmful; they may be required to allow suitable clearance, as for a spandrel girder over a plate glass show window where excessive deflection would crack the glass; they may be needed as a check upon

a loading test made to determine the quality of material, workmanship, or design; or they may be investigated to prevent excessive deflection from cracking the plastered ceiling below or the plaster of a superimposed partition wall. In the first two cases only a rough value is needed as extra clearance can be allowed. In the third case only the differential instantaneous elastic deflection under certain increments of load is necessary; for the last case the total ultimate deflection allowing for plastic flow and shrinkage is desired.

About the most satisfactory method is to use $D = WL^3/kEI$ for a homogeneous beam, taking W as the superimposed load, L the span center to center of bearings, k the deflection factor from the ordinary elastic theory for the given conditions of load and end restraint, I the moment of inertia of the gross concrete section, including as flange width that recommended by the codes for stress computations and adjusting E for the quality and age of the concrete with due allowance for shrinkage and plastic flow effects, using values that tests and experiments show are reasonable for deflection (not stress) computations and that have given good results. For purely elastic deformations due to a short-time live load, values of E may be taken as 3,000,000 psi, corresponding to values of $n = 10$; for long-time ranges, including plastic flow, lower values may be taken with $E = 1{,}000{,}000$ psi ($n = 30$). In this last case only such live load would be included as is likely to be permanently in place.

Example 14-9. Determine the probable full load mid-span deflection for purposes of camber of the girder shown in Ex. 14-2.

Solution. It can readily be shown that the total center deflection of a simple span is some 85 per cent affected by the central half and only less than 15 per cent affected by the outer quarters; also the 4 in. thick topping over the ribs acts as flange in addition to that specifically provided. Therefore this girder is idealized as a 36 by 75 in. beam with an 89 by 16 in. total flange. The moment of inertia of the gross section is computed:

$$\begin{aligned} \text{Flange area} &= 89 \times 16 = 1424 \text{ sq in.} \\ \text{Stem area} &= 36 \times 59 = \underline{2124} \\ \text{Total area} &= 3548 \text{ sq in.} \end{aligned}$$

$$kd = 8 + \frac{2124}{3548} \times 37.5 = 30.5 \text{ in.}$$

$$I_{\text{flange}} = 1424\left(\frac{256}{12} + 22.5^2\right) = 751{,}000 \text{ in.}^4$$

$$I_{\text{stem}} = 2124\left(\frac{59^2}{12} + 15^2\right) = \underline{1{,}095{,}000}$$

$$I_{\text{girder}} = 1{,}846{,}000 \text{ in.}^4$$

Elastic deflection for live load only: $\dfrac{6000 \times 70^4 \times 1728}{76.8 \times 3{,}000{,}000 \times 1{,}846{,}000} = 0.59$ in.

Elastic, shrinkage and plastic deflection for dead load only: $\dfrac{5500 \times 70^4 \times 1728}{76.8 \times 1{,}000{,}000 \times 1{,}846{,}000} = 1.61$

Total = 2.20 in.

On the basis of these figures the girder might be cambered 3 in. at mid-span. Upon stripping the forms only the elastic dead load deflection plus some of the shrinkage deformation will take place and will result in a total deflection of about ¾ in., leaving 2¼ in. of crown still in the girder. With the passage of time the deflection due to shrinkage will increase and the effects of plastic flow will become apparent so that a year after stripping the deflection may well be about 1¾ in. The dead load deflections should stabilize somewhere in this neighborhood. The deflection due to live load will seldom if ever be realized unless every square foot of tributary floor area is fully loaded to the designed intensity.*

* For comparison the deflection of this same tee-beam has been computed by Turneaure and Maurer's method:

$$\frac{b'}{b} = 0.405 \quad \frac{t}{d} = 0.232 \quad p = 0.0112$$

$$k = \frac{\dfrac{b'}{b} - \dfrac{b'}{b}\left(\dfrac{t}{d}\right)^2 + \dfrac{t^2}{d} + 2pn}{2\left(\dfrac{b'}{b} - \dfrac{b'}{b}\dfrac{t}{d} + \dfrac{t}{d} + pn\right)};$$

$$I_o = \tfrac{1}{3}\left[k^3 - \left(1 - \frac{b'}{b}\right)\left(k - \frac{t}{d}\right)^3 + \frac{b'}{b}(1-k)^3 + 3pn(1-k)^2\right]bd^3$$

For live load only, $n = 8$, $k = 0.49$ and $I_o = 2{,}250{,}000$ in.^{4}, using $E_c = 3{,}750{,}000$ psi, $D_l = 0.39$ in.; for dead load only ($n = 40$), $k = 0.67$, $I_o = 4{,}010{,}000$ in.^{4} and, using $E_c = 750{,}000$ psi, $D_d = 0.99$ in. Hence the total live and dead deflection $= 0.39 + 0.99 = 1.38$ in.

Using George A. Maney's formula, with $n = 8$, $D = \dfrac{1}{76.8}\dfrac{L^2}{d}\left(\dfrac{nf_c + f_s}{E_s}\right) = 1.4$ in. under full live load, which represents only the elastic deflection with no allowance for shrinkage or plastic flow.

CHAPTER XV

FOUNDATIONS AND FOOTINGS

15-1. The design of individual members needs to be extended to cover several specific constructions before starting the details of a complete building. One of the most important parts of any structure is the foundation or footings upon which it rests. This chapter considers the bearing capacity of soils and the design of individual and combined footings. Additional applications are shown in Chapter XVII.

15-2. Substructures. Nearly all engineering structures consist of two parts, the superstructure above and the substructure below the level of the ground. The foundation is that portion of the substructure whose function is the distribution of the load to the earth. In order to reduce the bearing pressure to proper limits it is necessary to spread out or enlarge that part of the foundation which is in immediate contact with the earth. The spread part of any foundation unit is called a footing.

Concrete is in universal use for the foundation of all types of superstructures. It is the function of this chapter to consider briefly the application of the simple principles of reinforced concrete design, already outlined, to the relatively massive members used as the supports of buildings. Detailed consideration of the carrying capacities of different soils and the methods for determining them is beyond the scope of this text.

15-3. Foundations of Buildings. The foundation of a building consists usually of spread footings beneath the interior columns and either local footings or continuous wall footings beneath the exterior columns, the loads being transmitted to the soil at a level only a short distance below the basement floor. When the bearing stratum lies so far below the usual footing level that the cost of excavating open pits is prohibitive, it becomes necessary either to use piles which rest upon the harder underlying material and act as columns, or to drive shafts or caissons, 3 ft or more in diameter depending upon the depth and manner of sinking, which are sometimes enlarged at the bottom and filled with concrete, forming a supporting pier.

When the supporting stratum of soil is of low bearing capacity for a considerable depth, it was until recently, usually considered necessary

to increase this capacity, and also the area of distribution of the loads, by means of piles which transferred their load to the surrounding soil by friction. The present judgment increasingly is against this use of piles. As an alternative, on soft soil a single bearing surface may be employed for the whole structure, forming a floor or raft upon which all the basement walls and columns rest. The design of such a mat foundation is the same as that of a beam and girder or a flat slab floor.

The bottom of a footing should always be below the level affected by frost action, a distance which varies with the locality. It should also be remembered that the deeper the footing the greater the consolidation of the soil and consequently the greater its bearing power.

15-4. Soil Resistance. All soils are compressible and so all structures settle to a greater or less degree unless they are founded on solid rock. The allowable bearing pressure on a soil is set at such a value that settlement will be limited to a reasonable amount, a small fraction of an inch or several inches, according to the locality, the nature of the soil, and the type of structure.

The settlement of all the different parts of a building must be closely the same in order that cracking of walls and plaster and structural damage to the frame be avoided. This becomes a matter of increasing importance and difficulty with the poorer soils where the settlements to be expected, and consequently the variations in settlement, are greater. By far the larger part of all settlement is caused by the dead load of a structure, a factor which can be computed with great accuracy. Live loads have practically no effect on settlement unless they are long continued, but of course they have the same effect on the stresses in the footings as any other load. It is important, accordingly, that the unit bearing pressure on the soil under all footings of a building under dead load be such that equal settlement of all parts may occur. If the bearing capacity of the earth under one part of a structure differs from that under the rest, careful studies of the relative capacities must be made and bearing pressures used that will result in equal settlement. In some types of buildings where the live loads are certain of realization for long-continued periods, a portion of the live load is added to the dead load and footings are designed for proportional unit pressures under the combination. A fact to be kept in mind when deciding upon the fraction of live load to be so used, and the portions of the structure where live load concentrations are to be expected, is that a large proportion of the total settlement usually has taken place by the time the building is ready for occupancy. Another factor to be considered here is the reduction to be made in live load on girders and columns.

It has been customary to determine soil capacity to support vertical

loads within allowable deformations or settlements by experience with local soils or by tests made by placing increments of load on top of a vertical 12 by 12 timber in a shallow pit and measuring settlements. Obviously experience is a sound guide but the method of judging bearing

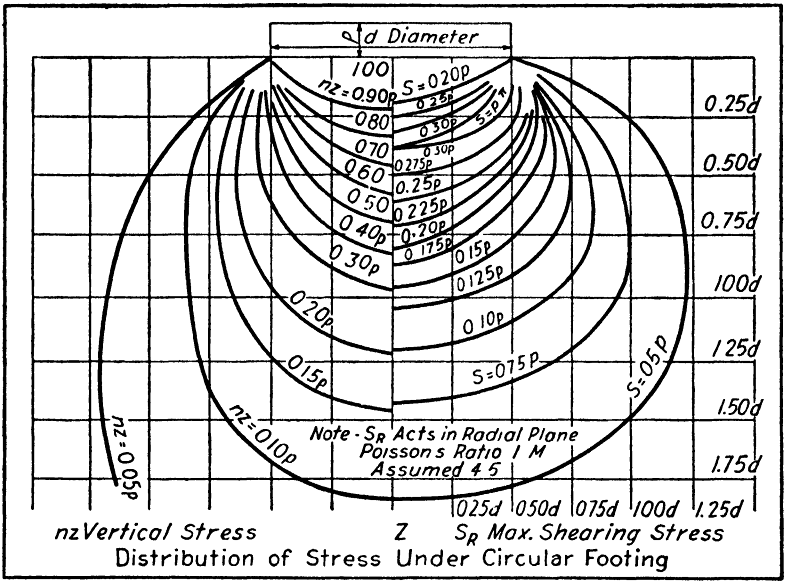


FIG. 15–1 (from *Journal of the Boston Society of Civil Engineers*, July, 1934).

capacity by the action of a single square foot of loaded area may be entirely misleading and inadequate. The extent of the loaded area determines the depth to which important stress effects are transferred, as is shown by the stress contours of Fig. 15–1. A small loaded area affects the bearing strata only to a small depth; a large area affects the strata to a relatively greater depth. Since only rarely will it occur that the supporting beds of material beneath a foundation are homogeneous for a great depth, a wide foundation will carry stress effects to depths where the material may vary importantly from the upper levels reached by loading tests. Where a soft layer underlies one that is hard the unit load determined as proper for the test area may give sufficient stress at lower levels to cause excessive settlements due to the yielding of the deep bed. It is plain, therefore, that the only ways to determine bearing capacity where experience is lacking as guide are (*a*) to make exploratory borings with the collection of undisturbed soil samples to sufficient depth and to test these samples in the laboratory to determine

action under stress, and (b) to make a series of load tests at various depths.

Reference has already been made to the great advance in understanding of soil action made within the last few years following the pioneering studies of Dr. Karl Terzaghi,* which initiated wide activity in this field of exploration. Of the outcome of these investigations the remark of Dr. Glennon Gilboy† is significant: "Foundation problems cannot be resolved into the simple task of assigning to a given soil a certain bearing value per unit area. On the contrary a study of the action of soil as a supporting material involves mechanics of a high order of difficulty and a vast amount of research will have to be performed before anything resembling a simple solution can come into being." It is accordingly easy to understand why many controversial points remain and why competent engineers differ in their manner of handling foundation problems. In this text it is impossible to give even a brief summary of the new science of soil mechanics as related to foundation design. The references previously given will direct the reader further.

Soil consists of "the unconsolidated products of rock disintegration "‡ and is usually classified on the basis of particle size: gravel and sand at one extreme, and colloidal clay at the other. The relatively coarse particles of sand and gravel do not tend to cohere; clay is strongly marked by the coherence of the extremely small particles. The action of these two materials under load varies markedly and there is some controversy regarding the nature of the phenomena and their significance for engineering practice.

The bulb of pressure shown in Fig. 15-1 gives the ratio of vertical pressure intensities at various depths beneath the footing in terms of the average directly at the footing base. However, this base intensity will not be uniform but will vary from the edges to a maximum at the center for a foundation on granular material. If the footing is on clay and is very stiff in itself the pressure here may approach uniformity or may be a maximum at the edges and relatively low in the center.

Under the group of spread footings supporting a building the individual bulbs of pressure tend to coalesce and at some depth the group acts more or less as a unified bearing surface with a single pressure bulb. The variation of pressure above described for a single footing tends to be

* Charles Terzaghi, The Science of Foundations, Trans., A.S.C.E., Vol. 93, 1929; C. A. Hogentogler, Engineering Properties of Soils, McGraw-Hill Book Co., 1937. For other references see footnote, p. 147.

† Trans., A.S.C.E., Vol. 98, p. 232.

‡ C. C. Williams, "Foundations," in Civil Engineering Handbook (Urquhart), McGraw-Hill Book Co., 1934.

repeated for the group. This is shown by the fact, quite commonly neglected in practice, that for a building on sand with equal pressure intensities under all footings the settlement at the interior columns will be somewhat less than for the wall columns; for a building on clay the reverse will be true.

These and other phenomena too complicated to permit of description here are not reflected in current design procedure, although some specialists engaged for the foundation design of important structures go minutely into the situation here so faintly suggested. Current procedure is to determine a safe bearing pressure for the soil at hand and use that value as hereafter described. However, the modern engineer no longer uses such values haphazardly and he is alert to see that all construction procedures avoid action on the supporting strata which might affect their bearing capacity adversely.

The actual procedure in computing footings is as follows: The bearing area of the footing under the column with the smallest proportion of dead load (one of the interior columns) is calculated by dividing the total load on the soil (column load, total live and dead, plus the weight of the footing) by the allowable bearing pressure for the earth. Next the unit pressure under this footing is determined for dead load only (or for dead load plus one-half the live load, the usual proportion chosen when live load is included). Then all other footing areas are determined by dividing their dead load totals by this computed dead load unit bearing. *Query*: Why not use dead load bearing intensity under a wall column as criterion? Would the area for an interior footing thus determined be sufficient for the total load, live plus dead?

All footings should settle evenly without tipping and this requires a symmetrical distribution of pressure on the base. Though the distribution of pressure on the bearing area of a footing may vary as just discussed, it is assumed that if the resultant of all the applied loads passes through the center of gravity of the bearing area, the distribution figure will at least be symmetrical and there will be no tendency for the footing to rotate. This is in conformity with known facts for footings that are regular polygons. Such an assumption would not be justified in the case of a footing having a reentrant angle that removes a good portion of the bearing area, as the centroid of the bearing area might well lie outside the area itself. Such a condition would occur on an irregular plot of ground where a mat type of foundation was used. In such a case more accurate studies along the lines previously suggested would be required. To prevent rotation, the footing under a single column which carries only vertical loads should be concentric with the column. A footing of this sort is shown in Fig. 15-6.

A combined footing is one that supports two or more columns. The line of action of the resultant of the dead column loads (or dead loads plus a fraction of the live) should pass through the centroid of the base of the combined footing. A common type is illustrated in Fig. 15-2.

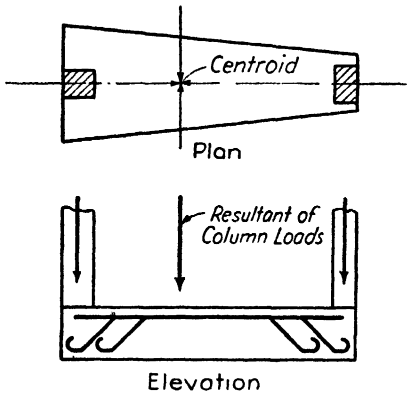


FIG. 15-2

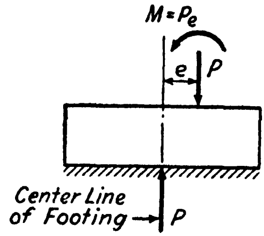


FIG. 15-3

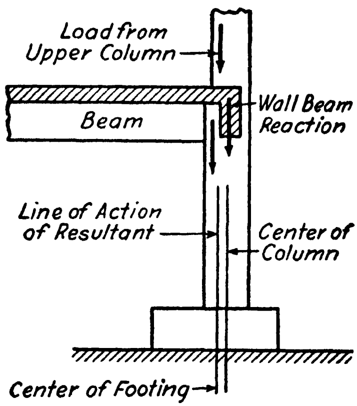


FIG. 15-4

A wall column is usually subjected to bending as well as to direct compression and so its footing should not be concentric with the column. This moment can be estimated or computed and the footing centered so that the resultant of an assumed uniform base pressure and the axial column load form a couple equal and opposite to the moment. This is illustrated in Fig. 15-3. A more common method is to estimate the lines of action of the several loads carried by the lower column above the footing and find the line of action of their resultant, which is the required center line of the footing. This is illustrated by Fig. 15-4.

Beam reactions are usually assumed to act at from one-quarter to one-third the depth of the column from the inside face, or a distance equal to one-half the depth of the beam if that is less. Any error of assumption that places the center of the footing inside the actual resultant load results in a tendency to tip so that the column leans out from the building, causing cracks, so it is desirable that the load resultant in Fig. 15-4 act a little inside the center of the footing; this results in tipping that tends to force the building together. Most conservative designers place the center of gravity of corner footings 3 or 4 in. outside of the resultant of the load so that all corner columns tend to tip in against the building from both directions.

15-5. Bearing Capacity of Soils. A table of safe bearing values of soils can be only a very approximate guide as to the proper value to choose in any particular case unless the table has been constructed from the records of experience of the given locality and covers very definitely the conditions at the site. There are many factors that affect bearing value, such as those of composition, moisture content, and degree of confinement laterally. Up to the present we have had no standard of measure by which it was possible to give an accurate description of all the essential properties of soil. At least the stress-strain curve, the compressibility, elasticity, and stability* must be known if we are to have any competent idea of how a soil will behave under loads.

In considering any specification for soil pressures it is necessary also to consider the allowable live load reductions. In some building codes extreme conservatism in one of these matters is offset by liberality in the other. In the Code of the City of Boston extremely large reductions of the column loads are permitted but the allowable soil values are not unusual. The Boston rules are reprinted below as they give a much better definition than is usual for the various kinds of soil.

CLASSIFICATION OF FOUNDATION-BEARING MATERIAL

Section 2904, Boston Building Code

(1) **Rocks:**

Shale;— A laminated, fine-textured, soft rock composed of consolidated clay or silt, which cannot be molded without the addition of water, but which can be reduced to a plastic condition by moderate grinding and mixing with water.

Slate;— A dense, very fine-textured, soft rock which is readily split along cleavage planes into thin sheets and which cannot be reduced to a plastic condition by moderate grinding and mixing with water.

Schist;— A fine-textured, laminated rock with a more or less wavy cleavage, containing mica or other flaky minerals.

* Hogentogler, *op. cit.*, p. 167.

(2) **Granular earth:**

Gravel; — An uncemented mixture of mineral grains one-quarter inch or more in diameter.

Sand; — A type of earth possessing practically no cohesion when dry, and consisting of mineral grains smaller than one-quarter inch in diameter.

Coarse Sand; — A sand consisting chiefly of grains which will be retained on a 65-mesh sieve.

Fine Sand; — A sand consisting chiefly of grains which will pass a 65-mesh sieve.

Compact Gravel, Compact Sand; — Deposits requiring picking for removal and offering high resistance to penetration by excavating tools.

Loose Gravel, Loose Sand; — Deposits readily removable by shoveling only.

(3) **Cohesive earth:**

Hardpan; — A thoroughly compact mixture of clay, sand, gravel, and boulders, for example boulder clay; or a cemented mixture of sand or of sand and gravel, with or without boulders, and difficult to remove by picking.

Clay; — A fine-grained, inorganic earth possessing sufficient cohesion when dry to form hard lumps which cannot readily be pulverized by the fingers.

Hard Clay; — A clay requiring picking for removal, a fresh sample of which cannot be molded in the fingers, or can be molded only with the greatest difficulty.

Medium Clay; — A clay which can be removed by spading, a fresh sample of which can be molded by a substantial pressure of the fingers.

Soft Clay; — A clay which, when freshly sampled, can be molded under relatively slight pressure of the fingers.

Rock Flour (Inorganic Silt); — A fine-grained, inorganic earth consisting chiefly of grains which will pass a 200-mesh sieve, and possessing sufficient cohesion when dry to form lumps which can readily be pulverized with the fingers.

Foundation Bearing Values

Section 2904 & 2905, Boston Building Code

Class	*Material*	*Allowable Bearing Value tons per sq. ft.*
1	Massive bedrock without laminations, such as granite, diorite, and other granitic rocks; and also gneiss, trap rock, folsite and thoroughly cemented conglomerates, such as the Roxbury Puddingstone, all in sound condition (Sound condition allows some cracks)	100

Class	*Material*	*Allowable Bearing Value tons per sq. ft.*
2	Laminated rocks such as slate and schist, in sound condition (some cracks allowed)	35
3	Shale in sound condition (Some cracks allowed)	10
4	Residual deposits of shattered or broken bedrock of any kind except shale	10
5	Hardpan	10
6	Gravel, sand-gravel mixtures, compact	5
7	Gravel, sand-gravel mixtures, loose; sand, coarse, compact	4
8	Sand, coarse, loose; sand, fine, compact	3
9	Sand, fine, loose	1
10	Hard Clay	6
11	Medium Clay	4
12	Soft Clay	1
13	Rock flour or any deposit of unusual character not provided for herein	Value to be fixed by the Commissioner

(*b*) The tabulated bearing values for rocks of Classes 1 to 3 inclusive shall apply where the loaded area is less than two feet below the lowest adjacent surface of sound rock. Where such depth is two feet or more these values may be increased ten per cent for each foot of such depth, up to a maximum of twice the tabulated values.

(*c*) The allowable bearing values of materials of Classes 4 to 9 inclusive may exceed the tabulated values by two and one-half per cent for each foot of depth of the loaded area below the lowest ground surface immediately adjacent, but shall not exceed twice the tabulated values. For areas of foundations smaller than three feet in least lateral dimension, the allowable bearing values shall be one-third of the tabulated values, multiplied by the least lateral dimension in feet.

(*d*) The tabulated bearing values for Classes 10 to 12 inclusive apply only to pressures directly under individual footings, walls, and piers. When structures are founded on or are underlain by deposits of these classes, the total load over the area of any one bay or other major portion of the structure, minus the weight of excavated material, divided by the area, shall not exceed one-half the tabulated bearing values.

(*c*) Where the bearing materials directly under a foundation overlie a stratum having smaller allowable bearing values, these smaller values shall not be exceeded at the level of such stratum. Computation of the vertical pressure in the bearing materials at a given depth below a foundation shall be made on the assumption that the load is spread uniformly on a horizontal plane under the foundation and extending one-half that depth beyond the edges thereof; but such planes under adjacent foundations shall not over lap.

For all important structures studies should be made of the foundation material by means of borings and test loads.

To guard against unequal settlement when part of the footings of a structure rest on rock, it is common to specify that the soil value for the remaining footings be reduced one-half.

Basement floors often rest directly on the soil. Although their load is relatively light there will be large settlement on compressible soils and the resulting failure of the floor may be expensive to repair although not dangerous to the building. If the soil is quite compressible or if cracking of the slab is serious, as in the case of waterproofed structures or the insulated slabs of cold storage warehouses, basement floors are often designed as self-supporting two-way or flat slabs spanning from footing to footing, properly reinforced and using the subgrade only as a temporary form.

In multi-story construction the footings become quite large so that a considerable area of basement floor and backfill rests directly on top of them. Some designers increase the footing area by including the reduced live load on the basement floor, the weight of the floor itself, and the weight of backfill on top of the footings in the load for which the footing is designed. This load is transmitted directly through the footing and affects only the soil pressure. The authors prefer to accomplish the same thing by using a safe bearing pressure that has already been reduced by a suitable allowance for loads on the basement floor. This simplifies the computations and produces comparable results.

Footings and floors are sometimes subject to uplift due to water pressure. This seldom needs to be considered in footing design except for hydraulic structures. It is often of importance in designing basement and pit floors. In designing for water pressure when a non-flexible waterproofing is to be used, it is essential that steel be used to prevent any stress cracks, considering the floor or the pit as a monolith.

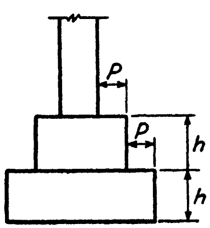


FIG. 15-5

15-6. Footings of Plain Concrete. For large footings reinforcement is usually an economy, but plain concrete is used with advantage for the footings of many small structures. The ratio of projection p to height of course (h, Fig. 15-5) is commonly set by rule of thumb, such as $\frac{1}{2}$ for all soil values. A projection of 7 in. horizontal per foot vertical is often used. This method is not logical since the plain concrete section acts like a cantilever beam and the allowable ratio of offset to height is a function of the load,

which for the lowest course is the soil pressure. This ratio is ordinarily limited by the tension in the concrete, frequently taken as $0.03\,f_c'$ (J.C. 867*a*), which gives ratios of projection to height varying from ½ to 1 for the usual range of soil presssures.

15–7. Reinforced Concrete Footings. Experiments at the University of Illinois under the direction of Professor Arthur N. Talbot* from 1908 to 1912 are the basis of modern practice in footing design.

Wall footings are simple cantilever slabs projecting on each side of the wall. Footings for isolated piers are square or rectangular slabs, concentric with the column, made uniform in thickness or with a sloping or stepped top. Such a slab is essentially a system of radial cantilevers projecting from the pier and is designed as a two-way cantilever with reinforcement parallel to the sides. All footings are relatively short, heavily loaded beams and therefore shear and bond are of proportionately greater importance than in ordinary slabs.

Shear, on a vertical surface which is the extension of the faces of the column down through the footing, has been much used in the past as a criterion for the total depth required for footings under columns and piers. This shear is called punching shear and is often regarded as pure shear unaccompanied by tension. The allowed value commonly was 6 per cent of the ultimate strength of the concrete. The use of punching shear in footing design is not mentioned in the latest A.C.I. and J.C. reports and it is quite generally recognized that former practice was unnecessarily conservative in this regard.

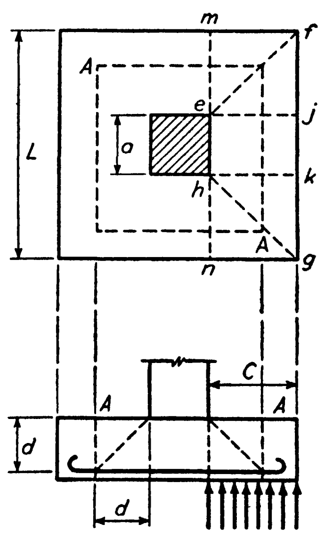


FIG. 15–6

The experiments at the University of Illinois proved that the critical section for diagonal tension lies at a distance from the face of the wall or column equal to the depth from the top of the footing to the steel (plane AA, Fig. 15–6). For this reason shear as a measure of diagonal tension is measured on a vertical surface at this section. Since it is difficult and expensive to use diagonal tension reinforcement in footings the diagonal tension shear is kept low. If this shear is kept within $0.02f_c'$ to avoid diagonal tension reinforcement it is usually the factor which determines the minimum depth of footing; hence the following relation

* Bul. 67, Univ. of Ill. Engineering Experiment Station

between the depth of a flat-topped footing and its other proportions is useful.* The total shear on four sides of the block determined by section AA in Fig. 15–6 is $w[L^2 - (a + 2d)^2]$, and $v_c = \frac{w[L^2 - (a + 2d)^2]\text{psi}}{4(a + 2d)jd144}$, L, a, and d being in feet. Let $j = 7/8$, $a/L = k$, and $w/504v_c = C$, then:

$$\frac{d}{L} = \frac{\sqrt{2C + 4C^2 + \frac{k^2}{4}} - \frac{k}{2}(1 + 4C)}{2 + 4C} \qquad [15\text{–}1]$$

In wall footings the critical section for bending is at the face of the wall. For column footings the critical section for moment also is at the face of the supported member except in the case when a metal base plate is used, when the theoretical section of maximum moment is taken as halfway between the face of the column and the edge of the plate (J.C., 866*a*). The bending moment to be resisted by the steel crossing the *mn* plane in the column footing, shown in Fig. 15–6, is that due to the upward pressure on the trapezoid *efgh*.† The assumption of uniform pressure over the whole area results in an unnecessarily large moment and it is proper to assume non-uniform pressure, taking the center of pressure on the area *ejkh* a distance $c/2$ from *mn* and that on the triangles *efj* and *hgk* $0.6c$ from *mn*. The moment thus computed acts on a beam the limit of whose width is the width of the column or pier, a, plus twice the depth, d, of the footing, plus one-half the remaining width of the footing. The steel must be placed in this width with uniform spacing and the concrete stress computed on the exact shape of the concrete section having this width at the face of the pier. If the footing is wider than $a + 2d$ it may be desirable to use additional reinforcement outside the effective width. Each step in a stepped footing is a possible critical section.

Footings with sloping tops contain less concrete than the other types but are troublesome and uneconomical on account of the formwork for the top, except for slopes flatter than 2 to 1, which may be poured without forms by using dry mixtures. Builders prefer the stepped top or the single slab. The difficulty with the stepped footing is the tendency to pour the concrete in as many separate operations as there are steps. Unless the pouring proceeds with sufficient continuity so that the footing

* Turneaure and Maurer, Principles of Reinforced Concrete Construction, John Wiley & Sons, Inc., 4th ed., 1935, p. 217.

† J.C. 863 recommends for simplicity the substitution of a simple rectangular strip carrying only its portion of the upward soil pressure for the trapezoid here considered, but the trapezoidal method agrees very closely with the Univ. of Ill. tests, *op. cit.*

forms a single homogeneous mass of concrete each step must be designed independently.

Bond stresses usually are very high in footings. When straight rods are used for reinforcement, without hooks at the ends for anchorage, bond usually controls the selection of the steel, the amount required sometimes being twice that for resisting moment. The shear for bond computations is that at the face of the wall or pier. The allowable bond stress for unanchored bars in a spread footing is obviously less than in an ordinary beam since the tension in the bottom tends to pull the concrete away at right angles from the steel. (See J.C. 878, Table 7.) Many designers require that footing rods always be hooked.

The problem of transferring the load from a highly stressed column with spiral reinforcement to a footing or pier of plain concrete is often difficult. The loaded area can usually be taken as the whole cross section of the column, including fireproofing, and this loaded area is confined laterally by the mass of the pier or footing outside it. The more effective this lateral support is, the larger the local compression can be, as is shown by the formula recommended by the Joint Committee. (See Art. 870.) Some designers make a practice of using spiral reinforcement in the top of piers and footings. The load from the column reinforcement is transferred to the pier or footing by means of dowels or by use of a metal base plate. An equal number of rods of the same size as the column reinforcement are required for dowels. At the bottom of the column they must carry a stress equal to that in the column reinforcement in the upper part of the column and so must extend above and into the footing a distance sufficient to develop this stress in bond.

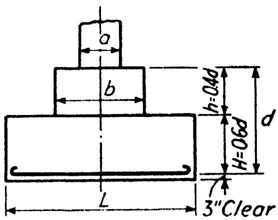


Fig. 15–7

On account of varying soil conditions and the presence of different levels in the basement floor due to special construction, such as pits for elevators or machinery, it is not common for all footings in a building to have their bases at the same elevation. (To avoid excessive lateral pressure against the backfill it is desirable that no footing come within a 45° frustum below any other footing.) It is not convenient to have a variety of lengths for the basement columns; the necessity for this can be avoided by using plain concrete pedestals on all footings, extending to the underside of the basement floor. These piers have larger cross sections than the columns they support and so their use results in more economical footings.

The following proportions for stepped footings are recommended by Taylor. Thompson, and Smulski* (Fig. 15-7):

Ratio Side of Pedestal to Side of Base a/L	*Width of Top Block* b
0.10	$0.36L$
0.15	$0.38L$
0.20	$0.40L$
0.25	$0.45L$
0.30	$0.50L$
0.35	$0.52L$
0.40	$0.55L$

Ratio of depth of top block to total depth (d), 0.4.

The top block should be at least 4 in. wider all around than the pedestal in order that there may be room for setting the forms. Similarly the pedestal must be larger than the column which it supports.

15-8. Design of an Interior Column Footing. (Computation Sheet IF1.) The footing here designed is that required under a typical interior column of the building designed in Chapter XVII. The data are assembled at the top of the sheet. The sketch there shown was built up step by step as the computations proceeded.

This is a stepped footing, top block cast integrally with the base. The weight of an interior column footing can be approximated as 5 to 7 per cent of the load it carries, or a rough estimate of its size and thickness will give a good indication of a trial weight, to be checked when the design is completed. Great precision in the matter of weight is not necessary.

For a footing of these proportions the depth is commonly determined by keeping the diagonal tension shear low enough to avoid web reinforcement, and so a computation was made of the required depth in the case of a flat-topped footing. Actually a greater thickness is required because the critical section for diagonal tension, as shown on the sketch, is at a point of lesser depth than that at the face of the column. A later check of the intensity of this shear shows that it is within the allowable limits. The steel is protected by about 3 in. of concrete as required by J.C. 506*a*. The shear and moment computations are made as described in Art. 15-7, which differs somewhat from J.C. 865-6. The student should check this design by the J.C. Code to see what variations result. There are two critical sections for moment in this footing, that at the face of the column and that at the edge of the cap or upper block. The second was first considered as the other was seen by inspection to be less critical. It is thus a matter of indifference in this case whether the cap

* Concrete Plain and Reinforced, 4th ed., Vol. 1, p. 490.

COMPUTATIONS FOR INTERIOR COLUMN FOOTING — *Sheet IF1*

Loads:	Stresses	Column	Footing
Reduced LL = 136 000# Dead Load = 144 000 Total = 280 000#	$f'_c = 2000$ psi $f_c = 900$ psi $f_s = 20000$ psi $v_c = \begin{cases} 40 \text{ psi} \\ 60 \text{ S A} \end{cases}$ $n = 15$ $u = 75$ psi (113 with SA)	19"x19" 10-1"ϕ Soil = 5000 psf	B2 B3 B4

Total Load = 280 000#
Wt Ftg (6%) = 16 000
296 000#

Area of Footing $= \frac{296\,000}{5000} = 59.2$ sf
$= (7'\text{-}9")^2 = 60.1$ sf

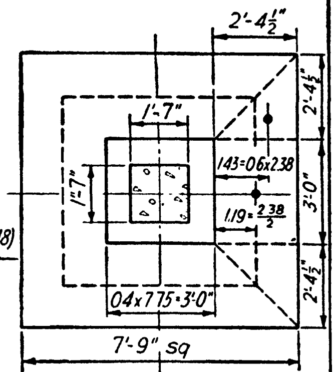
Depth: Diagonal Tension

$k = \frac{1.58}{7.75} = 0.204 \quad C = \frac{5000}{504 \times 40} = 0.248$

$$\frac{d}{L} = \sqrt{\frac{2 \times 0.248 + 4(0.248)^2 + \frac{(0.204)^2}{4} - \frac{0.204}{2}(1 + 4 \times 0.248)}{2 + 4 \times 0.248}}$$

$= 0.223 \qquad d = 21"$

Try $h = 8"$, $d - h = 16"$

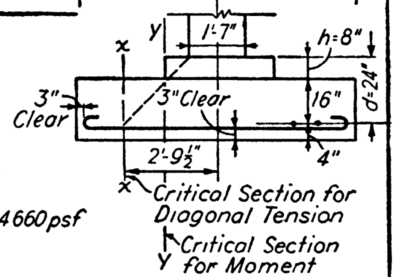


Base: 7'-9" sq x 1'-8"

Cap 3'-0" sq x 8"

Weight of Footing:
$(7.75)^2 \times 1.67 @ 150 = 15\,000$#
$(3.0)^2 \times 0.67 @ 150 = 900$
15 900#
Total Load = 280 000#
60.1) 295 900
4 930 psf

Net Soil Pressure $= \frac{280\,000}{60.1} = 4660$ psf



Check Shear: $V_x = (7.75^2 - 5.58^2)\,4660 = 134\,800$#

$v_c = \frac{V}{bjd} = \frac{134\,800}{4 \times 67 \times \frac{7}{8} \times 16} = 36 \text{ psi} < 40 \text{ psi}$

Moment · Edge of Cap: My
[4660 x 3 x 2.38 = 33 300 lb] 1.19 = 39 600 lb-ft
[4660 x (2.38)² = 26 400 "] 1.43 = 37 800 " "
59 700 lb — 77 400 lb-ft ≈ 929 000 lb-in.

Effective Width = 1.58 + 4.00 + 1.09 = 6.67 ft

$R = \frac{929\,000}{6.67 \times 12 \times (16)^2} = 46 < 157 \qquad A_s = \frac{929\,000}{17500 \times 16} = 3.32$ □" in 6.67 ft
$= 3.86$ □" in 7.75 ft
13-⅝ ϕ = 4.03 □" Each Way
16-½ □ = 4.00 □" " "

19-½ □

Check Bond: Sect y-y No. ½ □ Req = $n = \frac{V}{\Sigma ujd} = \frac{59\,700}{2 \times 113 \times \frac{7}{8} \times 16} = 19$ — Ends Hooked

SHEET IF1

is cast with the base or not since it satisfies the requirements for a flat-topped concrete footing.

The shear used in calculating the bond stress is the same as the upward load acting on the trapezoid used in determining the bending moment. It is assumed that this is the shear bringing bond stresses to one end of one set of rods.

15–9. Footings on Piles. Concrete footings are used with both wood and concrete piles. The design of an interior footing with pile supports proceeds on exactly similar lines to those outlined above. The piles bring concentrated reactions to the base which must be brought into the footing by bearing. The moments to be resisted are

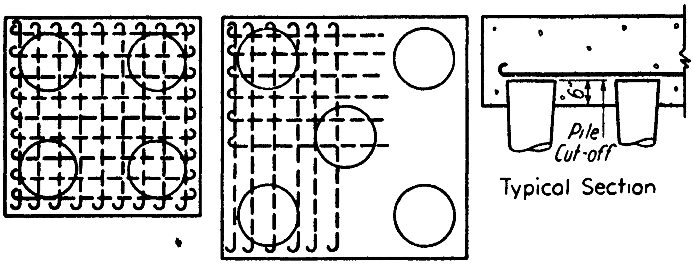


FIG. 15–8

computed as before except that the forces acting through the piles are often assumed to be definitely located with lever arms measured perpendicularly to the face of the column or pier. As a matter of fact, a variation of a foot in the location of individual pile heads is not uncommon in field work. For this reason some engineers design these footings as though the base pressure were uniformly distributed. The steel is placed in two layers and directions, as before for a rectangular footing, and not radially to each pile. The exact shape of the footing often deviates slightly from the rectangular but this is ignored in design. A few typical interior footings on piles are illustrated in Fig. 15–8.

15–10. Combined Footing. (Computation Sheets CF1–CF2 and Fig. 15–9.) The footing here designed is that supporting columns C2–D2 of the building designed in Chapter XVII. The first step in designing a combined footing to carry an exterior and an interior column is the determination of the area required for the base, which must be the sum of the areas required by the individual footings. For equal settlement it was decided to make the unit pressure under dead load plus one-half the live load the same as for the typical interior footing, i.e., 4050 psf. It is necessary to include the dead weight of the footing not yet designed in the total loads, which can be approximated as 12 to

COMPUTATION FOR COMBINED FOOTING — Sheet CF1

Footings C2-D2, C3-D3, C4-D4, C5-D5

Loads	Interior Col	Wall Col
Total LL =	166 000#	75 200
Reduced LL =	136 000	61 600
Dead Load =	144 000	117 100 Col; 3 300 Wall
Total Load =	280 000 +	182 000 = 462 000#

Specs: $f'_c = 2000$ psi, f_c 900 psi, $f_s = 20\,000$ psi, n 15

$v_c = \{40$ psi, 60 psi SA$\}$

$v = 120$ psi, $u = \{100$ psi, 150 SA$\}$

Soil = 5000 psf

Soil Values Typical Interior Footing

Check Pressures Interior Footing B2 on Sheet IF1

Reduced LL =	136 000#	Half LL =	83 000#
Dead Load =	144 000	Dead =	144 000
Active Total =	280 000	Footing =	16 000
Footing Weight =	16 000	Total =	243 000
Total =	296 000	7'-9" x 7'-9" = 60.1 sf	

Active $p = \frac{280\,000}{60.1} = 4660$ psf

$D + \text{Red. }LL = \frac{296\,000}{60.1} = 4930$

$D + \frac{1}{2}LL = \frac{243\,000}{60.1} = 4040$

Base Area Required

	Interior Col	Wall Col
Half LL =	83 000#	37 600#
Dead =	144 000	120 400
Total =	227 000 +	158 000 = 385 000

Int Col = 227 000#
Wall Col = 158 000
(12-14%) Wt Ftg = 53 000
$4040 \overline{)438\,000}$
108 sf

Size of Footing 4'-6" x 24'-0" x 3'-2"

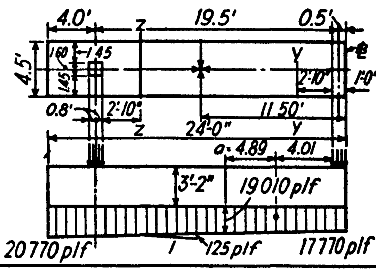
Dead + $\frac{1}{2}$ Live Load:

c of g $= \frac{227\,000}{385\,000} \times 19.5 = 11.50$ ft from ℄ Col

℄ Ftg = 12.0 ft from ℄ Col so footing = 24.0 ft long

Width of Ftg $= \frac{108}{24} = 4.5$ ft

Bearing under reduced live load $= \frac{462\,000 + 53\,000}{108} = 4770 < 4930$



Shear and Moment Curves

Dead + Reduced Live Load:

c. of g. $= \frac{280\,000}{462\,000} \times 19.5 = 11.81$ ft

eccentricity: $e = 0.31$ ft

pressure: $f = \frac{P}{L}\left(1 \pm 6\frac{e}{L}\right) =$

$\frac{462\,000}{24}\left(1 \pm \frac{6 \times 0.31}{24.0}\right) = 19\,270(1 \pm 0.078) =$

20 770 lb/ft Int ⇨ 4620 psf < 4660

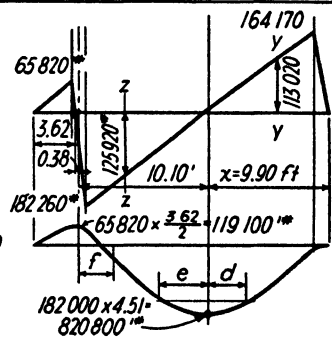
17 770 " Wall ⇨ 3950 psf < 4660

Zero Shear: $62.5x^2 + 17\,770x = 182\,000$

$x = 9.90$ ft

Moment Arm:

$a = \frac{9.90}{3}\left[\frac{19\,010 + 2 \times 17\,770}{19\,010 + 17\,770}\right] = 4.89$ ft



Depth

$d = \frac{V}{bjv} = \frac{182\,260}{54 \times \frac{7}{8} \times 120} = 32''+,$ and at Sect zz the required depth would be less

$d = \sqrt{\frac{M}{Rb}} = \sqrt{\frac{820\,800}{157 \times 4.5}} = 34.1''$

Cover = 3.9"
Depth = 38.0"

Weight

Weight = $4.5 \times 24.0 \times 3.17 \times 150 = 51\,400$ lb

SHEET CF1

14 per cent of the total applied load; or a rough preliminary computation of the footing can be run off sufficient to establish its size closely enough to give a reasonable approximation of its weight.

With the base area computed, the center of gravity of the soil pressure should coincide with that of the loads and, as the footing itself is prismatic, its weight can here be left out. The footing is extended sufficiently beyond the interior column to place the centroid in the desired position and the width is computed to produce the required area.

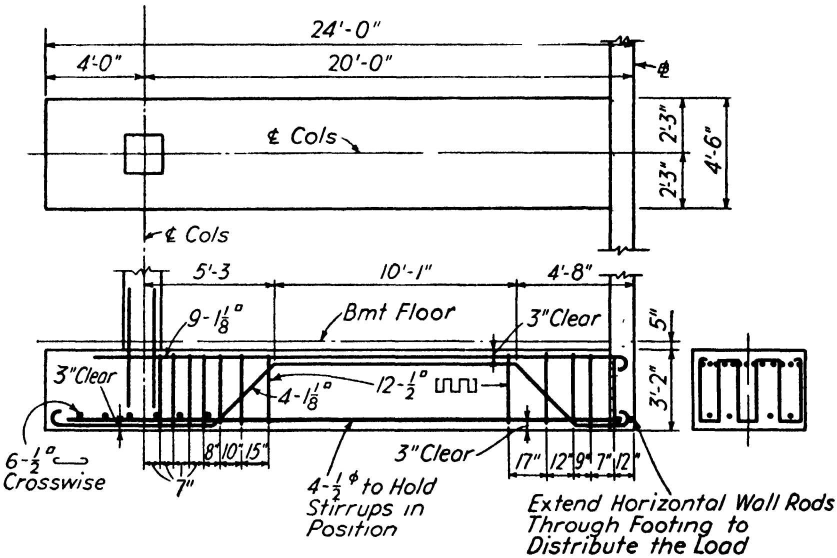


FIG. 15–9

As the total reduced live load will produce higher stresses in the footing than the half live load used in proportioning the base area, the center of gravity, eccentricity, and soil pressure are determined and shear and moment curves are drawn for this latter condition. The dotted portions show the values if the widths of the columns are neglected.

The combined footings designed on Computation Sheets CF1 and CF2 were proportioned to have, under the condition of full dead plus half live load, uniform upward soil pressures of an intensity equal to that under similar loading for the typical interior footings designed on Computation Sheet IF1, thus assuring approximately equal settlements under the designer's assumption that the actual live load for average normal conditions will be 50 per cent of the maximum live load used in proportioning the floor system. When the full reduced live load is realized the maximum soil pressure (taking account of eccentricity)

COMPUTATION FOR COMBINED FOOTING Sheet CF2

Top Steel

$A_s = \frac{M}{f_s jd} = \frac{820\,800 \times 12}{17\,5000 \times 34.1} = 16.5^{\square''}$ $\quad 13\text{-}1\tfrac{1}{8}^{\square} = 16.51^{\square''}$

$v = \frac{V}{bjd} = \frac{182\,260}{54 \times \frac{7}{8} \times 34.1} = 113\,psi < 120$ $\quad N = \frac{vb}{ou} = \frac{113 \times 54}{4.5 \times 150} = \frac{9\ Str}{4\ Bt}$

$v_z = \frac{125\,920}{54 \times \frac{7}{8} \times 34.1} = 78 > 60 \therefore$ Stirrups Req'd

Bottom Steel; Int. Col.

$A_s = \frac{M}{f_s jd} = \frac{119\,100 \times 12}{17\,500 \times 34.1} = 2.40^{\square''}$ $\quad 4\text{-}1\tfrac{1}{8}^{\square} = 5.08^{\square''}$

$u = \frac{V}{\Sigma o jd} = \frac{65\,820}{18.0 \times \frac{7}{8} \times 34.1} = 123\,psi$ - S A

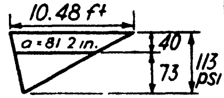
Stirrups Int. Col.

Int. Col to Right $\quad v = 113\,psi \quad a = \frac{73}{113} \times 10\ 48 \times 12 = 81.2\ in.$

$A_v = \frac{81.2 \times 54 \times 73}{2 \times 16\,000} = 10.0^{\square''}$

$Max\ Dia = \frac{4uL}{f_v} = \frac{4 \times 150 \times 17.1}{16\,000} = 0.64\ in.$

$or = \frac{4 \times 150 \times 9}{6000} = 0.90\ in.$



Use $7\text{-}\frac{1}{2}^{\square''}$ ⊓⊓⊓ $= 10.5^{\square''}$ Spaced from ℄ Int Col 7, 7, 7, 7, 8, 10, 15

Int Col to Left $\quad v = \frac{V}{bjd} = \frac{65\,820}{54 \times \frac{7}{8} \times 34.1} = 41\,psi$, None Req'd

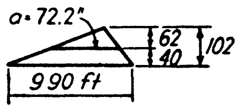
Stirrups Ext Col.

Ext Col $\quad v_Y = \frac{V}{bjd} = \frac{113\,020}{54 \times \frac{7}{8} \times 34.1} = 70.2 > 60\,psi$; Stirrups Req'd

$v = \frac{V}{bjd} = \frac{164\,170}{54 \times \frac{7}{8} \times 34.1} = 102\,psi$

$a = \frac{62}{102} \times 9.90 \times 12 = 72.2\,in$

$A_v = \frac{72.2 \times 54 \times 62}{2 \times 16\,000} = 7.55^{\square''}$



$5\text{-}\frac{1}{2}^{\square}$ ⊓⊓⊓ $= 7.5^{\square''}$ Spaced from ℄ Col. 6, 7, 9, 12, 17

Bending Bars

Right of Zero Shear Assume Parabola : $d = 9.90\sqrt{\frac{4}{13}} = 5.49\ ft$

Check : $M_d = 182\,000 \times 3.91 - \left[17\,770 + \frac{4.41 \times 125}{2}\right]\frac{(4\,41)^2}{3}\left(\frac{53\,880}{36\,110}\right)$

$= 538\,000 < \frac{9}{13} \times 820\,800 = 568\,000$

Bend Down @ 4'-8" from ℄

Left of Zero Shear Assume Parabola : $e = 10.10\sqrt{\frac{4}{13}} = 5.62\,ft$

Try 15'-0" from ℄ : $M_e = 182\,000 \times 14.5 - \left[17\,770 + \frac{15 \times 125}{2}\right]\frac{(15)^2}{3}\left(\frac{57\,060}{37\,420}\right)$

$= 506\,000 < \frac{9}{13} \times 820\,800 = 568\,000$

Bend Down 14'-9" from ℄

Bent Up at Int. Col. $f = 2'\text{-}6''$ $M_f = -280\,000 \times 2.5 + \left[20\,770 + \frac{6.5 \times 125}{2}\right]\frac{(6.5)^2}{3}\left(\frac{61\,480}{40\,710}\right)$

$= -267\,000 < 0$

Cross Rods - Int Col

Projection = 1.45 ft with depth = 2.83 ft Shear O.K. no Steel Req'd,

but $A_s = \frac{M}{f_s jd} = \frac{1.45 \times 280\,000 \times 0.725 \times 12}{4.5 \times 17\,500 \times (34.1 - 1)} = 1.35^{\square''}$

Suggests $6\text{-}\frac{1}{2}^{\square''}$ ⊔ Crosswise

NON-RECTANGULAR COMBINED FOOTING — Sheet CF3

Data as on CF1 Except no Projection Past Face of Int. Col. — **DATA**

c of g loads $(D+\frac{1}{2}LL) = 11.50$ from ℄ $= 12.00$ from ℄

$(b+B)\frac{20.8}{2} = 108;\ b+B = 10.38$

$\frac{20.8}{3}\left(\frac{b+2B}{b+B}\right) = 12.00;\ 5.08b = 1.86B$

$b = 2.79$ Use $2'\text{-}9\frac{1}{2}'' = 2.79$; $B = 7.59$ Use $7'\text{-}7'' = 7.58$

$a = \frac{20.8}{3}\left(\frac{2.79+2\times 7.58}{2.79+7.58}\right) = 12.00$

$A = \left[\frac{2.79+7.58}{2}\right]20.8 = 107.85$

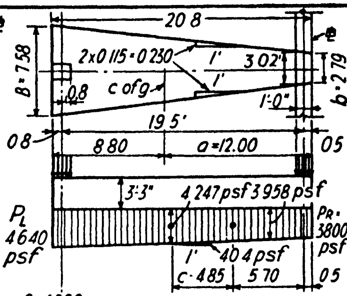
$I = \frac{20.8^3(7.58^2+4\times 7.58\times 2.79+2.79^2)}{36\times 10.37} = 3612\ ft^4$

Active Pressure; Dead + Red. LL·

$P/A = \frac{462000}{107.85} = 4283,\ e = 0.31$ (See Sheet CF1)

$P_L = P/A + \frac{Mc}{I} = 4283 + \frac{462000\times 0.31\times 8.80}{3612} = 4640\ psf < 4660$

$P_R = P/A - \frac{Mc}{I}\ 4283 - \frac{462000\times 0.31\times 12.00}{3612} = 3880\ psf < 4660$



Soil Pressure· Dead + Red LL

$4640 + 3\frac{1}{4}\times 150 = 5128\ psf > 5000$

$3800 + 3\frac{1}{4}\times 150 = 4288\ psf < 5000$

Size of Footing: $20.8' \times \begin{cases}2'\text{-}9\frac{1}{2}''\\ 7'\text{-}7''\end{cases}$

$V_m = 280.0 - \frac{1.6}{6}[7.58\times 4.64 + 4\times 7.40\times 4.61 + 7.21\times 4.58] = 225400\#$

$V_x = 0 = 280.0 - \frac{x}{6}[7.58\times 4.64 + 4(7.58 - \frac{0.23x}{2})(4.64 - \frac{0.0404x}{2}) + (7.58-0.23x)(4.64-0.0404x)];\ x = 9.75$

$V_w = 182.0 - \frac{1.0}{6}[2.79\times 3.80 + 4\times 3.82\times 2.91 + 3.84\times 3.02] = 170900$

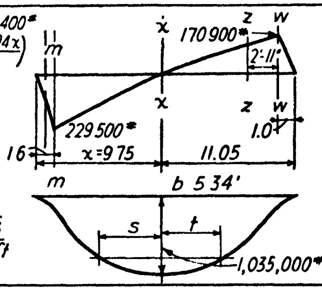
Compute C $2.79\times 11.05\times 4.024 = 124.1\times\frac{11.05}{3}\left(\frac{11847}{8047}\right) = 674$

(Centroid of pressure) $2.55\times 3.80\times\frac{11.05}{2} = 53.6\times\frac{11.05}{3} = 197$

$2.55\times 0.447\times\frac{11.05}{3} = 4.2\times\frac{11.05}{4} = 11.6$

181.9 -------- 882.6 / $4.86\ ft$

$M_{max} = M_x = 182000\times 5.69 = 1{,}035{,}000'\#$



Shear & Moment Curves

$d = \frac{V_w}{bjd} = \frac{170900}{3.02\times 12\times\frac{7}{8}\times 120} = 45''$ $\quad d = \sqrt{\frac{M}{Rb}} = \sqrt{\frac{1{,}035{,}000}{157\times 5.34}} = 35.1 + 3.9$ Cover $= 39''$

$v_w = \frac{170900}{3.02\times 12\times\frac{7}{8}\times 35.1} = 154\ psi > 120\ psi$ but see V_z

$V_z = 182.0 - \frac{3.92}{6}[2.79\times 3.8 + 4\times 3.24\times 3.88 + 3.69\times 3.96] = 132500\#$

$v_z = \frac{132500}{3.96\times 12\times\frac{7}{8}\times 35.1} = 91\ psi > 60\ psi$ ∴ Stirrups required

Depth 3'-3"

$A_s = \frac{M}{f_s jd} = \frac{1{,}035{,}000\times 12}{17500\times 35.1} = 20.2\square''$ $\quad 13\text{-}1\frac{1}{4}\square = 20.3$

$N = \frac{v_w b}{ou} = \frac{154\times 3.02\times 12}{5\times 150} = 8$ str, $\frac{5}{13}$ bent

Bending down bars:

	Factor	Dist	S	Dist	t
1 Center	$\sqrt{1/13}$	8.15'	2'-3"	10.05'	2'-10"
Next 2	$\sqrt{3/13}$		3'-11"		4'-10"
End 2	$\sqrt{5/13}$		5'-0"		6'-2"

Steel $8\text{-}1\frac{1}{4}\square$ Str, $5\text{-}1\frac{1}{4}$ Bt, Hook Ends

---

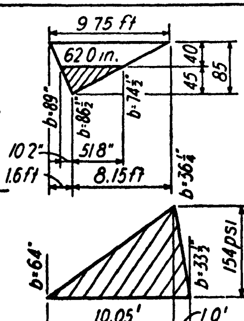
$V_m = \frac{V_m}{bjd} = \frac{225400}{7.21\times 12\times\frac{7}{8}\times 35.1} = 85\ psi > 60\ psi$ ∴ Stirrups required

$A_{vL} = \left(\frac{86\frac{1}{2}\times 10.2\times 45}{2} + \frac{2\frac{1}{2}\times 10.2\times 45}{2\times 3} + \frac{74\frac{1}{2}\times 51.8\times 45}{2} + \frac{12\times 51.8\times 45}{2\times 3}\right)/16000 = 7.0\square''$ $\quad 5\text{-}1\frac{1}{4}\square''$ Bent bars @ 45° $= 7.80$ Stagger bends

Since $v_w = 154 > 120$; A_v is for entire shear

$A_{vR} = \left(\frac{36\frac{1}{4}\times 120\frac{1}{2}\times 154}{2} + \frac{27\frac{3}{4}\times 120\frac{1}{2}\times 154}{2\times 3} + \frac{33\frac{1}{2}\times 12\times 154}{2} + \frac{2\frac{3}{4}\times 12\times 154}{2\times 3}\right)/16000 = 28.4\square''$

$5\text{-}1\frac{1}{4}$ Bent bars @ 45° $= 7.80 + 34\text{-}\frac{5}{8}''$ ⊔ @ 2 1/1 $= 28.9\square''$



Stirrups

exceeds by 2 or 3 per cent the allowable 5000 psf. Although it is possible to widen the footings to keep this maximum pressure within that value, then, under the normal condition of dead plus half live load, these combined footings would no longer be in balance with the interior footings. The designer felt that for this building the latter criterion should govern. In all footing designs judgment is involved in establishing the amount of live load most likely to be continuously applied; the footing is designed for that condition with a check to see that the results under other possible conditions are within reason.

The computations for depth, steel areas, and stirrup requirements are similar to designs already made. Three inches of protection in the clear is allowed for both the top and bottom steel, as shown on Fig. 15–9. For the main reinforcement the minimum number of bars of the largest size permitted by bond was chosen so that when in place in the top of the footing there will be as little interference as possible with the pouring of the concrete. The spacing of stirrups is determined from the shear curves shown, neglecting the slight curvature and placing a stirrup near the centroid of each of the equal volumes into which the excess shear prism is divided. Several loops of relatively heavy material were used; they were chosen to keep within the requirement of developing the stirrup in bond.

Although 60 psi is permissible with special anchorage and no stirrups would be used if the shear intensity is within this limit, yet in proportioning stirrups the authors prefer in such case to use only 40 psi on the concrete, not taking advantage of the increase allowed by special anchorage, because there is always the possibility that changes may occur during the preparation of a design; for example, the special anchorage may be done away with. Since the cost of a few extra stirrups is a negligible item, it is safer to use this lower value. The bond stress on the top steel is at the allowable, and the remaining reinforcing bars are bent down for reinforcement at the end. The bond stress on this bottom steel is such as to require special anchorage so at the outer end the bars are hooked.

Had there been some interference that prevented the projection past the face of the interior column it would have been possible to use a trapezoidal instead of a rectangular bearing area, for this footing, proportioning the widths at the two ends to make the centroid of the trapezoid coincide with that of the loads. This design is illustrated on Computation Sheet CF3 and in Fig. 15–10. The computation of shear and moment curves is somewhat more tedious because of the variations in width but no new principles are involved. Note that the contents of the pressure volumes are computed by the prismoidal formula.

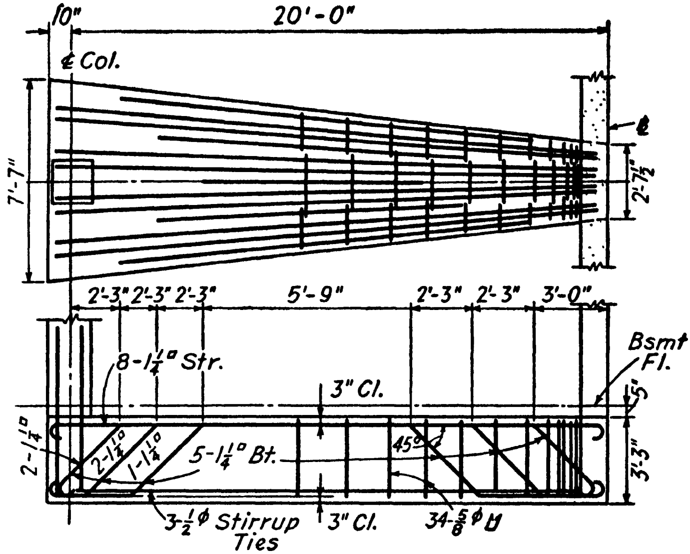
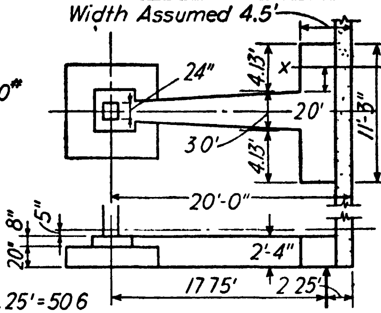


Fig. 15–10

15–11. Connected Footings. In place of a combined footing in a situation where a projection beyond the wall is impossible, a wall column footing is often used, carrying the wall column near its outside edge and connected with the nearest interior column footing by a beam, called a strap, which counterbalances the eccentricity and maintains approximately uniform pressure under the outer footing. A strap footing (often called a cantilever footing) of this type is designed on Computation Sheet SF1, and is illustrated in Fig. 15–11. The footing block under the wall column has to support not only the accumulated loads from above but also the amount of uplift developed at the interior column to maintain equilibrium against rotation. Some designers add an allowance for the weight of the strap, whereas others assume that there will be sufficient bearing under the strap to support its own weight. In any event precautions are taken to minimize this bearing as much as possible, by loosening the ground, and, in some cases, by sloping the bottom of the strap so that it tends to cut into the soil. To reinforce for the positive moment in the strap which might occur under considerable differential settlement some designers recommend reinforcement in the bottom made arbitrarily equal to one-third of the area of the top steel. Others omit such bottom steel, believing that the structure should function more nearly to the assumptions or a different type of design should be selected.

COMPUTATION OF STRAP FOOTING — Sheet SF1

Data Same as on CF1 but Using Strap Footing	Data
Interior Ftg Designed on IF-1 Wall Col. Half LL = 37 600# Dead L = 117 100 Wall = 16 300 } 171000# Wt Ftg = 18 000 Uplift at Int Col. 189 000 $= \frac{1.75}{17.75} \times 171\,000 = 16\,900$ 205 900# Soil Pressure Including Weight of Footing = 4040 psf (DL + ½LL) See Sheet CF1 $Area = \frac{205\,900}{4040} = 50.9$ Use 4.5' × 11.25' = 50.6  	Bearing Area x
Neglect Wall & Design as Cantilever: Assume Projection Beyond Edge of Strap = 5'-0" D L = 133 400# Red L L = 61 600 } 195,000# $P = \frac{232\,200}{50.6} =$ { 4590 psf < 4930 (D.L.+ Red. L L) See CF1 Wt Ftg = 18 000 Uplift = 19 200 232 200 V = 4590 × 5 = 22 950#/ft M = 22 950 × 2.5 = 57380#'/' Try 28" Depth d = 24", Same as Interior Footing $v = \frac{V}{bjd} = \frac{22\,950}{12 \times \frac{7}{8} \times 24} = 91\,psi$ $R = \frac{M}{bd^2} = \frac{57\,380}{24^2} = 99.6 < 157$	Exterior Footing 4'-6" × 11'-3" × 2'-4" Deep
Neglect Weight of Strap M = 195 000 × 1.75 = 341 000'# $V = \frac{M}{L} = \frac{341\,000}{17.75} = 19\,200$# (Uplift) Try 28" Depth, d = 24" $b = \frac{M}{Rd^2} = \frac{341\,000 \times 12}{157 \times (24)^2} = 45.2"$ on ℄ Ext Col $b = \frac{V}{vjd} = \frac{19\,200}{40 \times \frac{7}{8} \times 24} = 22.9"$ Throughout Make Strap 28" Deep × { 24" Wide on Edge of Int Ftg; 45" " " ℄ Ext Col = 36" Wide at Edge of Ext Ftg $A_s = \frac{M}{f_s jd} = \frac{341\,000 \times 12}{17500 \times 24} = 9.7\square''$ 10-1□ = 10.0 □" Top No. of Bars for Bond, Left End $= \frac{Vb}{ou} = \frac{40 \times 24}{4 \times 100} = 3$ Top (Cut 7 Short) 3 @ $\frac{3}{10}$ × 19.5 + 3'-4" = 9'-2" from ℄ Ext Col 2 @ $\frac{5}{10}$ × 19.5 + 3'-4" = 13'-1" " " " " 2 @ $\frac{7}{10}$ × 19.5 + 3'-4" = 17'-0" " " " " 3 @ 19.5 + 1'-0" = 20'-6" " " " "	Strap 24" } × 28" 45" 10-1□ Top 3-20'-6" 2-17'-0" 2-13'-1" 3-9'-2" from ℄ Ext Col
$M = \frac{4590 \times 4.13^2 \times 12}{2} = 470\,000$"# 12-¾φ = 5.28□" Bott. $A_s = \frac{M}{f_s jd} = \frac{470\,000 \times 4.5}{17500 \times 24} = 5.04\square''$ $u = \frac{vb}{\Sigma o} = \frac{75 \times 54}{12 \times 2.36} = 143\,psi < 150$ $v = \frac{V}{bjd} = \frac{4590 \times 4.13}{12 \times \frac{7}{8} \times 24} = 75$ Hook all Bottom Rods for S.A. $v_x = \frac{V_x}{bjd} = \frac{4590 \times 2.13}{12 \times \frac{7}{8} \times 24} =$ { 39 < 40 so no Stirrups Req'd	Steel in Ext Ftg
Check Weight: 11.25 × 4.5 × 2.33 @ 150 = 17 700# < 18 000 Assumed	Weight

Computation Sheet SF2 shows the design of another type of connected footing, used with caisson piers of plain concrete. In this design the same columns and loads were taken as before, and the condition

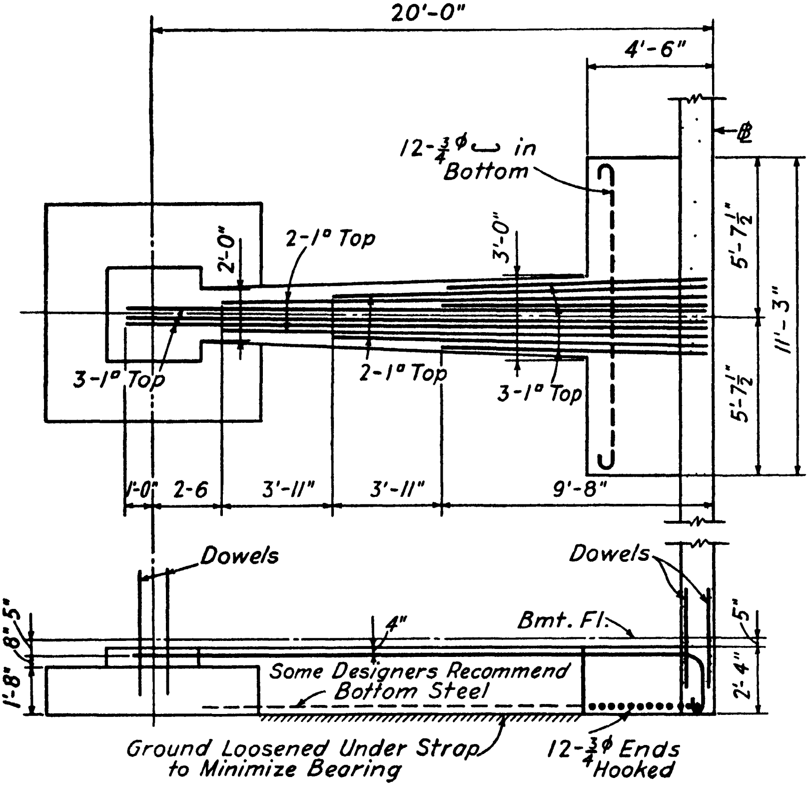


Fig. 15–11

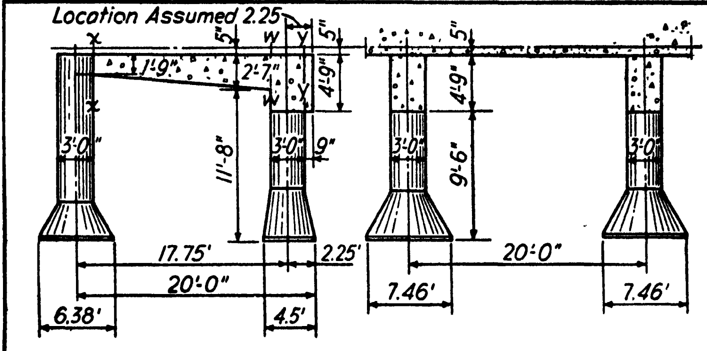
assumed of a considerable distance between the bottom of the footings and the basement floor. The location of the exterior pier was assumed so that it would center on the center of the bearing block of the similar design on Computation Sheet SF1. This pier was enlarged at the bottom to give an oval bearing area entirely within the lot.

Instead of using isolated bearing blocks, as in Fig. 15–11, frequently a continuous beam footing is extended the full length of the property line of the necessary width, strapped back at each column to take the eccentric moment.

Many variations are possible for unusual conditions and considerable ingenuity has been shown in selecting bearing areas best suited to the

CAISSON STRAP FOOTINGS — Sheet SF2

Data Same as on CF1 Except Soil Pressure = 9000 psf Under Caissons	Data



Bearing Areas

	Int. Col.	Wall Col	
Reduced LL	136,000	61,600	211 700
Dead Load	144,000	117,100	
Wall	—	33,000	
Caisson & Strap	28,000	30,000	
Uplift $= \frac{211\,700 \times 1.75}{17.75} =$	$-20{,}800$	$20{,}800$	
	$9000 \,\overline{)287{,}200}$	$9000 \,\overline{)262{,}500}$	
	32.0 sq ft = 6.38' ft dia	29.2 sq ft = 4.5 x 7.46 Oval	2.96, 2.25

Strap — $36" \times \begin{Bmatrix} 21" \\ 31" \end{Bmatrix}$

Left End	Right End
Take $b = 36"$	Take $b = 36"$
$V_x = \text{Uplift} = 20\,800$	$M_w = 20\,800 \times 16.25 = 338\,000'^{\#}$
$d = \frac{V}{bjv} = \frac{20\,800}{36 \times \frac{7}{8} \times 40} = 16.5"$	$d = \sqrt{\frac{M}{Rb}} = \sqrt{\frac{338\,000 \times 12}{157 \times 36}} = 26.8$
4.5 Cover	4.2 Cover
Strap = 36" x 21" — 21.0" Total	Strap = 36" x 31" — 31.0 Total

36" x 57" at Sect YY

V_y at wall column < 211,700, say 200 000#

$d = \frac{V}{bjv} = \frac{200\,000}{36 \times \frac{7}{8} \times 120} = 53"$

4" Cover

57" Total (with Stirrups)

Strap = 36" x 57" to right of caisson only

Steel — 3-1⅛"□-21'-0" Top; 3-1⅛"□-13'-11"; 2-1⅛"□-8'-10"

$M_w = 338\,000'^{\#}$

$A_s = \frac{M}{f_s jd} = \frac{338\,000 \times 12}{17500 \times 27} = 8.60\,\square"$ $7\text{-}1\tfrac{1}{8}"\square = 8.89\,\square"$

No. for bond left end $= N = \frac{vb}{\Sigma ou} = \frac{40 \times 36}{4.5 \times 150} = 3$ Str Top Full Length

2 Top $(\frac{4}{7} \times 17.75 + 3.75) = 13.89$ from ℄ Ext Col

2 Top $(\frac{2}{7} \times 17.75 + 3.75) = 8.82$ " " " "

Stirrups Right End

$A_v = \frac{\text{excess shear vol}}{16\,000} = \text{approx.}\ \frac{0.60 \times 24 \times 36 \times 80}{16\,000} = 2.60\,\square"$

3-½"□ ⊔⊔ = 3.0□"

@ 6", 12", 18" to right of ℄ of caisson

24", 27", 80 psi, 40 psi

Bott. Steel

Some designers use

$A = \frac{wL^2}{8 f_s jd}$ for weight of strap only $= \frac{1000 \times 17.75^2 \times 1.5}{17500 \times 22.2} = 1.21\,\square"$ 3-¾"φ = 1.32□"

particular problem. Sometimes several columns are founded on the same bearing block whose area is sufficient to support the group and whose center of gravity coincides as nearly as practicable with that of the group of columns supported.

15–12. Foundation Walls. Foundation walls have at least three different functions: (*a*) they act as distributing members supporting vertical loads; (*b*) when required they serve as retaining walls to hold back the earth banks; and (*c*) they are frost walls to protect floor slabs that rest on the ground.

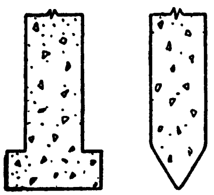


Fig. 15–12

The action of foundation walls in distributing vertical loads varies with different types of buildings. In skeleton frame construction of many stories the exterior walls above grade are often carried on the wall columns down to their independent footings so that the foundation walls have little vertical superimposed load in addition to their own weight. In such cases they either distribute their weight over the earth by means of small footing courses (Fig. 15–12*a*) or are reinforced to span as beams between column footings, in which event some designers give the bottom of the wall a wedge shape to avoid bearing (Fig. 15–12*b*).

In low frame buildings the wall columns may rest on top of the foundation walls which, if necessary, are stiffened with pilasters. The bearing area under such walls must be sufficient to support the reaction at the base of the wall columns as well as all floor or wall loads and the weight of the foundation wall itself. This bearing area may be of uniform width, requiring the wall to spread the column loads for its whole length, or it may be widened at the columns and decreased between. In either case the foundation wall is a vertical beam having concentrated column loads and uniform floor and wall loads on top, and an approximately uniform upward soil pressure on the bottom; hence it is designed as a beam for flexure, shear, and diagonal tension. Buildings of one and two or even more stories are ordinarily made wall-bearing, the foundation walls carrying all floors, roof, exterior walls, and their own weight, usually considered as uniformly applied along the full length of the wall.

Foundation walls around excavated areas hold up the earth banks and must be designed to resist lateral earth pressure as described in Chapter X. When the height of bank exceeds about 10 ft, or even less in cases of heavy superimposed loads, special provisions are required to resist this thrust. Walls may be designed to span vertically from basement to first-floor slab, but in such cases no backfill can be placed

until the first floor is set and the basement slab is sufficiently hardened to resist pressure. Walls may span horizontally between wall columns which would then be required to act as vertical beams spanning from footing to first floor construction. Sometimes the bottom story is so high as to preclude the spanning of either wall or columns vertically and some form of self-supporting retaining wall is economical. The thrust on the building frame at floor level from an earth-bearing basement wall must not be overlooked by the designer.

Foundation walls should extend down below frost line (3½ to 5 ft) to prevent heaving of the walls themselves by frost action and also to act as a cut-off wall to protect the enclosed slab that rests on the ground.

Example 15-1. Design a foundation wall for the building shown on Fig. 17-2 from Columns D2 to D5. Specifications, 1940 J.C.: $f'_c = 3000$ psi; $f_s = 20{,}000$ psi; $n = 10$.

Solution. (See Computation Sheet FW1) This wall spans vertically 11 ft and resists earth pressure increasing with the depth. A surcharge of 300 psf is applied on top of the ground to represent the allowance for trucks or any material stored next to the building. The lateral earth pressure is computed as in Chapter X. The resultant pressure is evaluated and located. The top and bottom reactions are found. The point of zero shear is located and the maximum bending moment is found at this point. The amount of vertical steel to resist this moment is about the minimum amount that would be required for temperature and shrinkage alone, so ½ in. round rods at 16 in. c to c will be used horizontally and vertically on the inside face. Many designers would use a similar mat in the exterior face above grade to prevent temperature cracks and surface crazing. The wall is also checked to span as a beam between column footings carrying the wall, sash, and a portion of floor. Although a couple of ½-in. rods in the bottom would be sufficient it is considered good practice to use two ⅝ or ¾ in. round rods in the top and in the bottom to form a good distributing beam.

Example 15-2. Design a foundation wall for the building shown on Fig. 17-2 along the track side from Columns A2 to A5. Specifications as for Ex. 15-1. Cooper E50 track loading. (See Sheet FW1.)

Solution. The difference from Ex. 15-1 is in the track load. Note that the load is assumed to spread within a 45° prism from the ends of the ties. Where this line strikes the outside of the wall there is a sudden increase in the pressure line, the value being obtained from the mean intensity of vertical pressure on a horizontal plane at this elevation. The remainder of the computations are similar to Ex. 15-1; the amount of reinforcement being less than for the sides away from the track, rods are used to meet the requirements of Ex. 15-1.

Example 15-3. Design a continuous wall footing course for the building shown on Fig. 17-2 from Columns D2 to D5 as an alternate for the individual column footings. Specifications as for Ex. 15-1. (See Sheet FW1.)

Solution. The individual footing design is a better arrangement, but this example illustrates wall footings using conditions and loads that are familiar. The longitudinal rods at top and bottom of the wall and the mat of reinforce-

FOUNDATION WALLS — Sheet FW1

Wall Col D2-D5

Specifications and Data as on Sheet CF1

$p = wh\frac{1-\sin\phi}{1+\sin\phi} = 100h\frac{1-\sin 33°40'}{1+\sin 33°40'} = 28.6h$

$P = 7.25\left(\frac{86+293}{2}\right) = 1370$ lb

$y = \frac{7.25}{3}\ \frac{293+2\times86}{293+86} = 2.96$ ft

$R_T = 370$ lb $R_B = 1000$ lb

$x_o(86+14.3x_o) = 370$ $x_o^2 + 6x_o = 25.9$

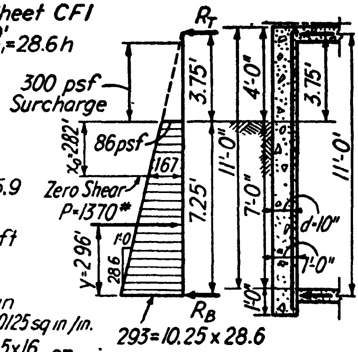
$x_o = 2.82$ ft

Arm $= 3.75 + \frac{2.82}{3}\left(\frac{86+2\times167}{86+167}\right) = 5.31$ ft

$M = 370\times5.31 = 1970$ lb-in/in

$A_s = \frac{M}{f_s jd} = \frac{1970}{17500\times10} = 0.0113$ sq in./in ½φ@16" c/c 0.0125 sq in/in.

$v = \frac{V}{bjd} = \frac{1000}{12\times\frac{7}{8}\times10} = 9.5$ psi $u = \frac{vb}{\Sigma o} = \frac{9.5\times16}{1.57} = 97$ psi



For Vert Load:

Sash	= 0.09
Spand	= 0.19
Sill	= 0.06
Floor	= 0.81
Wall	= 1.80
	2.95

$L = 17.3$ $\frac{WL}{12}$ $M = 883$ k-in.

$A_s = \frac{M}{f_s jd} = \frac{883}{17.5\times140} = 0.36$□"

Use 2-⅝φ Top - 2-⅝φ Bottom = 0.62□" Each

12" Thick ½φ-16"c/c

2-⅝φ Top
2-⅝φ Bott

Wall Cols A2-A5

$p_v = \frac{50000}{5(8+8)} = 625$ psf

$p_h = 0.286\times625 = 179$ psf

$R_T = \frac{750\times2.42+492\times1.38}{11} = 227$ lb

$R_B = 1015$ lb

$14.3x_o^2 = 227$ $x_o = 3.98'$

Arm $= 3.75 + \frac{2}{3}\times3.98 = 6.40$ ft

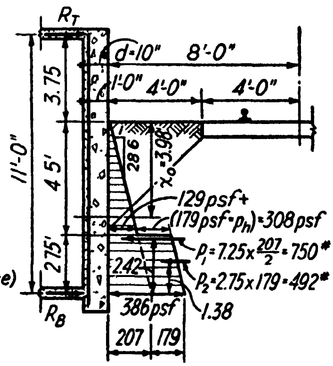
$M = 227\times6.40 = 1450$ lb-in/in.

$A_s = \frac{M}{f_s jd} = \frac{1450}{17500\times10} = 0.0083$ lb-in/in.

½φ@ 16"c/c = 0.0125 sq in/in (From First Case)

$u = \frac{V}{\Sigma o jd} = \frac{1015\times1.33}{1.57\times\frac{7}{8}\times10} = 98$ psi

Vertical Load Similar to Preceding Case



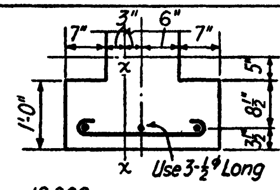
Alternate Wall Footing D2-D5

Column Load
Including Wt of Bmt Col = 178 700 lb
−Wt of Col = −5 200
Wt on Top of Wall = 173 500

$\frac{173500}{20}$ = 8680 plf
Wall = 11×150 = 1650
Ftg = 2½×150 = 380

$\left.\right\}\frac{10330}{2.17} = 4760$ psf

5000)10 710
2.14 say 2'-2" = 2.17'



$M_x = \frac{10710}{2}(6\frac{1}{2}-3) = 18800$ lb-in.

$A_s = \frac{M}{f_s jd} = \frac{18800}{17500\times8.5} = 0.127$ sq in./ft

⅜φ Rod @ 5½"c/c = 0.24 sq in./ft

Use ⅜φ-5½"c/c S.A.

Spacing for Bond: $u = \frac{V}{\Sigma o jd} = \frac{V}{(o\times\frac{12}{s})jd}$ ∴ $s = \frac{uojd\times12}{V} = \frac{150\times1.18\times\frac{7}{8}\times8\frac{1}{2}\times12}{4760\times0.58} = 5.68"$

ment required to resist lateral earth pressure are sufficient to spread the column load practically uniformly throughout the length of the wall, and the bearing on the earth is taken as of uniform intensity for the length of the building. With the small projection of footing course some designers would omit the cross-reinforcement, but with the soil pressure here used and the importance of this footing course to the safety of the entire structure the small amount of reinforcement is justified.

CHAPTER XVI

SLAB CONSTRUCTIONS
RIBBED, TWO-WAY, FLAT, AND STAIR SLABS

16–1. All structures for human occupancy require some sort of floor system which in the case of reinforced concrete is an integral part of the frame. The design of uniformly loaded one-way slabs as a series of rectangular beams has been discussed in Chapter XIII, and some explanation of the spreading of concentrated loads widthwise was given in Chapter XI. This chapter deals with the more common variations that have been developed for particular purposes. For one-way spans of 10 or 12 to 25 or 30 ft, with live loads up to about 150 psf, the ribbed-slab (concrete joist) floor is economical. When supporting beams are available on all four sides of approximately square panels, two-way slabs with bands of steel in two directions at right angles to each other can be used. Beams can be omitted entirely by the use of flat slabs supported directly by the columns. For loads above 125 psf and on approximately square panels with spans from 16 or 18 to 25 or 26 ft, flat slabs are economical. Stair slabs involve a few special requirements. All these slab systems are discussed in this chapter.

16–2. Ribbed Slabs. Solid concrete slabs such as have been described in preceding articles are heavy and uneconomical for long spans and light loads. Their excessive weight is due to the large mass of concrete in the tension part which is much greater than is required to resist diagonal tension and shearing stresses. In lightweight floors this difficulty is met by forming the floor slab of a series of tee-shaped joists, the space between the stems being filled by hollow tile of clay, gypsum, or pressed steel, or else left empty by the removal of the forms. The thin flanges (usually 2 or 2½ in. thick) of the tee-beams, known as topping, serve to transmit the loads to the ribs and at the same time act as compression flanges for the joists.

The details of a design using clay-tile fillers are illustrated in the following example, after which a similar design is prepared using removable steel forms.

Example 16–1. Design a clay-tile and joist slab for a fully continuous span ($M = wL^2/12$) of 20 ft c to c of concrete beams, to carry a live load of 125 psf with a 1-in. granolithic finish on top and ¾-in. plaster ceiling. This

RIBBED SLAB	Sheet RS1

Data

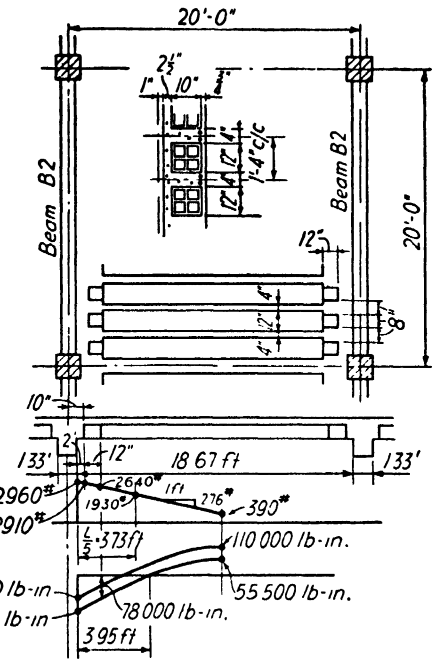
Data.
$LL = 125$ psf
$f'_c = 3000$ psi
$f_s = 20000$ psi
$n = 10$

Loads
$LL = 125$ psf
1" Fin = 13
$10 + 2\frac{1}{2} = 92$
Ceiling = 8
238 psf
× 1.33
317 plf

$V_e = 317 \times 9.33 = 2960$ lb

$V_{\text{C}} = 125 \times 1.33 = \frac{933}{4} = 390$ lb.

$\frac{WL}{12}$ $M = 317 \times (18.67)^2$
$= 111\,000$ lb-in.



Shear and Moment Curves

$20' @ 238 = 4760$ plf
Beam = 260
5020 plf

$L = 19.0$ ft
Try 16 × 22 in.
$T = 20$ in.

$W = 95.4$ k
$v = 170$ psi

$\frac{WL}{12}$ $M = 1810$ k-in
$R = 226 < 236$

Beam B2

At edge of flange $v = \frac{2910}{8 \times \frac{7}{8} \times 11\frac{1}{4}} = 37$ psi, at offset $v = \frac{2640}{4 \times \frac{7}{8} \times 11\frac{1}{4}} = 67 > 60$ S A

$t/d = 2\frac{1}{2}/11\frac{1}{4} = 0.22$ Max R allowable = 194 psi [Table A-1] $j = 0.93 \pm$

$A_s = \frac{110\,000}{20\,000 \times 0.93 \times 11\frac{1}{4}} = 0.525$ □"/Joist $\left\{\begin{matrix} 1 - \frac{1}{2}\text{□}\ S \\ 1 - \frac{5}{8}\phi\ B \end{matrix}\right\} = 0.56$ □"/Joist $R = \frac{M}{bd^2} = \frac{110\,000}{16 \times (11\frac{1}{4})^2} =$
$54 < 194$

$p = \frac{0.56}{16 \times 11\frac{1}{4}} = 0.00311$ $j = 0.926$ [Fig A-4]

At fifth point $u_1 = \frac{1930}{20 \times \frac{7}{8} \times 11\frac{1}{4}} = 98 < 150$ psi At support $u = \frac{2960}{2 \times 1.97 \times \frac{7}{8} \times 11\frac{1}{4}} = 77 < 150$ psi

Shear o.k
f_c = o k
$\left.\begin{matrix}\frac{1}{2}\text{□}\ S \\ \frac{5}{8}\phi\ B\end{matrix}\right\}$
u = o k

$R' = \frac{110\,000}{8 \times (11\frac{1}{4})^2} = 109 < 236$ at support

$R' = \frac{78\,000}{4 \times (11\frac{1}{4})^2} = 154 < 236$ at offset

−M = o k.

$A_t = 0.0025 \times 2\frac{1}{2} \times 12 = 0.075$ □"/ft $\frac{1}{4}\phi @ 8" = 0.075$ □" ft

Temp Steel

---

1 - 10" Tile = 38 plf
$1\frac{1}{3}$ ft $2\frac{1}{2}$" Topping = 42
4" × 10" Rib = 42
$1\frac{1}{3}$ 122 plf
92 psf

Check Weight

Special Anchorage requirement because shear at offset is greater than 60 (J C 828)

$10000 \times \frac{\pi \times d^2}{4} = \pi \times d \times l \times 150$ $l = 16\frac{2}{3}\,d$
say 9"

S A

Sheet RS1

might be an alternative design for the floor on Fig. 17–2. Specifications, 1940 J.C.: $f'_c = 3000$ psi; $f_s = 20{,}000$ psi; $n = 10$.

Solution. (See Computation Sheet RS1.) The load consists of the given live load, weight of floor finish, ceiling plaster, and an allowance for the dead weight of the slab itself which can be approximated from available tables and checked after the design is completed. The stems of the joists are taken as

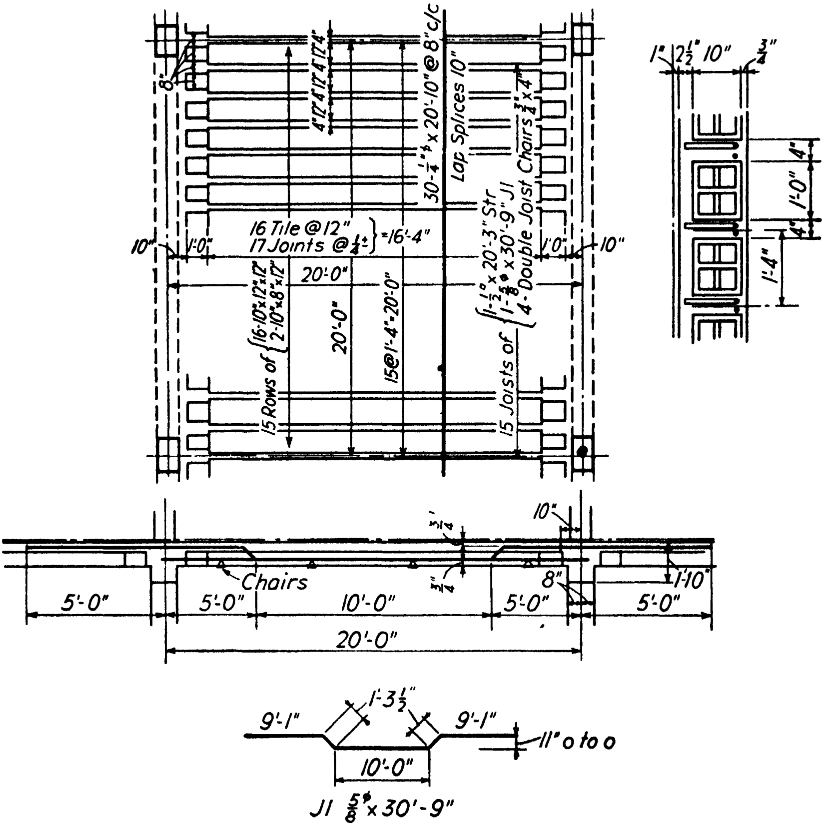


Fig. 16–1

4 in. wide for the greater part of their length and 8 in. wide at the ends, thus furnishing increased section for resisting diagonal tension and negative moment at the supports. The 4-in. width is the usual standard and also the minimum practicable to accommodate the reinforcing steel. In this design the tiles stop 10 in. from the center line of the supporting beams, thus providing the 20-in. flange width which the rough computations of beam B2 show to be necessary. The design of this beam is here carried only just far enough to establish its size. In computing resistance to diagonal tension the shell of the tile may be considered as increasing the thickness of the web by one-half its own thickness (J.C. 817*c*), usually taken as 1 in. for computations. The tile should not be assumed to add to the resisting section otherwise. Bond

is checked on positive reinforcement at the fifth point of the span and on negative steel at the face of the support, and is within the allowable. A brief computation of R' for negative moment at the edge of the beam flange and again at the offset in the width of rib shows that the bottom of the joist is amply wide to keep the negative compression low. Temperature steel is provided to equal 0.0025 of the volume of the topping. Finally, the weight of slab is computed as a check on the assumption originally made. The computations on RS1 are slightly abbreviated but should be clear to anyone who has carefully read Chapter XIII. A detail drawing of this slab is shown in Fig. 16–1.

The use of removable steel forms or "pans" results in a lighter slab and so lighter supports. According to U. S. Department of Commerce's "Simplified Practice Recommendation R87–29," standard depths are 6, 8, 10, 12, and 14 in.; standard widths are 20 and 30 in. with special narrow filler widths of 10 and 15 in. and 12 and 16 in. To afford additional resistance to diagonal tension and additional compressive area for negative moment, tapered end forms are available decreasing 2 in. of width on either side in 3 ft of length. These forms are erected on suitable centering and after the concrete has set the forms and centering are completely removed. The underside of the slab is straight and true with ribs at intervals and for many purposes is left without a plastered ceiling, though a ceiling of metal lath and plaster or of accoustical fiber board is easily applied by hanging from support wires embedded in the slab at the time of pouring. The design computations are carried through the same as for any tee-beam, the only limitations being the requirement of working to standard sizes of forms.

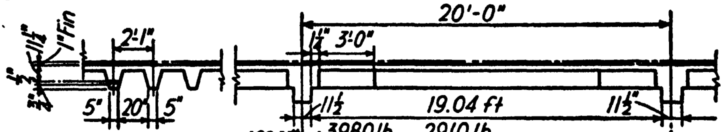
Example 16–2. Design a ribbed slab to use removable steel forms for a fully continuous span ($M = wL^2/12$) of 20 ft c to c of concrete beams that are 11½ in. wide, to carry a live load of 125 psf with a 1 in. granolithic finish. This is a typical interior panel of the building designed in Chapter XVII. Specifications, 1940 J.C.: $f'_c = 3000$ psi; $f_s = 20{,}000$ psi; $n = 10$.

Solution. (See Computation Sheet RS2.) The load consists of the given live load, weight of floor finish, and an assumed allowance for the slab weight taken from available tables and checked after the design is completed. No plastered ceiling will be used. With joists 25 in. on centers, multiplying the load by 2.08 gives the weight per lineal foot of joist. Shear and fully continuous moment are readily evaluated. $R = M/bd^2$ is only about one-quarter of the allowable value, indicating a very low compressive stress at mid-span. R' at the support exceeds the 236 allowable, showing the need for either double reinforcement or tapered end pans. Since the diagonal tension shear exceeds 60 psi but is less than the 90 psi allowed with the special anchorage that would result if the bottom rods were extended for double reinforcement, the use of either tapered end forms or extended bottom rods is indicated. Use tapered end forms in this design.

It is necessary to establish the concrete size for beam B2 before computing the shear at the edge of the flange and at the start of the flaring taper. The design of the beam is here carried only far enough to determine the stem size and width of flange.

The shearing intensities on the 9 in. width of joist at the edge of the flange and on the 5 in. width of joist at the start of the taper are within the allowable.

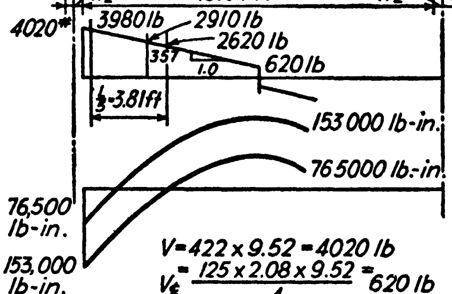
RIBBED SLAB — Sheet RS2



Data

Data:
LL = 125 psf
f'_c = 3000 psi
f_s = 20 000 psi
n = 10

Loads
Live = 125 psf
1" Fin = 13
$10+2\frac{1}{2}$ (T) = 65
203 psf
× 2.08
422 plf



Shear & Moment Curves

$V = 422 \times 9.52 = 4020$ lb

$V_{\text{℄}} = \dfrac{125 \times 2.08 \times 9.52}{4} = 620$ lb

$\dfrac{WL}{12}$ $M = 422 \times (19.04)^2 = 153{,}000$ lb-in

$10+2\frac{1}{2}$

$R = \dfrac{M}{bd^2} = \dfrac{153\,000}{25 \times (11\frac{1}{4})^2} = 48\frac{1}{2}$ psi (f'_c is Low)

$R' = \dfrac{153\,000}{5 \times (11\frac{1}{4})^2} = 242 > 236$ {Double Reinf or Tapered Ends

$V = \dfrac{4020}{5 \times \frac{7}{8} \times 11\frac{1}{4}} = 82$ psi > 60 psi {SA or Tapered Ends

Tapered Ends

20' @ 203 = 4060 plf
Beam = 260
4320 plf

$L = 19.0$ $W = 82.1$ k $\frac{WL}{12}$ $M = 1560$ k-in..
Try $11\frac{1}{2}$" × 25" $v = 181$ psi. $R = 230$
$T = 14$" $d_v = 23$"; $d_m = 22$" (2 Layers)

Beam B2

At Edge of Flange $v = \dfrac{3980}{9 \times \frac{7}{8} \times 11\frac{1}{4}} = 45$ psi

At Start of Flare $v = \dfrac{2910}{5 \times \frac{7}{8} \times 11\frac{1}{4}} = 59$ psi < 60 psi

Shear OK

$t/d = \dfrac{2\frac{1}{2}}{11\frac{1}{4}} = 0.22+$ $R = 194$ Allowable [Table A-1] $j = 0.92\pm$

$A_s = \dfrac{153\,000}{20\,000 \times 0.92 \times 11\frac{1}{4}} = 0.74$ $\dfrac{\text{sq. in.}}{\text{Joist}}$ $\left\{\begin{matrix} 1\text{-}\frac{5}{8}\phi\,S \\ 1\text{-}\frac{3}{4}\phi\,B \end{matrix}\right. = 0.75$ sq in.

$p = \dfrac{0.75}{25 \times 11\frac{1}{4}} = 0.00267$ $j = 0.935$ [Fig. A-4 / Fig. A-1]

$1\text{-}\frac{5}{8}\phi\,S$
$1\text{-}\frac{3}{4}\phi\,B$

$u_i = \dfrac{2620}{1.97 \times \frac{7}{8} \times 11\frac{1}{4}} = 136$ psi at Fifth Point < 150 psi

At Support $R' = \dfrac{153\,000}{9 \times (11\frac{1}{4})} = 135$ psi < 194

$u = \dfrac{4020}{2 \times 2.36 \times \frac{7}{8} \times 11\frac{1}{4}} = 87$ psi < 150 psi

---

Temp Steel: $A_t = 0.0025 \times 2\frac{1}{2} \times 12 = 0.075$ □"/ft $\frac{1}{4}\phi$ @ 8" c/c = 0.075 □"/ft — Temp. Steel

---

2.08' of $2\frac{1}{2}$" Topping = 65 plf
Joist – 10" × 6" av = 63
128 × 9.52 = 1220
Tap End = 65
$1285 \div (2.08 \times 9.52) = 65$ psf

Check Weight

For moment computations j will be greater than $\frac{7}{8}$; say about 0.92; the values of t/d and pn show that this is not a true tee-beam with the neutral axis below the underside of the flange, so that j is checked from Fig. A-1 as 0.935. This is greater than was assumed but not enough to affect the amount of reinforcement, so no correction was made in the computations. In checking diagonal tension shear and bond the customary value of $j = \frac{7}{8}$ was taken, as the method of analysis does not justify greater refinement.

Special variations of ribbed slab design to fit conditions frequently encountered in buildings are illustrated in Fig. 16-2. For the support of heavy partitions more heavily reinforced joists with or without extra

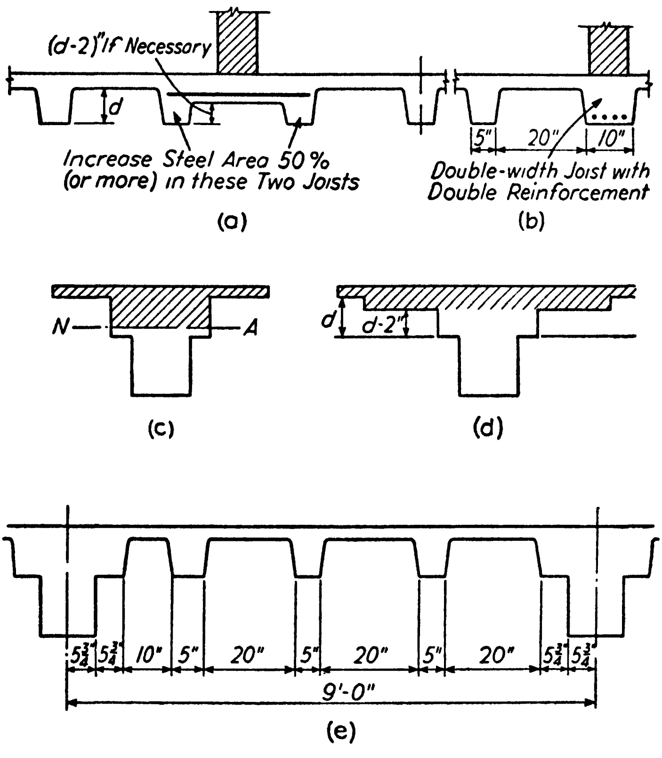


FIG. 16-2

thickness of topping are shown in (*a*); (*b*) indicates a double-width joist with double reinforcement. The former allows more variation in locating the partition and with a " pan " below permits easy passage for any pipes or conduits that are buried in the wall above. The use of the topping as a flange for the floor beam is shown in (*c*), which is improved in (*d*) by the use of a shallower end pan. In (*e*) is shown a pan of narrow width to fill out a given width of panel.

16-3. Two-Way Slabs. When panels are nearly square, say $L \lesseqgtr 1.5B$, it is frequently economical to place the reinforcing steel in two directions at right angles, making it possible to regard part of the load as being carried on the short span and the balance on the long span. This division of the load is a statically indeterminate problem that is very difficult of exact solution, but by means of certain simplifications it is possible to get some idea of the action from simple statics. Refer

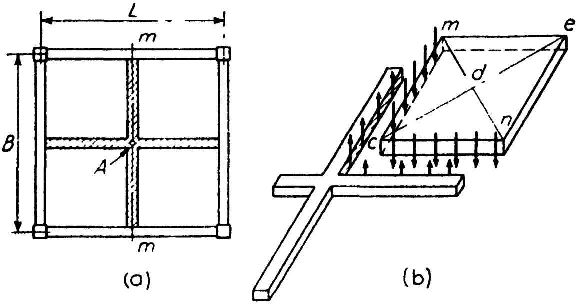


Fig. 16-3

to Fig. 16-3a, in which a cross-shaped portion of slab is isolated along the main center lines of the panel, which may be thought of as two strips each, say, a foot wide spanning, respectively, lengthwise and crosswise of the panel. Since the deflection of the center intersection, A, below a horizontal plane is a fixed quantity equal to $w_L L^4/kEI$ and simultaneously equal to $w_B B^4/kEI$, where w_L is the portion of load carried on the long span L, and w_B is the portion on the short span B, it follows that $w_L/w_B = L^4/B^4$. Since $w_L + w_B = w$, then $w_L = wL^4/(L^4 + B^4)$ and $w_B = wB^4/(L^4 + B^4)$. Such a simple analysis could be applied only to the two central strips because the deflected surface is one of double curvature and is not cylindrical.

Part of the difficulty with the above rough analysis is that it omits the effect of the four quadrants of slab removed from the panel in forming the central cross. Obviously shears exist along the cut edges, since the removed quadrants were helping hold up the cross. Also moments exist on the cutting planes, as one narrow strip of slab cannot deflect independently of its neighbors. Fig. 16-3b attempts to picture somewhat more thoroughly the action of a two-way slab. Two adjoining sides of each quadrant are supported on stiff beams and deflect very little. On the other two adjoining sides of each quadrant the shears just described act downward as the cross deflects and pull the quadrant down. A little thought indicates that the quadrant is supported at points m and n, and that a strip from m to n becomes a supporting beam. It is also clear that along line cde the quadrant tends to

cantilever, carrying the shear on the two edges, and exerting an upward pull on the supports at the corner *e*. Evidently reinforcement of fair amount is required in the bottom of the slab along the line *mn* to aid in the girder action, and other, perhaps lighter, reinforcement is needed along line *cde* in the top of the slab to reinforce the cantilever. This action of the quadrants results in an upward shear of considerable magnitude on the edges of the cross, reducing its deflection and bending moments. The action is further complicated by bending moments on the planes between the cross and quadrants and by possible restraints around the periphery of the panel.*

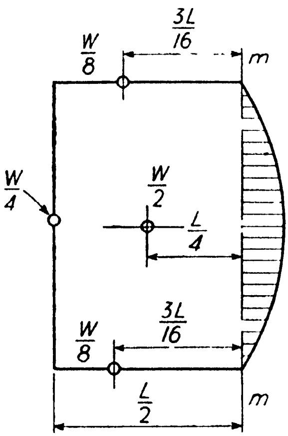


FIG. 16–4

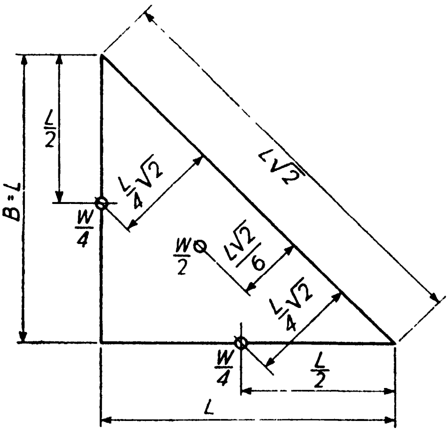


FIG. 16–5

* Attacks have been made on a single square panel freely supported on four sides, without continuity, carrying a uniform load, by dividing the entire panel into two equal halves either on a center line or a diagonal of the panel, and applying the principles of statics and of symmetry to the half panel, making reasonable assumptions as to the location of the line of action of a sufficient number of reaction forces to supply the unknowns. In Fig. 16–4 the total load $W/2$ on the half panel is concentrated $L/4$ from *mm*. By symmetry the left edge is supported by a reaction $W/4$ acting $L/2$ from *mm*, and the other two edges are each supported by a reaction $W/8$, whose arm depends upon the manner in which this reaction is distributed along the supporting beams. If this distribution is assumed to be parabolic, varying from zero at the column to a maximum at mid-span, the arm would be $3L/16$, and the moment equation is written $\frac{W}{4}\frac{L}{2} - \frac{W}{2}\frac{L}{4} + 2\left(\frac{W}{8}\frac{3L}{16}\right) = 3WL/64$ or $3wL^3/64$. This moment is not of uniform intensity along the plane *mm* but varies from zero at the supporting beams to a maximum at mid-span.

In Fig. 16–5 a similar line of reasoning gives $\frac{W}{2}\frac{L\sqrt{2}}{4} - \frac{W}{2}\frac{L\sqrt{2}}{6} = \frac{WL\sqrt{2}}{24}$ along a line of length $L\sqrt{2}$, or a mean value of $wL^2/24$ per foot. Such approximations are very naive, and serve only to indicate the presence of bending moments not only on the center-line sections of a panel but in inclined directions as well, and to illustrate the fact that a two-way slab has greater capacity than two one-way spans of similar properties.

A confirmation of the conclusions from the rough analogy of a cross and four quadrants is afforded by the manner in which isolated square and rectangular panels have cracked in tests under loads, as shown in

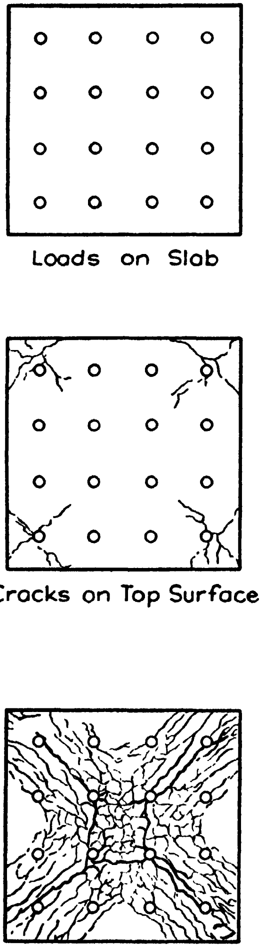


Fig. 16–6

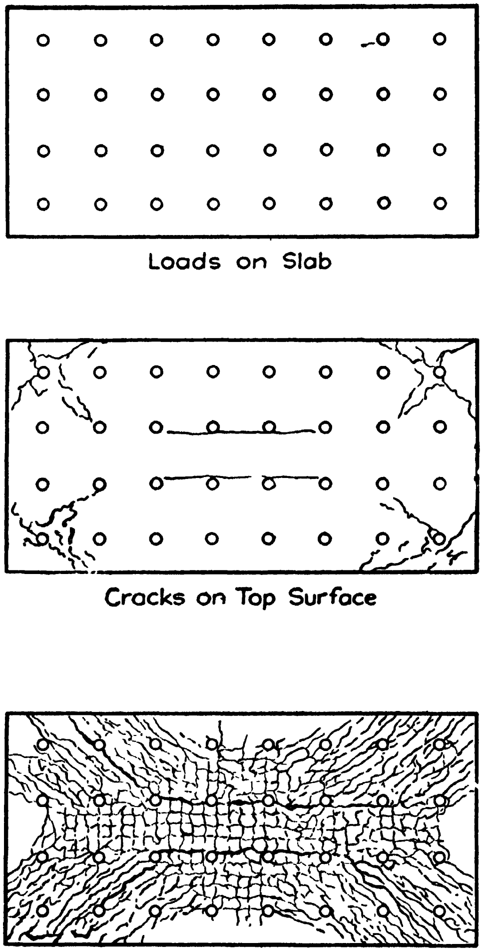


Fig. 16–7

Figs. 16–6 and 16–7. In both figures the original slab is shown with the manner of loading; under this is a diagram of the cracks that appeared in the top surface of the slab at or near failure, and in the bottom illustration is a diagram of the cracks in the underside of the same slabs at or near failure. These cracks give a fair indication of the dishing action that takes place. They show where tensions exist, and so where

special tension reinforcement is required. The reader should study these carefully for a better comprehension of plate action.

The 1928 A.C.I. Code recommended in Art. 713*d* that w_B be empirically taken equal to $w(L/B - \frac{1}{2})$ and $w_L = w - w_B$, but in 713*e* the Code suggested that the resulting bending moment used in proportioning the reinforcing steel be taken of this magnitude only for the center half of the panel and that the moment in the outer quarters be reduced 50 per cent, as shown in Fig. 16–8. For a square panel freely supported

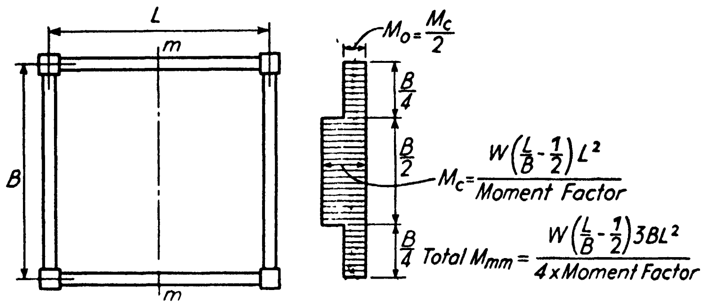


FIG. 16–8

on all four sides the total moment on *mm* becomes $W(\frac{1}{2})3L^3/(4 \times 8) = 3wL^3/64$, but tests and analyses show that this is on the safe side and that the true moment is perhaps 20 to 25 per cent less than this value.

This problem was analyzed by H. M. Westergaard and W. A. Slater in " Moments and Stresses in Slabs," Proc., A.C.I., 1921, and extended with bibliography by H. M. Westergaard in " Formulas for the Design of Rectangular Slabs and Supporting Girders," Proc., A.C.I., 1926, but the resulting equations are complicated for ordinary design purposes.* The student should read these references for a thorough understanding of two-way slabs.

The 1941 A.C.I. Code presents rules for the distribution of load on a two-way slab (Art. 709)† which at first inspection seem quite complex

* The reader may also consult J. A. Wise, " The Calculation of Flat Plates by the Elastic Web Theory," Proc., A.C.I., 1928; John R. Nichols, " Two Way Slabs in the Proposed Building Code for Boston and New England," and E. H. Uhler, " Design of Two-way Slabs on Beams — Report of Committee 302," both in Proc., A.C.I., May–June, 1934; Turneaure and Maurer, Principles of Reinforced Concrete Construction, 4th ed., 1935, John Wiley & Sons, Inc., pp. 203*ff*, and N. M. Newmark, " What Do We Know about Concrete Slabs? " *Civil Engineering*, Sept., 1940.

† The basis of the A.C.I. Code is explained in " Slabs Supported on Four Sides," by J. DiStasio and M. P. Van Buren, Proc., A.C.I., Jan.–Feb., 1936. Acquaintance with the rules is much facilitated by an article in the *Engineering News-Record* of Feb. 18, 1937, p. 268, " Simplified Computations for Two-Way Slabs."

but which are relatively simple of application once familiarity has been gained. These and other code rules the student can master independently of a textbook.

The 1940 J.C. Code attempts to recognize the various factors that affect the moments in such slabs and still obtain simple methods of computation. The coefficients suggested in Art. 811 of this Code are empirical but are based upon Westergaard's analyses.* For a single square panel freely supported on four sides without restraint on the edges the resulting moment becomes $wL^2/20$ per foot of width for the center half of the panel and two-thirds as much in the outer quarters, a mean of $wL^2/24$. If the panel is restrained on all four sides this reduces to $wL^2/48$.

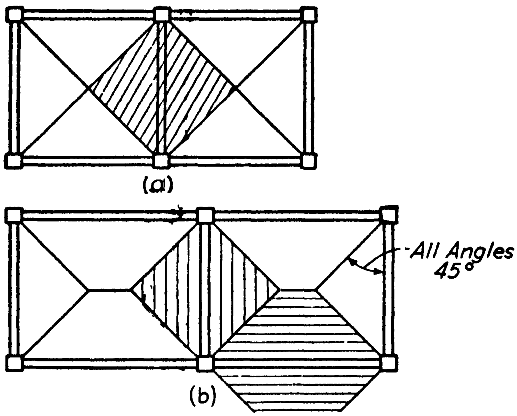


FIG. 16–9

The second problem in design is the distribution of the slab load to the supporting beams. An early recommendation was that each beam carry the load within the diagonals of the supported panels applied as a triangle with the vertex at mid-span as indicated on Fig. 16–9. The 1928 A.C.I. Code, in Art. 713*f*, recommended that supporting beams be designed for the reaction of the tributary strips of slab carrying the portion of the total load obtained from the $(L/B - \frac{1}{2})$ relationship without permitting any live load reduction and assuming that this load is applied uniformly throughout the span length of the beam. The 1940 J.C. Code recommends that the beams carry the portions of the load represented by the shaded triangle and trapezoid in Fig. 16–9*b*.

The applications of the 1940 J.C. recommendations are best understood from the following illustrations.

* *Op. cit.*

TWO WAY INTERIOR SLAB PANEL — Sheet TWS 1

Data

$f'_c = 3000$ psi
$f_s = 20000$ psi
$n = 10$
Code: 1940 J C

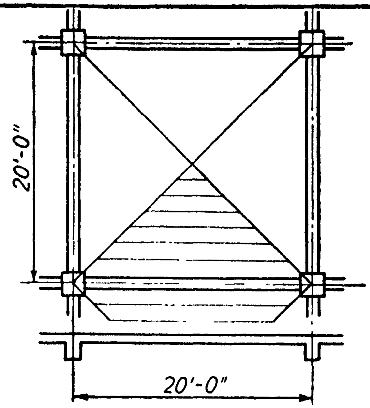
Live Load = 125 psf
1" Grano = 13
5½" Slab = 69
207 psf

$$\text{Min } t = \left[S + \frac{S}{m} - \frac{N}{10}\right]\frac{1}{72}\sqrt[3]{\frac{2500}{f'_c}}$$

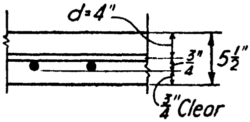
$$= \left[240 + 240 - 96\right]\frac{1}{72}\sqrt[3]{\frac{2500}{3000}}$$

= 5.02, Use 5½"

See TWS 2



Slab — 5½" Thick



$$v = \frac{207 \times 20^2}{4 \times 12 \times 19 \times \frac{7}{8} \times 4} = 26 \text{ psi}$$

$$u_1 = \frac{0.6 \times 26 \times 19 \times 12}{13 \times 1.57} = 174 > 150 < 225 \text{ SA}$$

$$u = \frac{26 \times 19 \times 12}{32 \times 1.57 + 9 \times 1.18} = 98 < 150$$

Length of Top Rods

$$2\left[\frac{0.11}{0.48} \times \frac{20 \times 12}{5} + 40 \times \frac{3}{8}\right] = 52''$$

Use 4'-6"

Table 5, J C 811 $+M = 0.025\, wS^2$ per 12" wide
$-M = 0.033\, wS^2$ " " "

$+M = 0.025 \times 207 \times \overline{20}^2 \times 12 = 24\,800$ lb-in./ft

$$R = \frac{24\,800}{12 \times 4^2} = 129 \text{ psi}$$

$$A_s = \frac{24\,800}{17500 \times 4} = 0.36\ \square''/\text{ft}$$

½φ-6½" c/c = 0.37 □"/ft Center Half
½φ-10" c/c = 0.24 " Outer Quarters
Bend alt bars for −M. Stagger Bends

$-M = 0.033 \times 207 \times 20^2 \times 12 = 32\,800$ lb-in./ft

$$R = \frac{32\,800}{12 \times 4^2} = 171 \text{ psi}$$

$$A_s = \frac{32\,800}{17500 \times 4} = 0.47\ \square''/\text{ft}$$

Truss Rods - ½φ @ 6½" c/c = 0.37 □"/ft
Top Rods - ⅜φ @ 12" c/c = 0.11
0.48 □"/ft

½"φ-6½" c/c Center Half
½φ-10" c/c Outer Quarters Alt. Bent.
⅜φ-12" c/c Top

Beams — 9⅝"×20"

$$\text{From Slab } W = \frac{2 \times 207 \times \overline{20}^2}{4} = 41\,400^{\#}$$

Beam = 3 000
44 400

Try 9⅝ × 20" (% end spans)

$$v = \frac{22000}{9\frac{5}{8} \times \frac{7}{8} \times 17} = 155 \text{ psi}$$

$$u_1 = \frac{0.60 \times 155 \times 9\frac{5}{8}}{2 \times 4.00} = 112 \text{ psi} < 150$$

$$u = \frac{155 \times 9\frac{5}{8}}{2 \times 4} = 186 > 150 < 225 \text{ SA}$$

$$a = \frac{99}{155} \times 114'' = 70''$$

$$A_v = \frac{9\frac{5}{8} \times 70 \times 95}{2 \times 16000} = 2.00\ \square''$$

V = 22 200

$$\frac{wL^2}{12}\quad M = \frac{4}{3} \times 44\,400 \times 19 = 1{,}125{,}000 \text{ lb-in.}$$

$$t/d = \frac{5\frac{1}{2}}{17} = 0.324 \quad \text{Max } R = 229$$

$$T = \frac{1{,}125{,}000}{229 \times 17^2} = 17''$$

$$A_s = \frac{1{,}250{,}000}{17500 \times 17} = 3.78\ \square''$$

2-1□S + 2-1□B = 4.00 □"
(2 Layers)

10 ∅-⅜φ = 1.98 □"

Spaced 2,4,4,5,5,6,7,8,8,8

2-1□S
2-1□B(SA)

10-⅜φ ⊔
2-4-4-5-5-6-7-8-8-8

SHEET TWS1

Example 16-3. Design a typical interior square panel 20 by 20 ft c to c of columns as a two-way slab with supporting beams to carry a 1-in. granolithic finish and a live load of 125 psf. This would be an alternative design of a panel for the building designed in Chapter XVII. Use 1940 J.C. Code: $f'_c = 3000$ psi; $f_s = 20{,}000$ psi; $n = 10$.

Solution. (See Computation Sheet TWS1.) The determination of minimum slab thickness proceeds directly from the equation in J.C. 814. The proper moment factors for a two-way slab continuous an all four sides are taken from Table 5, J.C. 811. The effective depth is obtained from a sketch, where the effect of two layers of steel crossing at right angles is shown. Although the effective depth one way of the panel is greater than that the other way by a bar diameter, it is customary to use the lesser depth throughout, as otherwise careful instructions and inspection would be required to insure placing steel as designed. Rods are first proportioned for the center half width of panel, and then the steel in the two outer quarters is chosen to provide two-thirds of this area. The Code permits varying the spacing across the quarter-strip but this results in steel placing that is too complicated for the ordinary job.

The supporting beams are designed in the customary fashion, the load being taken as that supported by the shaded area on the figure. Since the diagonal lines are taken from the exact center lines of the columns it may be contended that a small area of load is included that does not directly rest upon the beam, but there are so many empirical factors in the design of such panels that it seems conservative and simple to put one-quarter of a square panel on each of the supporting beams. Also note that this load is triangular, with a simple beam moment of $WL/6$ instead of the $WL/8$ of a uniform load; hence in the moment computation the total load is multiplied by $4/3$ to compensate for the triangular load distribution. The result is multiplied by the clear span and by 12 to reduce to inches and divided by 12 to obtain the moment for a fully continuous beam. In proportioning stirrups the shear curve is treated as a triangle such as would result from a uniform load. This is not correct as the real shear curve would be parabolic, concave downward on the left end with a greater contained area than a triangle It is doubtful if the empirical methods justify this refinement in handling shears, though the student should have no difficulty in doing so if required. A detail of this slab is shown in Fig. 16-10.

Example 16-4. Design a typical exterior panel of two-way slab construction for the condition shown on Computation Sheet TWS2. Also design a corner panel for the same building. Data and specifications as for Ex. 16-3. These would be side and corner panels for an alternative framing of the building designed in Chapter XVII.

Solution. The computations on Sheet TWS2 follow the same course as those on Sheet TWS-1 and but little explanation is necessary. Note that under the J.C. Code a panel lacking continuity on one or more sides has the amount of positive steel in *both* directions increased, not merely the steel in the direction which lacks continuity. Note also that the corner panel with a lack of continuity on two sides requires more positive steel than the side panel continuous on three sides. The J.C. Code recognizes the restraint on the non-continuous edges developed by the torsion in the spandrel beams, and extra top steel is required, which might be obtained by extending the outer legs of the spandrel stirrups to, and then bending out in, the top of the slab. The detail drawing of these panels is shown in Fig. 16-10.

TWO-WAY SIDE AND CORNER SLAB PANELS — Sheet TWS 2

Data

Design Data as on TWS1

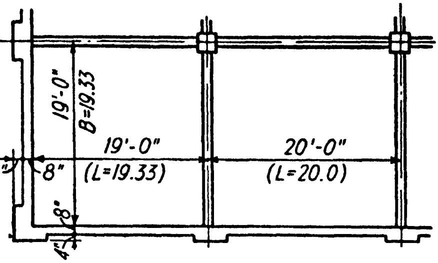
$$t = \left[S + \frac{S}{m} - \frac{N}{10}\right]\frac{1}{72}\sqrt[3]{\frac{2500}{f'_c}}$$

$$= \left[232 + \frac{232}{1} - \frac{464}{10}\right]\frac{1}{72}\sqrt[3]{\frac{2000}{3000}}$$

$= 5.46''$

$t = 5\frac{1}{2}''$

$d = 4''$



Side Panel

$m = \frac{19.33}{20} = 0.97$

$$+M = \begin{cases} 0.031 \times 207 \times \overline{20}^2 \times 12 = 30\,800 \text{ lb-in./ft} \\ 0.032 \times 207 \times \overline{19.33}^2 \times 12 = 29\,700 \text{ lb-in./ft} \end{cases}$$

Shear and Bond o.k from TWS 1

$A_s = \frac{30\,800}{17500 \times 4} = 0.44\ \square''/ft$

$\frac{1}{2}\phi$ @ $5\frac{1}{2}''$ c/c = 0.436 □"/ft center half

$\frac{1}{2}\phi$ @ $8\frac{1}{2}''$ c/c = 0.282 □"/ft outer quarters

$$-M = \begin{cases} 0.041 \times 207 \times \overline{20.0}^2 \times 12 = 40\,700 \text{ lb-in./ft} \\ 0.042 \times 207 \times \overline{19.33}^2 \times 12 = 39\,000 \text{ lb-in./ft} \end{cases}$$

$R = \frac{40\,700}{12 \times 4^2} = 212$ $\qquad A_s = \frac{40\,700}{17500 \times 4} = 0.581\ \square''/ft$

Length of Top Rods

$2\left[\frac{0.15}{0.586} \times \frac{20 \times 12}{5} + 40 \times \frac{1}{2}\right] = 5'\text{-}5''$

Truss Rods - $\frac{1}{2}\phi$ @ $5\frac{1}{2}''$ c/c = 0.436 □"/ft

Top Rods - $\frac{1}{2}\phi$ @ 16" c/c = 0.150

0.586

$-M = 0.021 \times 207 \times \overline{19.33}^2 \times 12 = 19\,500$ lb-in./ft

$A_s = \frac{19\,500}{17500 \times 4} = 0.297\ \square''/ft$

Truss Rods - $\frac{1}{2}\phi$ @ 11" c/c = 0.218 □"/ft

Top Rods - $\frac{3}{8}\phi$ @ 18" c/c = 0.073

0.291

Data (Side Panel): $\frac{1}{2}\phi$ @ $5\frac{1}{2}''$ center half; $\frac{1}{2}\phi$ @ $8\frac{1}{2}''$ outer quarters; $\frac{1}{2}\phi$ @ 16" c/c Top; $\frac{3}{8}\phi$ @ 18" c/c Spandrel

Corner Panel

$m = 1$ $\qquad +M = 0.037 \times 207 \times \overline{19.33}^2 \times 12 = 34\,400$ lb-in./ft

R = ✓ $\qquad A_s = \frac{34\,400}{17500 \times 4} = 0.49+\ \square''/ft$

Shear and Bond o.k from TWS 1

$\frac{1}{2}\phi$ @ 5" c/c = 0.48 □"/ft center half

$\frac{1}{2}\phi$ @ $7\frac{1}{2}''$ c/c = 0.32 □"/ft outer quarters

$-M = 0.049 \times 207 \times \overline{19.33}^2 \times 12 = 45\,500$ lb-in./ft

$R = \frac{45\,500}{12 \times 4^2} = 237 > (236 \text{ allowed})$ $\qquad A_s = \frac{45\,500}{17500 \times 4} = 0.65\ \square''/ft$

Truss Rods - $\frac{1}{2}\phi$ @ 5" c/c = 0.48

Top Rods - $\frac{1}{2}\phi$ @ 14" c/c = 0.17

0.65

$-M = 0.025 \times 207 \times \overline{19.33}^2 \times 12 = 23\,200$ lb-in./ft

R = ✓ $\qquad A_s = \frac{23\,200}{17500 \times 4} = 0.33+\ \square''/ft$

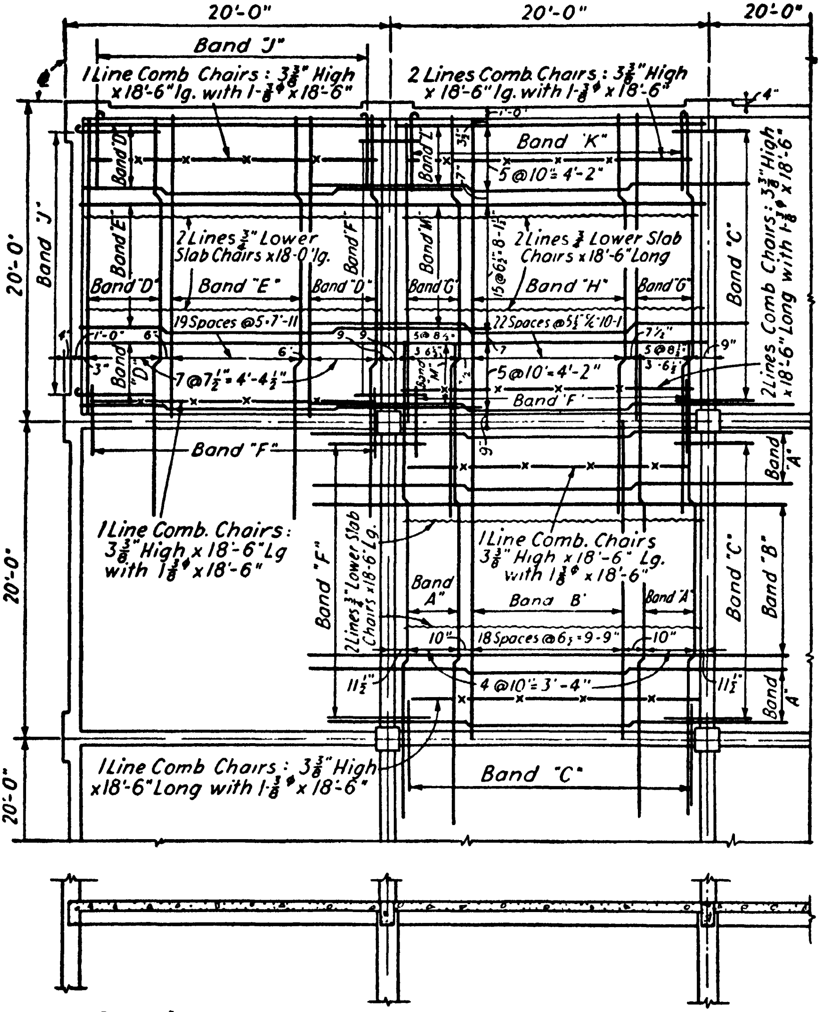
Truss Rods - $\frac{1}{2}\phi$ @ 10" c/c = 0.24 □"/ft

Top Rods - $\frac{3}{8}\phi$ @ 15" c/c = 0.09

0.33

Data (Corner Panel): $\frac{1}{2}\phi$ @ 5" c/c Center Half; $\frac{1}{2}\phi$ @ $7\frac{1}{2}''$ c/c Outer Quarters; $\frac{1}{2}\phi$ @ 14" c/c Top; $\frac{3}{8}\phi$ @ 15" c/c Spandrel

SHEET TWS2



Band "A"
3-½φ x 30'-3" TS1 Bent
2-½ x 20'-0" Straight

Band "B"
10-½φ x 20'-0" Straight
9-½φ x 30'-3" TS1 Bent

Band "C"
19-⅜φ x 4'-6" Top @ 12" %

Band "D"
4-½φ x 19'-6" Straight
4-½φ x 25'-8" TS 2 Bent

Band "E"
10-½φ x 19'-6" Straight
10-½φ x 25'-8" TS2 Bent

Band "F"
15-½φ 4'-6" Top @ 14" %

Band "G"
3-½φ x 19'-6" Straight
3-½φ x 25'-8" TS2 Bent

Band "H"
11-½φ x 19'-6" Straight
12-½φ x 25'-8" TS2 Bent

Band "J"
15-⅜φ x 5'-0" SP1 @ 15 %

Band "K"
13-⅜φ x 5'-0" SP1 @ 18" %

Band "L"
3-½φ x 20'-0" Straight
3-½φ x 30'-3" TS1 Bent

Band "M"
8-½φ x 20'-0" Straight
8-½φ x 30'-3" TS1 Bent

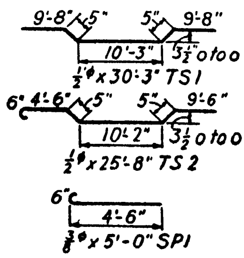


FIG. 16–10

16-4. Flat Slab Floors. The flat slab, first called the "mushroom floor," by the originator, Mr. C. A. P. Turner, is a type of construction distinctive to concrete. As can be seen on Fig. 16-11 these slabs have no supporting beams except at the margins, but rest directly on columns which are usually built with enlarged heads, called capitals. Often a portion of the slab about the column capital, called a drop panel, is made thicker than the rest of the floor. Because of their economy and

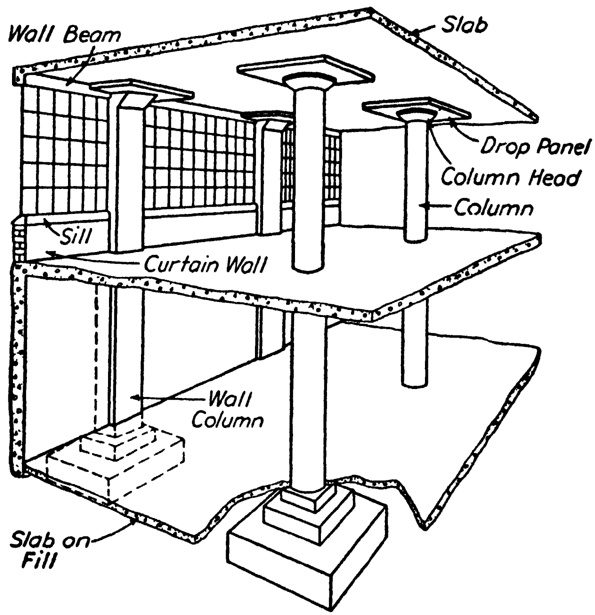


FIG. 16-11 (reprinted by permission from "Concrete Plain and Reinforced," Taylor, Thompson, and Smulski, Vol. I, John Wiley & Sons, Inc.)

other advantages flat slabs have largely replaced beam and girder construction in buildings adapted to their use.

Flat slab floors are suitable for use in buildings at least two and preferably three bays wide, where the column spacing can be made fairly regular, with panels from 17 to 30 or more feet each way and live loads of 100 psf or more. Forms are much less expensive than in beam and girder construction, which offsets savings in steel and concrete possible in the older type of design. The great saving, however, is in building height. A commercial building has a required clear story height which, added to the floor thickness, gives the floor-to-floor height. A flat slab will be one to two feet less in overall thickness, effecting a large saving in columns, walls, and partitions.

The flat ceiling of the girderless floor offers several advantages because of the absence of beams; the easier layout of sprinklers and of any other piping or shafting supported under the ceiling; easier artificial lighting; better day lighting with windows that extend to the ceiling; and better ventilation because of the absence of pockets in the ceiling.

Since corners are the most vulnerable parts of concrete masses exposed to fire, flat slab construction suffers far less in fire than do beam and girder buildings. The flat slab type of building is primarily adapted to industrial use — factories, warehouses, and garages — but because of low cost it is sometimes used very satisfactorily for stores, hotels, and office buildings. The main drawback to these latter uses is the difficulty of satisfactory architectural treatment of interiors;* however, there are now numerous buildings in which these difficulties have been reasonably well overcome.

There are four common systems of reinforcing flat slabs: the two-way, the four-way, the circumferential, and the three-way. The first three have columns at the corners of rectangular panels, and the last has the columns arranged at the apexes of equilateral triangles. The two-way system has reinforcement parallel to the column center lines both ways, the steel in the half panel centered on the columns being heavier than the intermediate bands between columns. The four-way system replaces the intermediate bands of the two-way with two lines of reinforcement along the diagonals of the panels. The circumferential (Smulski or S.M.I. system) uses hoops and radial rods centered on the columns and the intersection of the panel diagonals. The three-way system has bands parallel to the sides of triangular panels.

All flat slabs were originally patented systems but the fundamental patents have expired. The four-way was the original system of Mr. Turner. As originally built it had the disadvantage of four layers of steel over the columns which reduced the effective depth and made concreting difficult. The two-way steel arrangement does not come so near paralleling all lines of stress with rods as does the four-way, but it is simple to design and construct and seems in every way satisfactory. The circumferential and the three-way are somewhat more complicated in details and are designed by the patentees. The Smulski circumferential system probably arranges steel to take stress more directly than any other and often effects a saving in the weight of steel required.

The shape taken by a continuous, loaded flat slab, supported on points, is shown by the heavy lines of Fig. 16-12, study of which shows

* In the "flat ceiling" construction of W. A. Wheeler, Minneapolis, Minn., structural steel frames buried in the slab over the columns take the place of the drop panel and enlarged column capital.

where tension steel is required. Such a slab may be analyzed approximately by considering it to be, first, a beam spanning from AD to BC, and, second, a beam spanning from AB to CD. By supplying the

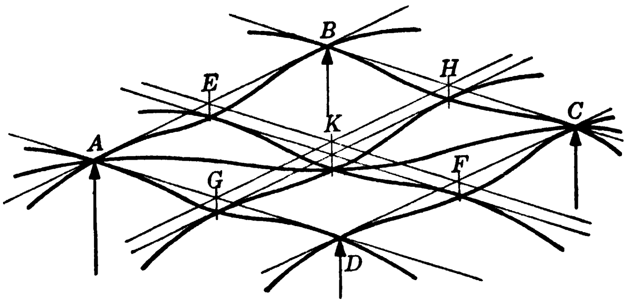


FIG. 16–12

steel required in each assumed beam the slab is safely reinforced in all directions.

The notation in Fig. 16–13 is that of the Joint Committee for the case in which the slab is considered as spanning from AD to BC, the breadth

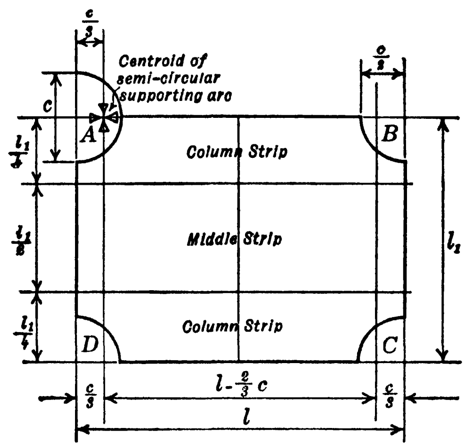


FIG. 16–13

of the assumed beam being l_1. The supports are along the quarter-circles that represent the partial outlines of the column capitals, and the span may be considered as the distance between the centroids of the supporting arcs, $l - 2c/3$. The total height of the parabolic moment curve of a uniformly loaded fixed-ended beam is $wL^2/8$, two-thirds of this being negative moment. Similarly, the theoretical height of the

moment curve,* for a uniformly loaded flat slab in the direction l, is

$$M_0 = \tfrac{1}{8}(wl_1)[(l - 2c/3)]^2 \qquad [16\text{-}1]$$

where wl_1 is the load in pounds per foot of length of beam. By interchanging l_1 and l in this equation the expression applies to the moment for the slab considered as a beam spanning in the l_1 direction.

These descriptions give a general idea of the deflection of a flat slab and the statical limitations to the total combined positive and negative moments across an entire panel width at mid-span and on a column center line, respectively. They do not shed any light upon the distribution of these moments throughout the slab, which, as the problem is statically indeterminate, can only be obtained from a study of the deformations and stresses. An accurate analysis becomes very involved because of two sets of factors, one of which merely adds tremendously to the amount of work in the analysis whereas the other may seriously affect the applicability of the results to a practical design problem. These factors include: (*a*) the three-dimensional nature of the problem, introducing many unknowns; (*b*) the correct value of Poisson's ratio which enters into any problem of three-dimensional elasticity (but wide variations in this, fortunately, will not affect the final results more than perhaps 15 to 20 per cent); (*c*) the relative stiffness of supporting columns (a flat slab building is a series of interlocked rigid frames at right angles to each other); (*d*) the stiffening of the column head section by the drop panel; (*e*) the tension resistance of the concrete (this is a source of considerable change in relative stiffness of parts of the slab and so affects the distribution between positive and negative moments); (*f*) a readjustment of stresses under load as experienced in load tests of actual structures, and the possibility of developing inclined supporting forces in the column cap with so-called " dome action."

The original development of flat slabs was an empirical one based upon statical considerations modified by the results of experience and tests. More recently elaborate mathematical attacks have been made upon the problem.† Although these in a general way justify our current practice they are difficult to follow because of the amount of mathematical work involved, and the results agree with tests of actual structures only roughly, partly because the researches solved only certain

* John R. Nichols, " Statical Limitations upon the Steel Requirements in Reinforced Concrete Flat Slab Floors," Trans., A.S.C.E., 1914.

† See H. M. Westergaard and W. A. Slater, " Moments and Stresses in Slabs," Proc., A.C.I., 1921; A. Nädai, Elästische Platten, Julius Springer, 1925; Dr. Ing. H. Marcus, Die Theorio elastischer Gewebe and ihre Anwendung auf die Berechnung biegsamer Platten, by Julius Springer, 1924; and Joseph A. Wise, " The Calculation of Flat Plates by the Elastic Web Method," Proc., A.C.I., 1928.

idealized problems and partly because of the various factors just discussed.

The 1940 J.C. Code prescribes quite definite rules for the design of regular panels in structures of three or more approximately equal spans in each direction, and is based upon the results of tests and experience modified by the above researches. For irregular panels and for partial panels which occur at stair, elevator, or other openings, the Code is less definite and the designer must take into account the probable deformation of the structure as an indication of the points at which tension reinforcement is required; in such cases familiarity with theoretical studies may be of assistance in visualizing requirements. The 1940 J.C. Code divides the panel into strips, a half-panel in width, called, respectively, the column strip and the middle strip, as indicated on Fig. 16–13, and tabulates moment coefficients in Art. 835, Table 6, for positive and negative moments in each of these strips for various conditions of restraint. Some adjustment of the distribution between positive and negative moment is provided for in Art. 836, depending upon the relative stiffnesses, but in any event the total moment must equal 0.09/0.125 of that given by equation 16–1. The Code further specifies taking into account bending in the supporting columns under unbalanced loading and particularly in wall columns. Torsion in the marginal beams should also be investigated. The application of the 1940 J.C. Code is best understood from the following illustrative examples.

Example 16–5. Design a two-way slab for a typical interior panel 20 by 20 ft c to c of columns to carry a live load of 125 psf and a 1-in. granolithic finish. Specifications, 1940 J.C.: $f'_c = 3000$ psi; $f_s = 20{,}000$ psi; $n = 10$. This is a typical interior panel of the building designed in Chapter XVII.

Solution. (See Computation Sheet FS1.) To allow for the dead weight of slab assume a total thickness of $\frac{1}{32}$ to $\frac{1}{40}$ of the column spacing. The minimum slab thickness is computed from J.C. 839*b* and confirms the choice. Earlier specifications frequently limited the minimum thickness to $L/32$ for floors and $L/40$ for roofs, but the later codes permit somewhat thinner slabs, though the designer must guard against excessive deflections and the difficulty of accommodating the different layers of steel. The J.C. formula for minimum slab thickness is solved here for an interior panel and on FS2 for side and corner panels. As the latter requires a thicker slab the $6\frac{1}{2}$-in. thickness will be used throughout. At the drop the thickness results from J.C. 839*c*. Note that, because of the irregular distribution of bending moment across the width of column strip, the drop thickness is made 20 per cent greater than the mean value obtained from $\sqrt{M/Rb}$. The width of drop panel comes from J.C. 841*c* as 6 ft 8 in. Diameter of column cap, from J C. 841*a*, is 4 ft 0 in., and, as explained in Chapter VI, this is made in multiples of 6 in. to suit standard steel column molds. For computing external moments and shears the total load on the panel is determined; the weight of the drop panel is included, but not that of the column capital. The total bending moment comes from

FLAT SLAB INTERIOR PANEL	*Sheet FS1*

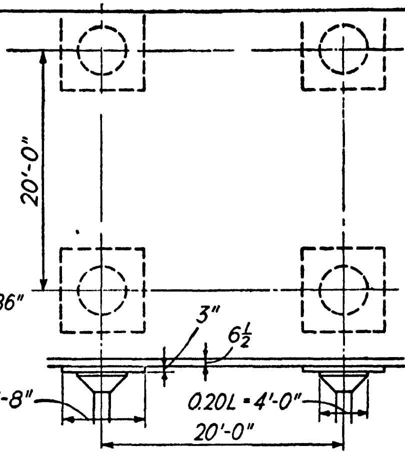
$f_c' = 3000$ psi
$f_s = 20\,000$ psi
$n = 10$

Code: 1940 J C
Live Load = 125 psf
1" Grano = 13
6½" Slab = 81
219 psf

Min $t = 0.026\,L\sqrt[3]{\frac{2500}{f_c'}}$

$= 0.026 \times 20 \times 12 \times \sqrt[3]{\frac{2500}{3000}} = 5.86''$

But See FS2 and
Use 6½" Slab



Data

$W = 20 \times 20$ @ 219   87 600
Drop $= 6.7^2 \times 38$ 1 700
89 300#

$M_o = 0.09\,(L - \frac{2}{3}C)^2 \frac{W}{L}$

$= 0.09\,(17.33)^2 \times 12 \times \frac{89\,300}{20}$

$= 1{,}450{,}000$ lb-in.

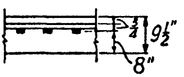
Column Strip:

$+M = 0.20\,M_o = 290\,000$ lb-in. $R = \frac{290\,000}{120 \times (5\frac{1}{2})^2} = 80 < 236$

$A_s = \frac{290\,000}{17500 \times 5\frac{1}{2}} = 3.02$ sq in $\left\{\begin{matrix} 4\text{-}\frac{1}{2}"\text{ø}S \\ 9\text{-}\frac{1}{2}"\text{ø}B \end{matrix}\right\} = 3.25$ sq in.

$-M = 0.50\,M_o = 725\,000$ lb-in. $t_1 = 1.2\sqrt{\frac{725000}{80 \times 236}} + 1\frac{1}{2}'' = 9''$ $R = \frac{725\,000}{80 \times (8)^2} =$ 142 < 236

Use 9½" Drop



$A_s = \frac{725\,000}{17500 \times 8} = 5.18$ sq in. $\left\{\begin{matrix} 18\text{-}\frac{1}{2}"\text{ø}B = 4.50 \text{ sq in.} \\ 3\text{-}\frac{1}{2}"\text{ø}\ T = 0.75 \end{matrix}\right.$
5.25 sq in.

Middle Strip:

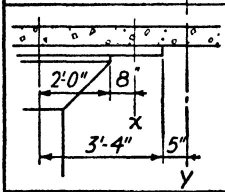
$+M = 0.15\,M_o$ 218 000 lb-in. $R = \checkmark$

$A_s = \frac{218\,000}{17500 \times 5} = 2.49$ sq in. $\left\{\begin{matrix} 5\text{-}\frac{1}{2}"\text{ø}S \\ 5\text{-}\frac{1}{2}"\text{ø}B \end{matrix}\right\} = 2.50$ sq in.

$-M =$ do.   do. $5\text{-}\frac{1}{2}"\text{ø}B + 5\text{-}\frac{1}{2}"\text{ø}B = 2.50$ sq in.

Slab
6½"
Drop
6'-8" sq × 3"
4-½"ø S
9-½"ø B
3-½"ø T

5-½"ø S
5-½"ø B



$V_x = 87\,600 - \pi\,\overline{2.67}^2 \times 219 = 82\,700$ lb

$v_x = \frac{82\,700}{\pi\,63 \times \frac{7}{8} \times 8} = 60$ psi < 90 psi

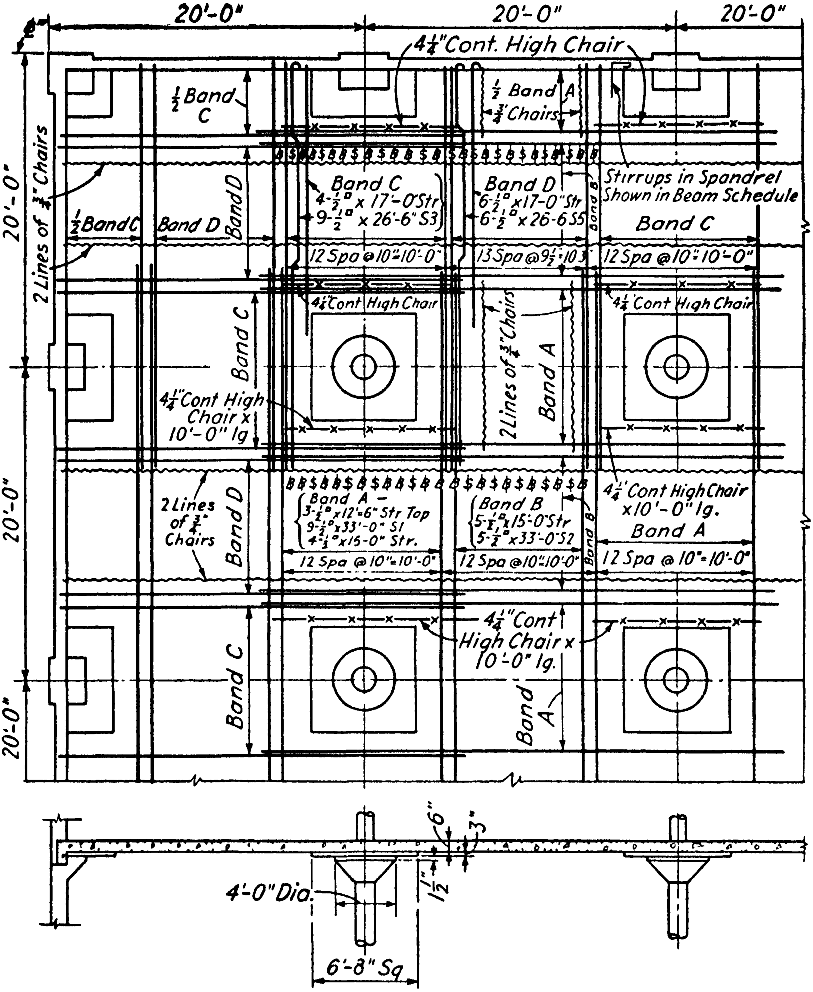
$V_y = 87\,600 - \overline{7.50}^2 \times 219 = 75\,300$ lb

$v_y = \frac{75\,300}{4 \times 92 \times \frac{7}{8} \times 5\frac{1}{2}} = 43$ psi < 90 psi

Shear
O.K

$u = \frac{82\,800}{4 \times 12 \times 2 \times \frac{7}{8} \times 8} = 123$ psi < 150 psi

Bond
O.K.



Placing Instructions

1. Place Lower Slab Bar Spacers Shown 〰〰〰
2. Straight Bars Only Bands A C D
3. Cont. High Chairs Bands A Close to Drop Panel Shown —x—x—
4. Bent Bars and Top Bars Bands A ↕ C
5. Bent Bars and Top Bars Bands A C ⟷
6. Straight Bars Only Bands B and D ⟷
7. High Chairs Bands B and D —x—x—
8. Bent Bars Bands B and D ⟷
9. Straight Bars Only Bands B and D ↕
10. High Chairs Bands B and D ↕
11. Bent Bars Bands B & D ↕

Fig. 16–14

J.C. 835*b*, Formula 9, and the distribution to positive and negative moments in the column and middle strips respectively follows from the values in J.C. 835, Table 6. In checking f_c the full width of strip is used except for negative moment in the column strip where J.C. 839*c* limits the effective width to that of the drop panel proper.

In selecting steel a bar size is chosen that will keep the bar spacing somewhere between t and $2t$ and whose bond stress will be within the allowable. For this case ½ in. squares seem suitable. A slightly greater bar size might be used at greater spacing with some saving in "size extras" but very much larger bars would be far apart and have high bond stress. For the column strip 40 per cent of the positive steel is run straight into the drop panel (J.C. 843) and the balance is bent up over each column and extended past the point of inflection of the adjoining span to reinforce for negative moment. Even then extra top rods are required to make up the steel area needed for negative moment in the column strip. For the middle strip positive and negative moments are equal, and bending up half of the bars from each side will just take care of the negative moment in this strip. Shear is checked on a vertical surface around the column cap and distant the effective drop thickness from it (J.C. 820*a*), and around the periphery of the drop panel and distant the effective slab thickness from it (J.C. 820*b*). These are both within the allowable; the bond is also. On page 349 this slab is detailed completely, and Fig. 16–14 should be referred to at this time for bending of bars, arrangement of spacers, etc.

Example 16–6. Design two-way flat slab side and corner panels for the conditions and specifications of Ex. 16–5. (See Sheet FS2 and Fig. 16–15.)

Solution. Slab thickness, sizes of column cap and drop panel, and the design of the continuous strips that run parallel to the wall are all taken from FS1; note that the exterior panel governs thickness. The non-continuous strips perpendicular to the wall have their moments increased in accordance with J.C. 838. The concrete compression is within the allowable in all cases. Drops will be used at the exterior columns (J.C. 841*d*) with brackets underneath (J.C. 841*b*). The amount of reinforcing steel is increased where necessary in non-continuous bands, in this example by adding to the number of bars.

Two types of spandrel beam are considered, the first 12 by 18 in. with slot for sash (d = 15.6 in.) (see page 265, paragraph *h*), the second a wide, flat beam flush with the exterior drop panels to improve the lighting by raising the sash. The load from the slab is taken as one-quarter of a panel (J.C. 840). The weights of a vertical wall section are taken from page 264. The bending moment is computed from $wL^2/10$ but the vertical shears are treated as for a simple span. (See page 247.) Torsional shear is a factor for marginal beams in flat slab construction and is provided for as in Examples 14–7 and 14–8. In this case the total applied torque is taken as the negative moment at the exterior end of a middle strip. (Many designers use 0 75 or 0.80 of this moment.) The balance of the computations will be clear if compared with Examples 14–7 and 14–8.

Bending must be provided for in the exterior columns (J.C. 849 and 859). The Code does not specify an exact amount. Earlier codes used $WL/60$ divided between the columns above and below the floor according to their relative stiffnesses, or 84,100 × 19.33 × (12/60) = 325 k-in. It would seem logical to use the sum of the negative moments at the end of a column and a

FLAT SLAB SIDE AND CORNER PANELS — *Sheet FS2*

For Sketches See Text

Specifications as on FS1

Min $t = 0.30 \times 19.33 \times 12 \times \sqrt[3]{\frac{2500}{3000}} = 6.54''$ Use $6\frac{1}{2}''$ Slab

Strips Parallel to Wall as on FS1

Data, $t = 6\frac{1}{2}''$, $t' = 9\frac{1}{2}''$, Cont. Strips from FS1

$W = 20 \times 19.33 \text{ @ } 219 = 84\,700$ lb $M_o = 0.09\left(19.33 - \frac{2\times4}{3}\right)^2 \times 12 \times \frac{86\,400}{19.33}$

Drop $= \underline{1\,700}$ $= 1{,}340\,000$ lb-in

$86\,400$ lb

Column Strip Perpendicular to Wall:

$+M = 1.15 \times 0.20 \times 1{,}340{,}000 = 308\,000$ lb-in. $d = 5\frac{1}{2}''$ R = ✓

$A_s = \frac{308\,000}{17500 \times 5\frac{1}{2}} = 3.20$ sq in. $4\text{-}\frac{1}{2}''S + 9\text{-}\frac{1}{2}''B = 3.25$ sq in.

$-M_{int} = 1.15 \times 0.50 \times 1{,}340{,}000 = 771\,000$ lb-in. $d = 8''$ $b = 80''$ $R = 151 < 236$

$A_s = \frac{771\,000}{17\,500 \times 8} = 5.51$ sq in. $18\text{-}\frac{1}{2}''B + 4\text{-}\frac{1}{2}''T = 5.50$ sq in.

$-M_{ext} = 0.80 \times 0.50 \times 1{,}340{,}000 = 536\,000$ lb-in. $d = 8''$ R = ✓

$A_s = \frac{536\,000}{17500 \times 8} = 3.83$ sq in. $9\text{-}\frac{1}{2}''B + 14\text{-}\frac{3}{8}\phi$ ⊓ $= 3.79$ sq in.
One Leg

Middle Strip Perpendicular to Wall:

$+M = 1.30 \times 0.15 \times 1{,}340{,}000 = 261\,000$ lb-in. $d = 5''$ R = ✓

$A_s = \frac{261\,000}{17\,500 \times 5} = 2.98$ sq in. $6\text{-}\frac{1}{2}''S + 6\text{-}\frac{1}{2}''B = 3.00$ sq in.

$-M_{int} = 1.30 \times 0.15 \times 1{,}340{,}000 = 261\,000$ lb-in. $d = 5\frac{1}{2}$ R = ✓

$A_s = \frac{261\,000}{17500 \times 5\frac{1}{2}} = 2.71$ sq in. $6\text{-}\frac{1}{2}''B + 6\text{-}\frac{1}{2}''B = 3.00$ sq in.

$-M_{ext} = 0.80 \times 0.15 \times 1{,}340{,}000 = 161\,000$ lb-in. $d = 5\frac{1}{2}''$ R = ✓

$A_s = \frac{161\,000}{17500 \times 5\frac{1}{2}} = 1.68$ sq in. $6\text{-}\frac{1}{2}''B + 10\text{-}\frac{3}{8}\phi$ ⊓ $= 2.60$ sq in.
One Leg

Shear and Bond O.K. (See FS1)

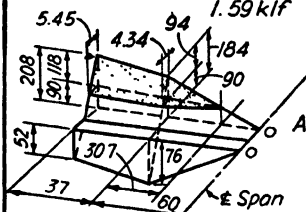
Non-Cont Strips

Column: $4\text{-}\frac{1}{2}''S$, $9\text{-}\frac{1}{2}''B$, $4\text{-}\frac{1}{2}''T$, $14\text{-}\frac{3}{8}\phi$ ⊓

Middle: $6\text{-}\frac{1}{2}''S$, $6\text{-}\frac{1}{2}''B$, $10\text{-}\frac{3}{8}\phi$ ⊓

Slab $= \frac{19.33}{4} \times 219 = 1.06$ klf

Sash	0.08	$L = 16.13$ ft	$W = 25.6$ k	$\frac{WL}{10} = 495$ ki
Sill	0.06	12 x 18	$v = 78$ psi	$A_s = 1.81$
Spandrel	0.19	$d = 15.6$	$u = 199$ SA	$2\text{-}\frac{3}{4}\phi S$ } = 1.88
Beam	0.20		$u_t = 133$ {SA	$1\text{-}1''B$ }
	1.59 klf		{% Torsion	
			$-u = 117$	$-R = 170$



$v_t = \frac{161\,000}{2}\left[\frac{15 \times 18 + 9 \times 12}{5 \times 18^2 \times 12^2}\right] = 130$ psi

$A_v = \frac{\left[\frac{5.45 \times 118 + 4.34 \times 94}{4} \times 37 + \frac{4.34 \times 94}{2 \times 3} \times 30.7\right]}{16000}$

$= 0.95$ sq in.

$11\phi\text{-}\frac{3}{8}\phi$ One Leg ⊓ $= 0.99$

2,4,4,5,6,6,7,8,8,8,8 & 12" to ℄
34¼

Beam FB1

12 x 18, $2\text{-}\frac{3}{4}\phi SA$, $1\text{-}1''B$, $11\text{-}\frac{3}{8}\phi$ ⊓ Ea. End 2,4,4,5,6,6,7, 8,8,8,8 & 12" to ℄

Use $2\text{-}\frac{5}{8}\phi$ One Each Top Corner

$L = 16.13$	$W = 25.6$	$M = 495$	$b = 43$
43 x 9	$v = 50$	$R = 235$	$A_s = 4.04$
$d = 7$	$V_t = \frac{161\,000}{2}\left[\frac{15 \times 9 + 9 \times 43}{5 \times 43^2 \times 9^2}\right]$		$5\text{-}\frac{3}{4}\phi S$ } = 4.40
			$5\text{-}\frac{3}{4}\phi B$ }
	$= 58$		$u = 182$ SA
	$v + v_t = 108$		$u_i = 110$
	$v_c = 90$ (SA)	$\frac{3}{8}\phi$ 12" c/c ⊓	

Alternative FB1

43 x 9, $5\text{-}\frac{3}{4}\phi SA$, $5\text{-}\frac{3}{4}\phi B$

Use $2\text{-}\frac{5}{8}\phi$ One Ea. Top Cor.

SHEET FS2

middle strip = 488 + 147 = 635 k-in. The computation of stresses would be similar to Ex. 13-6, page 272.

The detailing of these flat slab panels is shown on Fig. 16-14. For the continuous bands in the column strips 4-½ in. squares 15 ft 0 in. long straight represent one-quarter of all the positive steel in the band extended at least 20 bar diameters into each drop (J.C. 843). The embedment is actually 25 diameters to give an even foot cutting length. The 9-½ in. squares 33 ft 0 in. long bent represent over four-tenths of all the bars in the band (J.C. 843)

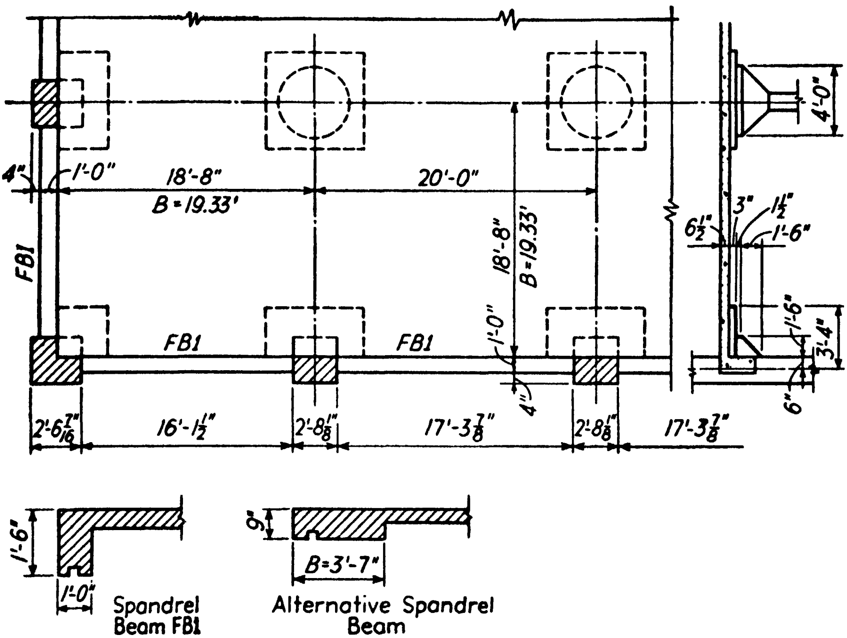


FIG. 16-15

and the length equals 20 ft 0 in. c to c of columns, plus 4 in. for trussing, plus 2 × 0.25 × 20 ft 0 in., or twice the distance from column center to point of inflection (J.C. 842), plus (2 × 20 ft 0 in.)/15 or two ends each of one-fifteenth of the span beyond the point of inflection (J.C. 843), a total of 33 ft 0 in. The length of the four ½ in. square 12 ft 6 in. long top bars over the column is determined as 2 × 0.25 × 20 ft 0 in. plus 2 × (20 ft 0 in.)/15 or 12 ft 8 in., the shorter length giving an even half foot and, as the bent bars extend the full distance, the difference of an inch or two is negligible. For the middle bands bars are alternately straight and bent, the positive and negative moments being equal. The lengths of bars are made the same as in the column strip to minimize the number of pieces to handle.

For the non-continuous bands additional bars are shown. At the spandrels the straight bars extend to within 2 in. of the outside face (J.C. 838*d*) but hooks are not needed as the bond is within the allowable. The drawing shows the location and spacing of straight and bent bars, the type and location of supporting chairs and placing instructions. These should be studied carefully

because by following the order given no bars will have to be slid under those already there. Each item will be merely set down on top of those already placed.

This method of detailing and scheduling bars in bands saves time for both the office and field; it is obvious that all bands of a similar mark have the same arrangement of bars. Ordinarily at the point where the bars in a given band are called for the total number of that type of bands on the entire floor will be given for convenience in ordering materials.

16-5. Irregularities in Flat Slabs. Nearly every large flat slab building has irregularities that present design problems of considerable complexity. It is not possible in this text to cover all the possibilities but two common difficulties will be noted: openings in the floors and varying outside bays. An opening of less than 2 ft on a side outside the drop panel can usually be taken care of simply by spreading the reinforcement to make place for it. In the drop panel even a small opening should be avoided, but if one is necessary, the concrete stress should be checked at the reduced section. In all cases of openings, corner reinforcement is desirable along the lines suggested on pages 277 and 283. In roofs, skylights are often needed. If they are less than half the panel width in maximum dimension it is sometimes possible, by centering them in the panel, to cut out the mid-section steel which would pass through the opening. This is possible because the skylight area carries little load. This should not be done in adjacent panels, and wherever it is done extra rods should be provided for the negative mid-section moment to replace the bent rods which are omitted. Where extra loads are applied around the edge of openings, as may often be the case with elevator or other shafts having a partition around them, beams should be used. As soon as a beam is put in, however, it takes slab load, owing to its greater stiffness, and often requires girders to carry its reactions to the columns. Beams are therefore a distinct disadvantage. Some designers take care of irregular openings by considering the column strips as beam systems supporting a two-way slab at right angles to them, represented by the central part of the mid-section. If a typical panel is analyzed on this basis it will be found that the moment coefficients reduced for full continuity, using the center-to-center distance l for the span, are on the safe side.

It sometimes happens that a flat slab building must cover an irregular lot. In such cases it is usually best to make the interior panels rectangular and take care of the variation in overall dimensions by end panels of varying span. The steel for such spans can be easily computed, but when the outside span is less than half the main span it should be noted that the entire short span may have negative moment. It is also necessary in detailing such cases to be sure that extra steel is used over

the first interior support to replace that lost by reducing the bent steel in the end panel. Sometimes the irregularity of outside spans is increased by an architectural treatment which requires a spacing of exterior columns varying from the typical. A slight lateral deflection of typical bands can be made but if the eccentricity of the exterior columns is more than about 10 per cent of the span the columns can be neglected and the slab rested on a stiff spandrel considered as a simple support. In special cases of flat slabs any designer not thoroughly experienced in this class of construction should work on a very conservative basis.

16–6. Stairs. No treatment of building design, however brief, would be complete without some description of stairs. In reinforced concrete construction stairs are inclined slabs with triangular treads formed on the upper surface. In Fig. 16–17*a* is shown an arrangement suitable for the flat slab building in these pages. The stairs shown are designed as simply supported slabs with a span equal to the horizontal distance between the floor beam, B7, and the landing beam, B8. The effective thickness of the inclined portion of the slab is the distance d shown in Fig. 16–16. In computing the dead weight per horizontal foot for use in design the inclined part should be considered to have the total thickness of t_1; the treads add 40 to 50 lb per horizontal square foot, an amount readily allowed for by increasing t_1 by t_2, i.e., one-half the rise.

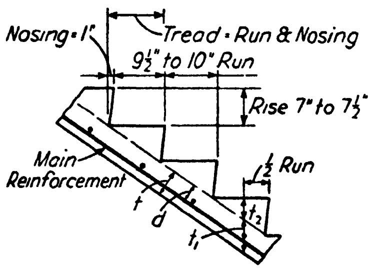


FIG. 16–16

The dimensions for rise and tread vary in any structure but should be kept as closely the same as possible for all stairs in any single building. The dimensions shown in Fig. 16–16 are common.

Stairs are usually poured after the floors are finished. The connection to the supporting floor beams is made by means of recessed joints and dowels as shown on Fig. 16–17*b*. The arrangement of the reinforcement in the stair shown in section should be studied. The main rods do not run from the lower floor beam to the exterior wall continuously in the bottom of the slab. The bars from the inclined slab extend into the top of the landing slab and run over to the wall where they are often hooked. There is quite likely to be negative moment in the landing. The bars in the bottom of the landing slab opposite this flight extend in their turn into the top of the inclined slab and then bend down for at least 40 bar diameters. If bars were ex-

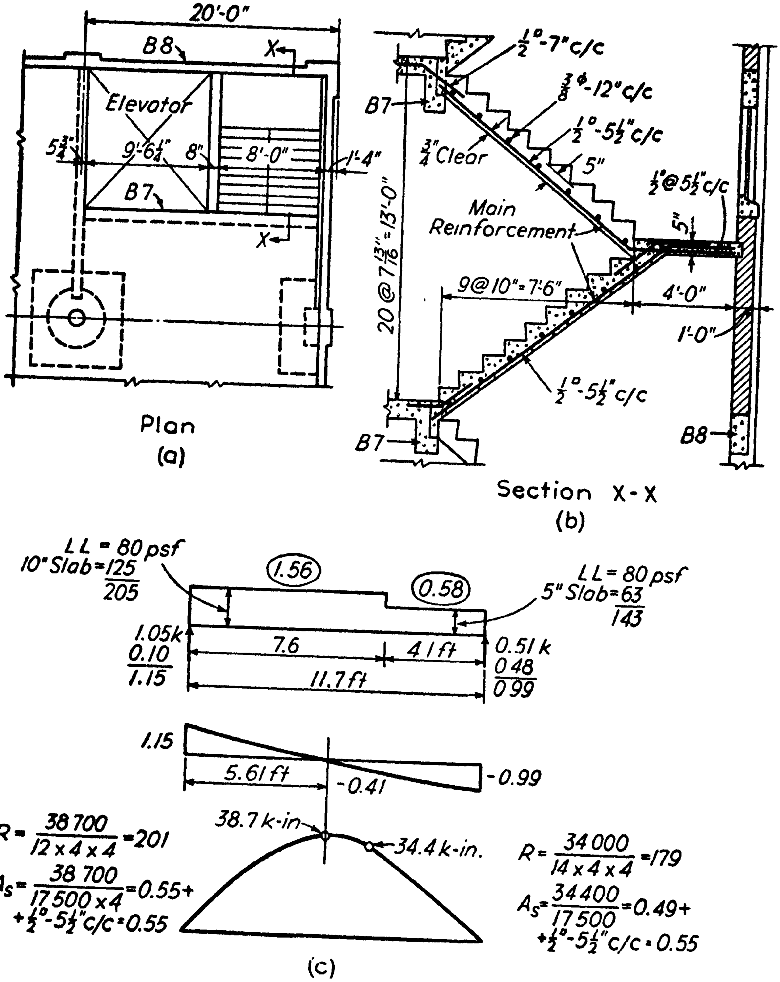


Fig. 16–17

tended continuously in the bottom of this slab around this reentrant angle they would exert a strong bursting pressure on the concrete as they tended to straighten under stress. For temperature reinforcement it is customary to place a single ⅜ in round bar under each tread.

Example 16–7. Design a stair slab for the condition shown on Fig. 16–17*a*. Specification, 1940 J.C.; $f'_c = 3000$ psi; $f_s = 20{,}000$ psi; $n = 10$.

Solution. The design computations are shown on Fig. 16–17*c*. The load curve is clear from the foregoing description. The live load is taken as 80 psf (various codes specify from 60 to 100 psf or more but in the absence of

code requirements 80 psf is a good average value). An assumed 5-in slab thickness at the heel of the stair normal to the soffit makes the total amount of concrete to allow for in the dead weight about 10 in. The landing slab is assumed 5 in. thick. The shear and moment curves follow directly, the moment curves being computed for freely supported ends. It is not ordinarily necessary to plot shear and moment curves but they are shown here for the student's benefit and to illustrate that stair slabs are designed as horizontal spans, but that they use the inclined effective depth. (The student should demonstrate that using the vertical load and the horizontal projection of span for a sloping beam is entirely equivalent to using the actual sloping span and the transverse load.) The slab depth is checked to keep f_c within the allowable and the reinforcing steel is proportioned as shown. In selecting dowels at B7, a moment of $WL/10$ should be provided for or 30.9 k-in., requiring ½ in. squares, 6¾ in. centers in the top of the slab as shown.

CHAPTER XVII

EXAMPLE OF BUILDING DESIGN

17-1. The rapid growth in favor of reinforced concrete as a material for building construction is due to its durability, its fire-resisting properties, and its relatively low cost. A reinforced concrete frame can almost always be built more cheaply than one of structural steel which is fireproofed. Usually it can be erected in less time after the completion of the plans than a steel structure which has to wait for the necessary shop work on the steel. "A floor a week" is a common standard for progress of erection by competent contractors when conditions are favorable.

In ordinary construction reinforced concrete is used for the entire frame, floors, columns, and footings. In tall buildings the columns are often made of structural steel encased in concrete to save the floor space that would be occupied by reinforced concrete columns. The latest column formulas and the use of high concrete strengths are gradually resulting in smaller reinforced concrete columns in tall buildings. Reinforced concrete is also much used for the floor slabs in steel frame buildings.

There are three main types of reinforced concrete floors:

(1) Solid slabs combined with beams and girders (Fig. 17-1).

(2) Ribbed slabs (concrete joist construction) with beams and girders (page 328).

(3) Flat slabs with slabs resting directly on the columns, made either uniform in thickness or with increased depth about the columns (Fig. 16-11).

17-2. General Problem. On the preceding pages theories have been developed for the design of solid slabs, rectangular beams, tee-beams, doubly reinforced beams, columns, stairs, walls, and footings. Computing moments and shears by the methods of continuity has been discussed. In this chapter it is proposed to apply these principles to the design of a simple building. Many practical problems arise in the application of the foregoing theories. Considerations other than strength or economy frequently govern the selection of members.

Structural computations are made for one of two purposes: to design a new structure of adequate size and strength for an owner's require-

ments, or to review the design of an existing structure. In making the design of a new structure it is necessary to establish the preliminary data, such as story heights, length and width of building, spacing of columns, location and arrangement of stairs and elevators, positioning

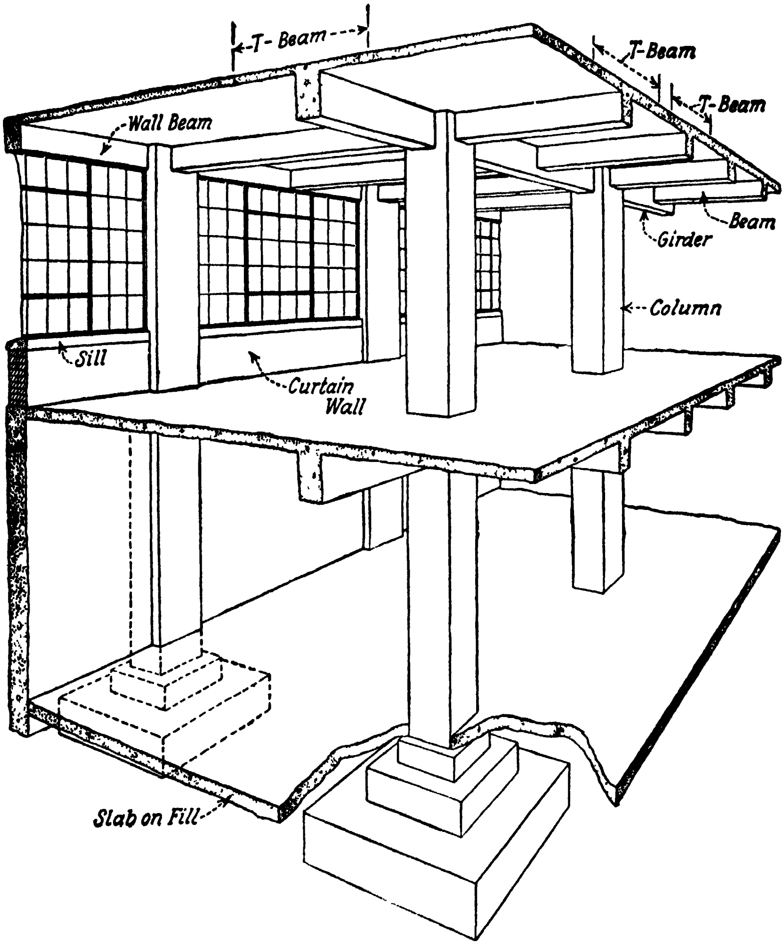


FIG. 17–1. Perspective View of Beam and Girder Skeleton (reprinted by permission from "Concrete Plain and Reinforced," Taylor, Thompson and Smulski, Vol. I. Published by John Wiley & Sons, Inc.).

of doors and windows, loads to be carried, stresses to be used, and so on. Since many hours of computation will be spent upon this design it is important that these data be worked out with great care. Too often the preliminary studies fail to take into account the contour of

the ground, some special needs of the occupant, code requirements for egress, safety, or sanitation, or various pieces of mechanical equipment, such as hoists, elevators, chutes, air conditioning ducts, etc., with the result that a good deal of the design work has to be done over.

When a consulting engineer is engaged by an owner to prepare plans and specifications for a structure he will first make a rough sketch of the building with the sizes of members proportioned in a general way and prepare an approximate cost estimate. These data are then discussed with the owner to check with his requirements, compared with state codes and city ordinances, and correlated with the various mechanical trades. When a satisfactory compromise between all of these is effected the design computations can be started.

In making the preliminary studies structural engineers often prepare cost estimates of various methods of framing using different column spacings and beam arrangements and compare different systems of floor construction. It is very embarrassing when a competitive design presents an equally good and much less expensive method of framing.

In the pages which follow the complete design of a modern commercial building is presented in essentially the form which the computation work actually assumes in the engineer's office, including:

(*a*) Preliminary sketches to establish data (Fig. 17-2).

(*b*) Memorandum of owner's and code requirements (Art. 17-3).

(*c*) Design computations, using arbitrary moment coefficients (Computation Sheets BG1-BG13).

(*d*) Detail drawings for the entire building (Chapter XXI).

Modern codes tend to require a more exact determination of bending moments than is given by the use of arbitrary coefficients and accordingly certain parts of this design are repeated:

(*e*) Design of typical floor slabs using three-moment equation, graphics, approximate moment distribution and 1940 J.C. coefficients. (Chapter XVIII.)

(*f*) Design of typical bent of columns and girders using P.C.A. approximate moment distribution, 1940 J.C. moment distribution, more exact moment distribution, and slope deflection. (Chapter XVIII.)

17-3. Data for Slab, Beam, and Girder Design. (Refer to Fig. 17-2 in reading the following.)

Owner's Requirements

Use of building: Light manufacturing, loft building.
Approximately 22,000 sq ft. of floor area
Located on railroad siding.

Equipped with stairs* and elevator (5-ton capacity).
Clear ceiling heights: Basement = 9 ft 0 in., upper floors = 11 ft 0 in.
Floor finish: Cement.
Walls: Brick or concrete with large steel sash.

Code Requirements

Exit facilities in case of fire.*
Toilet provisions.
Adequate lighting.
Ventilation.
1940 J.C. Code for methods of design.

Insurance Requirements

Fireproof construction (obtains advantageous rates on building and contents).
Sprinkler system to protect contents.
Fire doors on elevator hatch and stair towers.
Double exit facilities.

Designer's Suggestions

Building 60 ft by 100 ft. (The width was chosen because good natural daylighting can be projected to the center of a 60-ft building. The length was selected to make the floor area required.)

Three stories high with basement mostly below grade. (A higher building would be expensive for construction and operation. A lower building would have a lower first cost but the owner here states that the manufacturing processes permit gravity flow down through the building and that real estate available prevents spreading farther.)

Type of construction selected. Solid slab on beams and girders of reinforced concrete. (Fairly economical for this type of building and load, and quick and easy to build. For comparisons see Chapter XXII.)

Column spacings. 20 ft 0 in. each way. (The only choices crosswise would be 3 at 20 ft or 4 at 15 ft. Although the latter proved to be a trifle cheaper the difference did not warrant the obstruction of the additional columns. Lengthwise we might have used 5 at 20 ft, 6 at 16 ft 8 in. or 7 at 14 ft 3 in. For symmetrical panels and general all-round serviceaability we selected 5 at 20 ft.)

Live loads. Roof 40 psf; floors 125 psf; stairs 80 psf (roof and stairs set by codes; for the floors, 125 psf is adequate for light manufacturing). For the selection of live loads see State Codes and "Report of Building Code Committee, U. S. Department of Commerce," Nov. 1, 1924. Naturally the lower the live load the less the total cost, but most manufacturing buildings sooner or later have heavy loads imposed upon them. Even if the owner's occupancy is extremely light, too small a live load will affect adversely the resale value of the building.

* All codes require double exit facilities in opposite corners of the building. Ordinarily two stair towers would be provided. In this case we have assumed that a fire escape in one corner diagonally opposite the elevator would serve.

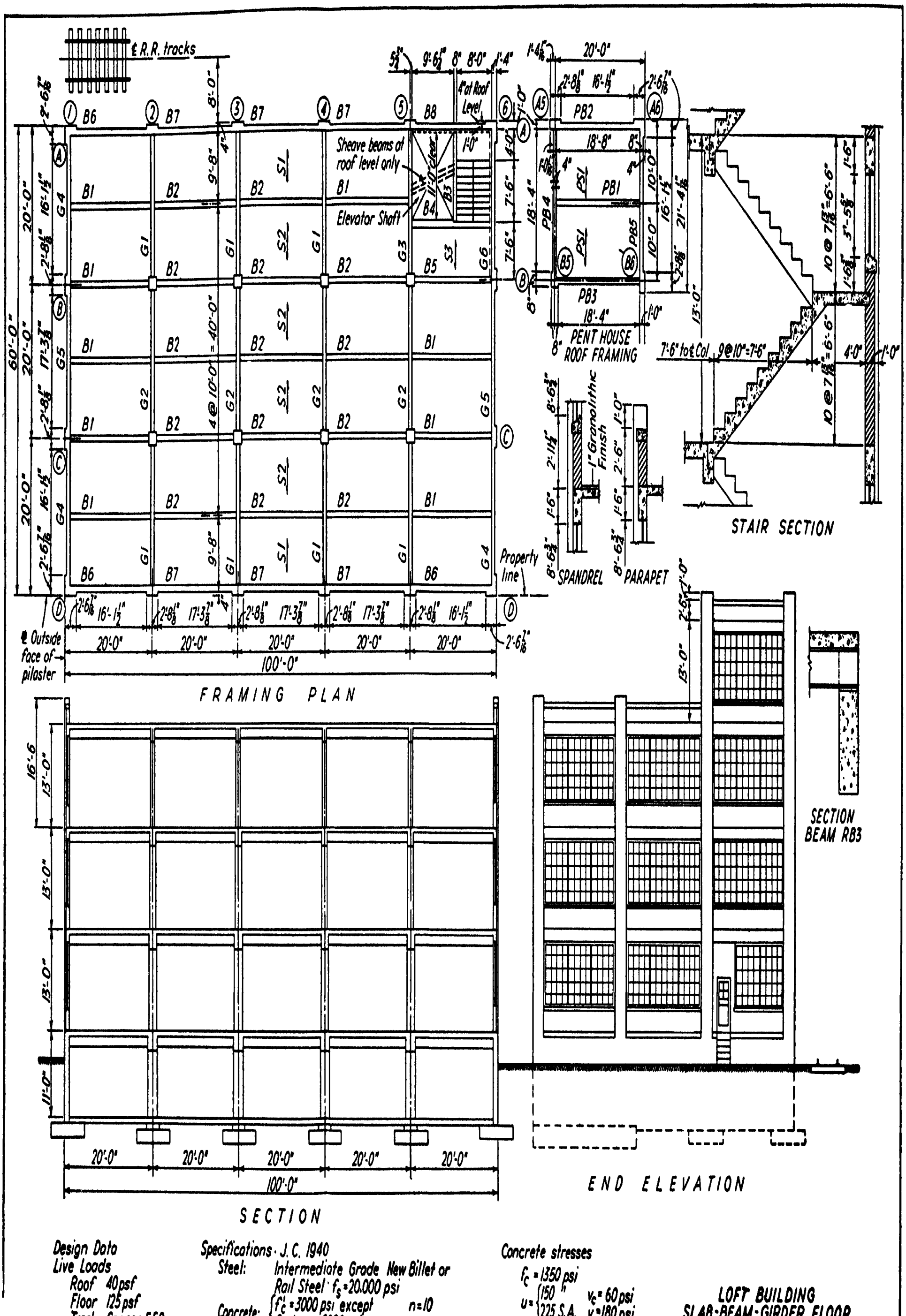

℄ R.R. tracks
Sheave beams at roof level only
Elevator Shaft
4" at Roof Level
℄ Outside face of pilaster
Property line
FRAMING PLAN
PENT HOUSE ROOF FRAMING
1" Granolithic Finish
SPANDREL
PARAPET
7'-6" to ℄ Col.
9 @ 10" = 7'-6"
STAIR SECTION
SECTION BEAM RB3
SECTION
END ELEVATION
Design Data
Live Loads
Roof 40 psf
Floor 125 psf
Track Cooper E50
Specifications · J. C. 1940
Steel: Intermediate Grade New Billet or Rail Steel · f_s = 20,000 psi
Concrete: f'_c = 3000 psi except footings = 2000 psi
n = 10
Concrete stresses
f_c = 1350 psi
u = 150 ″ / 225 S.A.
v_c = 60 psi
v = 180 psi
R = 236 psi
LOFT BUILDING
SLAB-BEAM-GIRDER FLOOR

Fig. 17-2

Story heights. Basement 11 ft 0 in., upper stories, 13 ft 0 in. (This allows about 2 ft for floor girders.)

Window openings. Sills about 3 ft above floor; piers about 2 ft 6 in. wide. Exact sizes of spandrel beams and columns to be made to fit standard sash as shown on detail drawings. (Concrete sizes are flexible but steel sash sizes are fixed.)

Property line. As the building is against the property line on one side it will there be necessary to use combined footings.

Soil bearing. On the basis of familiarity with the site, 5000 psf is selected as a safe value for the footings. (Loading tests of the soil are usually made unless designer is reasonably sure of safe values.)

Stresses. Ready-mixed concrete, bought on a strength specification, will be used. Cost analyses indicate that 3000 psi concrete is more economical than 2500 and is safely obtainable. For the footings only, 2000 psi concrete will be used as this may be job-mixed and control of quality is harder in early stages.

For reinforcing steel. Intermediate grade deformed new billet or rail steel bars will be specified.

Live load reduction. No reduction of live load will be taken on the beams as there is every likelihood of the entire panel being loaded. Although some reduction might be permissible on the girders, it was decided to take full load on these as well. Since not all floors will be loaded simultaneously the columns have live loads reduced 10 per cent under the third floor, 20 per cent under the second, and 30 per cent under the first.

17-4. Methods of Computation. In a design office doing any volume of business several designers will work on the same problem. It is important that office standards be adopted permitting the computers and detailers to follow the work readily. On buildings of ordinary size, the computations become a fairly thick book. It is not worthwhile to record all the formulas used nor to set down the factors that were multiplied and divided. By following the outline presented on the following pages any detailer can find the data he needs and any computer can reproduce the results from the figures recorded. This method is the result of many years of experience with different designers on all sorts of structures.

As has been noted, the first step is to prepare a sketch plan along the lines shown on Computation Sheet BG1 and developed on Fig. 17-2. This establishes dimensions, span lengths, story heights, and clearances. The spandrel beams, exterior columns, and parapets are indicated for sizes and loads. Next the front page of the computations, the head sheet, is filled out with the live loads, specification requirements, stresses, and moment factors. On this sheet each structural member is assigned a mark consisting of a prefix such as " S " for slabs, " B " for

beams, " G " for girders, and " C " for columns, followed by a serial number. Members are further prefixed " R " for roof, and " F " for floors (or, if the floors vary, with " 3 " for third floor, " 2 " for second floor, and so on.) Columns are prefixed with the number of the floor which they *support.* As members are worked out, dimensions that affect other members are transferred to the key plan. An index is accumulated which shows on which page of the computations the design of any particular member can be located. Frequently designers check off members on the key plan with different-colored pencils as the various floors are designed, detailed, and estimated.

Study the method of dimensioning from outside face of pilasters, taken as the building line, abbreviated " BL." This follows the common method of laying out by setting nails in batter boards to mark the building lines. Piano wires stretched between these nails make very real building lines from which all measurements are made. If the building steps back on upper floors continue to dimension to the same building lines, enclosing the building in a box. In this way there is no chance of columns or shafts not being directly over each other when the architectural treatment varies from story to story.

Note the marking of columns by lettered and numbered coordinates. This is simple and easy on a plain rectangular building. When the floor is irregular and columns do not line up an individual number for each column in numerical sequence, enclosed in a circle for identification, is preferable. No two columns ever have the same number, as they are the standard fixed points for locating all other details.

17-5. Precision. There is no possible value in carrying figures beyond the range of the ordinary 10 in. slide rule almost universally used for computing. Since loads are merely assumed in the first place to represent as well as possible the occupancy of the building, since allowable stresses are a matter of judgment, since there is a tolerance of 3 per cent in the rolling of reinforcing bars and an even greater tolerance in the concrete sizes, and since many assumptions are made in the mathematical theories, it should be obvious that hair-splitting of decimals is an impropriety. On the other hand, in using the slide rule no additional expense occurs in carrying as many significant figures as are available on the rule. Improved field methods are resulting in more accurate positioning of rods and closer adherence to concrete sizes. The improved quality of the concrete itself is so well known as to need no comment. As a matter of practical psychology, unless the computer sets some degree of precision to govern his work he becomes careless. The following standards work well:

Carry loads to the nearest 1 psf
10 plf
100 lb of concentration.

Take span lengths to nearest 0.01 ft (using relation ⅛ in. is 0.01 ft)
Record total loads and reactions to nearest 0.1 kip; moments to nearest 0.1 kip-in. if readable, or to limit of slide rule with large numbers.
Take individual bar areas to nearest 0.01 sq in. as individual bars vary that much from theoretical.
Concrete sizes are usually taken to ½ in., except that beam widths are often taken as 7⅝ in., 9⅝ in., and 11⅝ in. to suit standard dressed plank soffits.
Bar spacings are usually taken to nearest ½ in. because wire bar chair supports (Fig. 21–11) are often crimped at 1-in. intervals and the odd ½ in. is obtained by alternating long and short spacings.

17–6. Design of Slab, Beam, and Girder Building. Design and detail drawings for this building are shown and explained in Chapter XXI. Better understanding of the meaning of the following computations may be obtained by referring to these detail drawings from time to time. With the key plan completed and loads and specifications selected, we can start the computations. These are best made on quadrille ruled paper to facilitate arranging columns of figures and supplementary sketches. On Computation Sheets BG1 to BG13 are the design figures for the building shown on Fig. 17–2. These are made in the abbreviated form of office practice. As almost all designs are made at the designer's desk with curves and tables handy the computations here are made that way.

These computations represent actual office procedure, routine design made with the degree of care usually exercised, following upon approximate preliminary computations. Since the sizes of supporting members are not known accurately in advance these computations will be found to differ slightly from those which would be made were a higher degree of accuracy necessary.

In Chapter XIII the standard forms here used for the design of slabs, beams, and columns are explained in detail and illustrated by careful discussion of the working up of the computations for certain typical members. These abbreviated forms represent certain definite operations which are presented in full in Ex. 13–1 for a slab, in Ex. 13–2 for a beam, and in Ex. 13–4 for a column. The student should have made several complete unabbreviated computations for each sort of member before coming to this chapter, and as he reads he should have an appropriate unabbreviated example at his side for comparison as he

considers the abbreviated record for any particular piece. The remarks which follow are intended to explain choices of dimensions and other values, and the various decisions of detail necessary as the design proceeds.

The design of a reinforced concrete frame is something more than a direct mathematical solution of stress formulas. Considerable judgment and experience are required to obtain a well-balanced structure. Many sizes and combinations of members would be sufficiently strong but, to obtain a simple and easily constructed arrangement, the designer is continually making choices and decisions such as: between the use of deep, narrow beams which are ordinarily most economical, or wider and shallower ones to reduce story heights and possibly the cost of the entire structure; to keep all beams of the same concrete size for simplicity in formwork or to change size for various load and span conditions; whether or not to take into consideration such refinements as the displacing of stirrups by the bent-up portions of the tension steel, the inclusion of part of the bent-up bars in bond computations at the point of inflection, the use of all or a portion of the straight-bottom bars as compressive reinforcement for negative moment at the support and the anchorage of this steel, the choice between simple bar combinations of slightly deficient area and odd combinations that are in excess of the area required; whether or not to change the size of all beams because of a peak negative moment condition at one spot. This list could be extended indefinitely and it is in the meeting of theoretical stress requirements with solutions that are simple, economical, and practical that the designer does his best work. The preparation of design computations is a long and expensive task. It is unnecessary to consider refinements that would serve only to reduce the apparent stress but would not change the design. Factors that relieve stresses may be allowed for without direct computation: for example, if the bond stress at the free end of a run of continuous beam, computed on the basis of end shear as for a simple span, is somewhat higher than the code allowance, the designer may compensate by taking into account the reduced end reaction due to continuity. Comparison with previously designed members may indicate that the member under consideration is amply strong for some particular function and time is saved by checking off that item without detailed computation. Even if some minor readjustment might be possible, simplicity and economy may result from duplicating the previous detail.

The beam and girder design in this article is purposely arranged to illustrate these principles of choice and judgment. If the members

had been a few inches larger no such problems would arise, no opportunities would occur to explain the weighing of different factors, and the final building would have been considerably more expensive. The design of individual members has been explained in detail in Chapter XIII and the purpose here is to afford training in overcoming the difficulties that will arise in keeping the chosen sizes at certain critical points.

17–6A. Pent House Roof Slab. See Computation Sheets BG1 and BG2. (Read Ex. 13–1 in particular, noting the abbreviated office practice on page 244.) Roofs usually are drained; three methods are in vogue: (*a*) a level concrete roof slab subsequently built up with cinder concrete or gypsum fill to produce high points between the drains that take the water down either inside the building or in conductors on the outside, (*b*) a warped concrete roof slab whose upper and lower surfaces conform to the slopes from high to low points, and (*c*) a dead level roof on which most of the water finds its way to the drains and the balance evaporates. A decision must be made, as the weight of any fill is to be included in the loads: in this case a level roof is used. The live load comes from the design data. The allowance for roofing is for the insulation and the waterproof roofing surface. The weight of any fill would have been included here. The slab weight and thickness are assumed at first and *carefully checked* after the design is completed. Moment factors are from J.C., Appendix 3; span lengths from center to center of beams taken from the key plan on Sheet BG1 are used.

The design of temperature (or shrinkage) reinforcement is frequently left to the detailer, but the reinforcement should be between 0.002 and 0.0025 of the total slab volume, and preferably of small rods 12 or 15 in. c to c for ease in placing.

17–6B. Pent House Beams and Girders. (See Computation Sheet BG2 and read carefully Examples 13–2 and 13–3.) Loads are computed from the slab loads, hence the need for checking these as soon as the slabs are designed. The weight of parapet and coping is figured from the section on Fig. 17–2 as 2 ft of 8-in. wall at 80 psf plus 6 by 10 in. of coping at 150 pcf plus snow and ice on top. For the spandrel beams and girders the overall size is taken from the typical floors, thus keeping the same appearance on the outside but requiring padding the inside form ½ in. because the roof slab is that much thinner than the floor slab. The interior beam size is chosen so as to use the forms from the floors below with a minimum of refabrication. A typical floor beam is roughed out to establish dimensions. This has already been discussed (page 250) where a 9⅝ by 18 in. beam was selected. Because

of the ½ in. thinner roof slab a 9⅝ by 17½ in. beam is used here. Moments are computed* as for a simple beam. Ex. 13–3 illustrates the method of handling concentrated loads; for PB4 with symmetrical ends, the moments of the uniform and concentrated loads are figured separately and added. For PB5 the computation is: $[(7.46 \times 8.67) - (3.29 \times 8.67/2)]12 = [(7.46 - (3.29/2)]8.67 \times 12 = 5.815 \times 8.67 \times 12$, the factoring being done mentally and the last multiplication on a slide rule. The corner columns are relatively large and will offer considerable restraint. The spandrel girders afford some small torsional end restraint to the interior beam. No advantage is here taken of the reduction in positive moment, the beam sizes and amount of reinforcement seeming reasonable. In extreme cases these moments could be evaluated as illustrated in Chapter XVIII. Approximately one-half of the positive steel is bent up and anchored in the supports for negative moment with a capacity of $WL/16$ at ordinary working stresses or more if higher unit stresses are considered.

In determining effective depths 2 in. is deducted from the total depth for beams with one layer of steel, 3 in. for those probably having two layers (the spacing of bars is recorded for a check), and allowance is made in the spandrel beams for the sash slot described on page 265. J.C. 504 requires 2½ bar diameters spacing for round and 3 for square bars. The effective depth for shear and bond is taken the same as for moment even though with two layers of steel, one of which is bent up, the shear depth is greater. This is safe and in a critical case a revised computation could be made. To check the flexural stress in the concrete, $R = M/bd^2$ is computed or investigated in each case as it is just as important to have sufficient concrete as adequate steel.

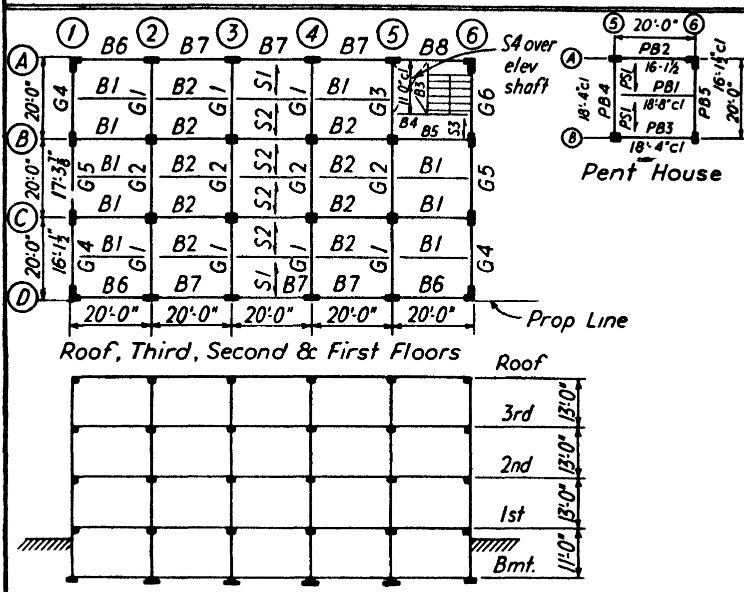
End shears are treated as for simple spans; this is also correct for any symmetrical system of end moments. Stirrups are proportioned to carry the entire excess shear above 60 psi on the concrete. No advantage is here taken of the bent-up tension steel, partly because of the possibility that the bent portion will not be in the field of high shearing stresses and partly because of the time and expense consumed in spacing stirrups around the bent portion of bar as compared with the relatively

* The computer is urged to familiarize himself with the CI scale of the slide rule, as much resetting is saved and increased accuracy gained by handling three or four factors at one time. It is often advantageous to repeat the formula mentally while setting the corresponding numerical values on the slide rule. The order of operations shown permits multiplying unit load by span for total load and again by span and moment coefficient to obtain the bending moment while still holding the setting to divide by bd^2 to obtain R, all in one continuous chain of operations, thus avoiding the minor discrepancies of recording and resetting values. Since standard units of feet, kips, kip-inches, etc., are maintained throughout it is not necessary to record the dimensions of the results.

BEAM AND GIRDER BUILDING Sheet BG1

60 x 100 Ft Loft Bldg For_____ At_____ By____ Date____ — *Title*

Layout



Specs

Loads: Roof = 40 psf
Stairs = 80"
Floors = 125"
Track = E50
Impact = 0%
Earth 28.6 pcf liquid
Surcharge = 3'-0"

L.L. Reduction:
Supporting 3rd – 10%
2nd – 20%
1st – 10%

Soil Pressure: 5000 psf (D L + Red. L. L)

Stresses		Superstructure	Footings
f_c'	=	3000 psi	2000 psi
f_s	=	20 000	20 000
n	=	10	15

Moments: Slabs by 1940 J.C.; Beams & Girders $\frac{wL}{8}$; $\frac{wL}{10}$; $\frac{wL}{12}$

Check by more exact methods later

Tied Columns – $f_c = 540$ (Y.P. Method)
$f_s = 12800$

Index

Prefix / Mark	PS	PB	P.H. Cols	RS	RB	RG	Roof Cols	FS	FB	FG	Main Cols	Walls	Ftgs
1	2	2	2	2	3	4	6	6	7	8	10 11	11	11-13
2		2		3	3	4		6	7	8			
3		2		3	3	5		7	7	9			
4		2		3	3	5			7	9			
5		2			4	5			7	9			
6					4	5			8	9			
7				Stairs on 3	4				8				
8					4				8				
9													
10													

Prefix all Computation Sheets BG__.

SHEET BG1

small cost of one or two extra stirrups. On heavy girders this approximation would not be used, the bent-up bars would be staggered, and stirrups spaced to carry only the excess shear above the capacity of the concrete plus the bent bars (see page 280). Note that v_c, taken as 60 psi, does not make any allowance for special anchorage and is on the safe side. When special anchorage is used this value may be raised to 90 psi, but often the need for anchorage is not determined when stirrups are being designed, and also using different values for v_c may cause difficulties if the design is changed from time to time during its preparation, sometimes by other than the original computer. The higher value is permissible but in a simple design of this type hardly expedient. To reinforce for torsional shear in the exterior beams as analyzed in Examples 14–7 and 14–8, stirrups are arbitrarily made ⅜ in. round 12 in. c to c for the full length without detailed computation.

17–6C. Pent House Roof Columns. (See Computation Sheet BG2 and read Examples 13–4 and 13–6.) Note that columns are designated by the level which they *support.* As far as possible columns are grouped together if they carry about the same load and can be of the same size. Loads are obtained as the sum of the end reactions of the supported beams plus an allowance for the dead weight of column, in which must be included the entire weight of the strip of slab shown cross-hatched on Fig. 17–3, since the beam reactions are figured for the clear span. The column sizes at this level are determined by the necessity of lining up with the columns below. They are all far too large for these loads but, even so, a minimum of approximately two-thirds of one per cent of vertical steel is used. (J.C. 851*a* recommends a minimum of 1 per cent when the column is working near capacity). The column ties are not designed but are left for the detailer to cover with a general note.

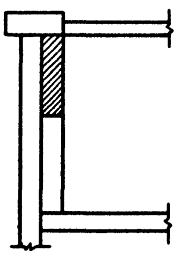


FIG. 17-3

As soon as the design of the pent house is completed it is reviewed to see that it is all consistent and easily framed, the index on Sheet BG1 is brought up to date so as to aid in finding any particular computation, and work proceeds to the main roof.

17–6D. Main Roof Slabs. (See Computation Sheets BG2 and BG3.) The design parallels that just completed for the pent house roof slabs. For the exterior ends the bond stress computed as for a simple slab is greater than 150 and less than 225 psi so special anchorage is called for. Reduction of the end reaction by continuity would probably bring the bond stress within the allowable but special anchorage is easily obtained. Though the span of RS3 is less than the others,

BEAM AND GIRDER BUILDING Sheet BG 2

PS1

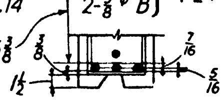
Live Load = 40 L_v = 9.27 +M = 0.075 x 50 x 10^2 = 375 −M = 0.125 x 90 x 10 = −1125
Roofing = 6 L_m = 10.00 0.10 x 40 x 10^2 = 400 +37½ x 3.21 = +120
3½" Slab = 44 d = 2½ 775 1005
90 R = 124 A_s = 0.0177 R = 161 A_s = 0.023
V = 0.417 v = 16 u = 122 ⅜ϕ @ 4½ = 0.0244 alt bt ⅜ϕ @ 4½ = 0.0244 alt bt
Temp Reinf = 0.002 x 3½ x 12 = 0.084 ; ⅜ϕ @ 15 = 0.088

PB1

9.74 x 90 = 0.877 V = 9.49 $+\frac{wL}{8}$ = 531 R = 40 ✓
Stem = 0.140 v = 73 A_s = 1.93 2 − ⅞ϕ S } = 2.08
1.017 a = 20 u = 1.29 2 − ¾ϕ B }
9⅝ x 17½ L = 18.67 A_v = 0.078 Spacing = 2⅛"
T = 56 d = 15½ t/d = 0.226 2 − ¼ ϕ 3, 8 = 0.10

PB2

4.3 x 90 = 0.387 V = 6.19 $+\frac{wL}{8}$ = 300 R = 56 ✓
Parapet = 0.230 v = 59 < 60 A_s = 1.14 2 − ⅝ϕ S } = 1.24
Beam = 0.150 Use − ⅜ ⊏ 12" c/c d = 15⅜ 2 − ⅝ϕ B }
0.767 full length
8 x 18 L = 16.13 u = 120
T = 24+ d = 15 t/d = 0.233

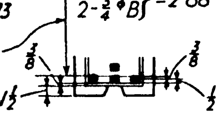


PB3

4.64 x 90 = 0.418 V = 7.32 $+\frac{wL}{8}$ = 403 R = 60
Ppt + Bm = 0.380 v = 70 A_s = 1.54 2 − ¾ϕ S } = 1.76
0.798 a = 16 u = 143 2 − ¾ϕ B }
8 x 18 L = 18.33 ⅜ ⊏ 3, 8, 12 full length (See PB2)
T = 26+ d = 15 t/d = 0.233

PB4

PB1 = 9.49 V = 78 $+\frac{wL}{8}$ = 714 R = 120
Ppt + Bm = 0.38 a = 60 A_s = 2.73 2 − 1□ S } = 2.88
~~9.33~~ ~~9.0~~ 4.75 A_v = 0.27 u = 78 2 − ¾ϕ B }
9.17 9.17 3.48 ⅜ϕ ⊏ 5, 8, 8, 8, 8, d = 15¼
18.33 8.23 8, 12 to ℄
8 x 18 T = 26½ d = 15 t/d = 0.233 R = 201



PB5

PB1 = 9.49 v = 78 $+\frac{wL}{8}$ = 605 R = ✓
3.29 Ppt + Bm = 0.38 See PB4 A_s = 2.31 2 − 1ϕ S } = 2.46
4.39 8.67 7.47 u = 100 2 − ¾ϕ B }
3.07 16.13 5.10
7.46 3.07 (See PB4)
8.77
8 x 18 T = 24 d = 15

P.H Cols

A5	A6	B5	B6
PB2 = 6.19	PB2 = 6.19	PB3 = 7.32	PB3 = 7.32
PB4 = 8.23	PB5 = 8.17	PB4 = 8.23	PB5 = 7.46
Col Etc. = 7.07	Col Etc = 10.59	Col Etc = 4.28	Col Etc = 6.62
21.49	24.95	19.83	21.40
12 x 32⅛ = 208	30 7/16 x 30 7/16 x 12 = 316	16 x 16 x 12 = 130	12 x 32⅛ } = 242
6 − ¾ϕ = 34	8 − ¾ϕ = 45	6 − ¾ϕ = 34	6 − ¾ϕ }
(0.68%) 242	(0.60%) 361	(1.1%) 164	

RS1

Live Load = 40 L_v = 8.60 +M = 0.080 x 50 x 9.33^2 = 348 −M = 0.110 x 50 x 9.33^2 = −479
Roofing = 6 L_m = 9.33 0.105 x 40 x 9.33^2 = 366 0.120 x 40 x 9.33^2 = −418
3½" Slab = 44 d = 2½ 714 +35 x 3.21 = +112
90 −785
V = 0.387 v = 15 u = 150 R = 114 A_s = 0.0163 R = ✓ A_s = 0.0179
⅜ϕ @ 6 = 0.0183 alt bt ⅜ϕ @ 6 = 0.0183 alt bt

SHEET BG2

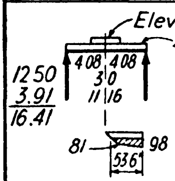
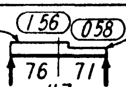
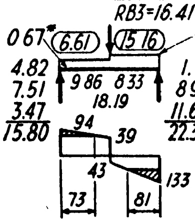
thus producing less shear and moment, the same design will be used throughout, but the bent rods of RS2 will be extended across the top of RS3 for the full length to resist any possible negative bending moment in the short span. The slab over the sheave beams will not be poured until after the elevator machinery is installed and will then be fitted around the bases of the machines and rest on the supporting beams. No rational design is possible but a two-way slab is customary, about 4 in. thick and reinforced with ½ in. round rods 6 or 8 in. on centers each way, cut in around the machine bases.

17–6E. Stairs. (See Computation Sheet BG3.) Before RB4 can be designed it is necessary to have the reaction of the stair slab. This has been worked out in Ex. 16–7 and but little explanation should be required. For suggestions on the detailing of stairs see Chapter XXI.

17–6F. Roof Beams. (See Computation Sheet BG3 and BG4 and read Examples 13–2 and 13–3.) The interior beams have the lowest numbers and are designed first. The size 9⅝ by 17½ in. is chosen to use the forms from the floors below, as explained for the pent house. Design computations follow the standard pattern of Chapter XIII, and negative moments involving the junction of two beams are checked as soon as the meeting beams are designed. For the beams supporting the elevator sheave beams a reaction of 25 kips over a 3-ft bearing is assumed. This could be approximated from the capacity of the elevator (5 tons), plus the weight of car (say 2 tons), plus an equal amount for the counterbalancing weights, plus a 100 per cent impact allowance on all the above, plus the weight of sheaves and that part of the operating machinery that rests on the sheave beams; or upon request elevator companies will prepare rough layout sketches suggesting the best arrangement of sheave beams and the down loads at each end. The beams and girders carrying these heavy loads have been deepened to 27 in. as the standard floor beam cannot be made adequate with any practicable amount of extra reinforcing steel, but these beams all occur in walls and shafts so no loss of head room results from the increased depth.

Note that the use of these deeper beams will create two types of special conditions: one, where deep and shallow beams meet over a column with large negative moments created in the shallow beams; the other, where the deep elevator beams frame into the shallow spandrels whose depth cannot be increased because of the window openings. These are discussed in Art. 17–6I.

Chapter XIII gave some suggestions on the handling of concentrated loads. These beams show how moments for irregular loading conditions can be quickly evaluated. Since RB3 is loaded symmetrically the

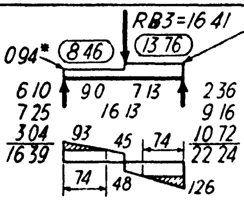
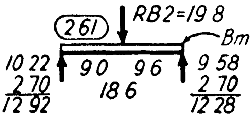
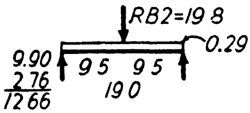
BEAM AND GIRDER BUILDING	Sheet BG3
$w=90$ $L_v=9.20$ $L_m=10.00$ $V=0.414$ $v=16$ $u=161$ SA $+M=0.046\times50\times10^2=230$; $0.085\times40\times10^2=340$; 570; $R=\checkmark$ $A_s=0.0130$; $\frac{3}{8}$ϕ @6 = 0.0183 alt bt $-M=0.080\times50\times10^2=-400$; $0.115\times40\times10^2=-460$; $+37\frac{1}{2}\times3.21=+120$; -740; $R=\checkmark$ $A_s=0.0169$; $\frac{3}{8}$ϕ @6 = 0.0183 alt bt	RS2
Live Load = 80; 1" Grano = 13; $3\frac{1}{2}$" Slab = 44; 137; $L_v=6.44$ $L_m=7.17$ $d=2\frac{1}{2}$ $V=0.441$ $v=17$ $u=171$ SA $+M=0.080\times57\times7.17^2=234$; $0.105\times80\times7.17^2=432$; 666; $R=\checkmark$ $A_s=0.0152$; $\frac{3}{8}$ϕ-6" c/c = 0.0183 alt bt; Extend Bent Bars of RS2 across top $-M=0.110\times57\times7.17^2=-322$; $0.120\times80\times7.17^2=-494$; $+41\times3.21=+132$; -684; $R=\checkmark$ $A_s=0.0156$; $\frac{3}{8}$ϕ-6" c/c = 0.0183 alt bt	RS3
Over sheave beams: 4" Slab - $\frac{1}{2}$"ϕ @ 8" c/c Ea way around Elev Mach	RS4
$10\times90=0.90$; Stem = 0.14; 1.04; $9\frac{5}{8}\times17\frac{1}{2}$ $L=18.52$; $T=55\frac{1}{2}$ $d=15\frac{1}{2}$ $t/d=0.226$ $V=9.63$ $v=74$ $a=21$ $A_v=0.089$ $2\text{-}\frac{1}{4}$ϕ 3,8 = 0.20 $\frac{wL}{10}=428$ $A_s=1.58$ $u=165$ SA $u_1=\checkmark$ $R=32$; $1\text{-}\frac{3}{4}$ϕ + $1\text{-}\frac{5}{8}$ϕ S, $2\text{-}\frac{3}{4}$ϕ B = 1.63; Spacing = 2.1"	RB1
$w=1.04$; $9\frac{5}{8}\times17\frac{1}{2}$ $L=19.04$ $d=15\frac{1}{2}$ $V=9.90$ $v=76$ $a=24$ $A_v=0.115$ $2\text{-}\frac{1}{4}$ϕ 3,8 = 0.20 $+\frac{wL}{12}=377$ $A_s=1.39$ $u_1=\checkmark$ $R=\checkmark$; $2\text{-}\frac{5}{8}$ϕ S, $2\text{-}\frac{3}{4}$ϕ B = 1.50	RB2
$-M=428$ $R=185$ $A_s=1.58$ $2\text{-}\frac{3}{4}$ϕ + $2\text{-}\frac{3}{4}$ϕ = 1.76 $u=78$ $u_1=93$	RB1-RB2
$-M=377$ $R=\checkmark$ $A_s=1.39$ $2\text{-}\frac{3}{4}$ϕ + $2\text{-}\frac{3}{4}$ϕ = 1.76 $u=\checkmark$ $u_1=\checkmark$	RB2-RB2
 $4.76\times100=0.476$; Bm = 0.224; 0.700; 8×27 $d=24$ $T=8$ $v=98$ $A_v=0.90$ $\frac{3}{8}$ϕ-6,12,12, 12,12 = 1.10 $+\frac{wL}{8}=(60.4+10.9)12=856$; $R=186$ $A_s=2.04$; $2\text{-}\frac{7}{8}$ϕ S + $2\text{-}\frac{3}{4}$ϕ B = 2.08; 2 layers; $u=143$ At rt end $\frac{16.41}{20}=0.82$; $1\text{-}\frac{3}{4}$ϕ ʋ = 0.88	RB3
Live Load = 80; 10" Slab = 125; 205  LL = 80; 5" Sl = 63; 143 1.05; 0.10; 1.15; 0.51; 0.48; 0.99; $d=4$"  Max $+\frac{wL}{8}=38.7$ $R=201$ $A_s=0.55$ $\frac{1}{2}$"@$5\frac{1}{2}$ = 0.55 Landing + M = 34.4 $R=179$ $A_s=0.49$ $\frac{1}{2}$"@$5\frac{1}{2}$ = 0.55	Stairs Ex 16-7
 RS3 = 0.441*; Stairs = 1.15; Bm = 0.23*; 1.82 8×27 $d=24$ $T=26\frac{1}{2}$ $+\frac{wL}{8}=1479$ $A_s=3.52$ $u=133$ $R=97$; $2\text{-}1$"ϕ S, $2\text{-}1$ϕ B = 3.58; 2 Layers $A_{vL}=0.62$ 4 $\frac{3}{8}$ϕ-6,15,22, 12,12 12 $A_{vR}=1.48$ 8 $\frac{3}{8}$ϕ-3,6,7,8,9,10,16, 12,12 At right end: $\frac{22.38}{20}=1.12$; $1\text{-}\frac{7}{8}$ϕ ʋ = 1.20 $v_{8\times18}=200$	RB4

maximum moment is the sum of two couples, one of 12.5 k $\times$ 4.83 ft, and the other of 3.91 k $\times$ 2.79 ft. For the stair slab, zero shear is 5.61 ft from the left reaction and the maximum moment is that of a couple of 1.15 k $\times$ 5.61/2 ft. For RB4 zero shear is under the concentration and maximum moment is 15.80 k $\times$ 9.86 ft $-$ 6.61 k $\times$ 9.86/2 ft, which can be factored into $[15.80 - (6.61/2)] \times 9.86$, so at one setting of a slide rule read 12.495 $\times$ 9.86 k-ft. It is best to keep these moments in kip-feet until the end and then multiply by proper coefficients, using 12 for $WL/8$ moments, 8/10 as much, or 9.6 for $WL/10$, and 8/12 as much or 8 for $WL/12$ moment. Thus the computer can ordinarily obtain the bending moment by direct slide-rule computation without writing down any figures but using the encircled summations on the load sketch. As illustrated in RG3, where intermediate figures are required, they are written on the computation sheet to assist the checker. The student would do well to verify each of these moment computations for practice.

The design of web reinforcement has been explained. Wherever the excess shear volume is a triangular prism the stirrups have been spaced either by the slide-rule method described on page 102 or by the use of Table A-1. In practical application of the former method it is usually sufficiently accurate to set the total number of stirrups opposite the distance a and then read the difference in spacings at $\sqrt{N - \frac{1}{2}}$, $\sqrt{N - 1\frac{1}{2}}$, etc., thus immediately locating each stirrup near the centroid of its excess shear volume. In the case of concentrated loads involving trapezoidal shear volumes the computer ordinarily establishes the spacing at each end and prorates between by eye, checking to see that the summation of spaces properly fills the shear volume. Where the stirrup spacing exceeds $d/2$ stirrups are arbitrarily added at this maximum spacing to cover the required distance.

Negative moments at the junctions of two members are checked as soon as the members are designed. The negative moments have been taken from whichever span produced the larger value, which is safe and is about as accurate a solution as any that can be obtained from arbitrary coefficients.* Frequently double reinforcement is required. The computations indicate whether all the bottom steel from both spans has been extended through the columns as compressive reinforcement or whether sufficient steel can be obtained from the bottom rods of one span only. This will affect the amount of extended length required on

* In the case of approximately equal spans, with uniform loading, the unit live load not exceeding three times the unit dead load, the A.C.I. code, 1940 (700–701), computes the negative moment as the moment factor times the unit load times the square of the mean of the two adjacent clear spans.

BEAM AND GIRDER BUILDING	Sheet BG4
RS3 = 0.44 $9\frac{5}{8} \times 17\frac{1}{2}$ V = 16.67 $+\frac{wL}{10}=741$ R=64 RS2 = 0 42 $d=14\frac{1}{2}$ v = 137 $A_s=2.92$ $\left.\begin{matrix}2\text{-}1^{\phi}S\\2\text{-}1^{\phi}B\end{matrix}\right\\}=3.16$ Wall = 0 72 $T=55\frac{1}{2}$ $a=62\frac{1}{2}$ u=210 SA 2 Layers Beam = $\underline{0\ 22}$ L=18.52 $A_v=1.10$ $u_1=$✓ 1 80 $6S\text{-}\frac{3}{8}^{\phi}$ 3,7,8,~~10~~,~~14~~,8,8,8	RB5
−M=741 R=366 p′=0.017 (89/16) $A'_s=1.32$ $2\text{-}\frac{5}{8}^{\phi}+2\text{-}1^{\phi}=2\ 20$ $d=14\frac{1}{2}$ d′/d=0.138 p=0 0215 $A_s=3.00$ $2\text{-}1^{\phi}+2\text{-}\frac{3}{4}^{\phi}+1\text{-}\frac{5}{8}^{\phi}$ Top=3 06	RB2-RB5
Parapet = 0 23 L=16 13 V=6 21 $+\frac{wL}{10}=241$ R=✓ RS1 = 0 39 8x18 d=15 6 v=57 $A_s=0\ 89$ $\left.\begin{matrix}2\text{-}\frac{1}{2}^{\phi}S\\2\text{-}\frac{1}{2}^{\square}B\end{matrix}\right\\}=0\ 90$ Beam = $\underline{0\ 15}$ T=24 + $\frac{3}{8}^{\phi}$ ⊔ 12″c/c u=145 Spacing = $1\frac{5}{8}$″ = 0 77	RB6
w = 0 77 L=17.32 V=6 67 v=✓ $+\frac{wL}{12}=231$ R=✓ $\left.\begin{matrix}2\text{-}\frac{1}{2}^{\phi}S\\2\text{-}\frac{1}{2}^{\phi}B\end{matrix}\right\\}=0\ 90$ 8x18 d=15 6 $\frac{3}{8}^{\phi}$ ⊔ 12″c/c $u_1=$✓ $A_s=0.85$	RB7
−M=241 d=16 R=98 $A_s=0.86$ $2\text{-}\frac{1}{2}^{\square}+2\text{-}\frac{1}{2}^{\square}=1.00$ u=60 $u_1=$✓	RB6-RB7
−M=231 d=16 R=✓ $A_s=0\ 83$ $2\text{-}\frac{1}{2}^{\square}+2\text{-}\frac{1}{2}^{\square}=1.00$ u=✓ $u_1=$✓	RB7-RB7
 Wall = 0 72* $+\frac{wL}{10}=1051$ R=360 Stair = 0 99 p′=0 0165 $\frac{92}{16}A'_s=0.78$ $2\text{-}\frac{3}{4}^{\phi}=0.88$ Beam = $\underline{0\ 22}$* p=0 021 $A_s=3\ 93$ $4\text{-}1^{\square}=4\ 00$ 1 93 Spacing 2.9″+ 12x18 d=15.6 d′/d=0 13 u=95 $A_{VL}=0\ 92$ $\frac{3}{8}^{\phi}$ 4,8,8 & 12 to ℄ $A_{VR}=1\ 78$ $\frac{3}{8}^{\phi}$ 2,4,5,5,5,6,7,~~9~~,~~11~~,8,8,8 & 12 to ℄	RB8
−M=1051 R=360 $A'_s=0.78$ $2\text{-}1^{\phi}+1\text{-}1\frac{1}{8}^{\square}=1.66$ See RB8 $A_s=3.93$ $2\text{-}\frac{1}{2}^{\square}+2\text{-}\frac{3}{4}^{\phi}+1\text{-}1\frac{1}{8}^{\square}$ Top=3.90	RB7-RB8
 $11\frac{1}{2} \times 20$ $v_L=71$ $+\frac{wL}{10}=1003$ d=18 $a_L=83$ R=56 $A_s=3.18$ $T=55\frac{1}{2}$ $A_{v_L}=0\ 23$ $2\text{-}1^{\phi}S+2\text{-}1^{\phi}B$ t/d=0 20 $8\text{-}\frac{1}{4}^{\phi}$ 5,7@9 =3.16 u=130 Spacing = 2 67″ $u_1=$✓	RG1
 $11\frac{1}{2} \times 20$ v = 70 $+\frac{wL}{12}=857$ d=18 a = 75 T=57 $A_v=$✓ R=✓ $A_s=2.72$ t/d=0 20 $7\text{-}\frac{1}{4}^{\phi}$ 5,6@9 $2\text{-}\frac{7}{8}^{\phi}S+2\text{-}1^{\phi}B=2\ 78$ $u_1=134$	RG2
−M=1003 R=269 d′/d=0 11 p=0.0153 $A_s=3\ 17$ $2\text{-}1^{\phi}+2\text{-}1^{\phi}=3.16$ u=66 d=18 p′=0.0038 $\frac{9.8}{16}A'_s=0.48$ $2\text{-}\frac{7}{8}^{\phi}=1.20$	RG1-RG2
−M=857 R=230 $A_s=2.72$ $2\text{-}1^{\phi}+2\text{-}1^{\phi}=3\ 16$ u=✓	RG2-RG2

SHEET BG4

the bottom bars and is needed by the detailer, as will be explained in Chapter XXI.

RB8 is increased in width along the stairwell. Because of the heavy moment involved, the 8-in. width is inadequate even with compressive reinforcement, and the depth is established because the uniformity in the steel sash fixes the bottom of the spandrel and appearance and continuity fix the top. As the projection of RB8 into the stairwell occurs above shoulder height and at the roof level, it will not reduce the effective width of stairs. Should some local code prohibit this projection, one step could be moved from the lower to the upper flight, changing the location of the top riser at the roof level.

17–6G. Roof Girders. (See Computation Sheets BG4 and BG5.) The roof girders carry so much less load than the floor girders that it does not appear economical to use the same forms as for the typical floors without any refabrication. A rough check of one member quickly verifies this. Therefore, in selecting a size for RG1 the width was maintained 11½ in. to keep the same soffit form but the depth was reduced to 20 in.; this involves cutting a 3½-in. strip from each side form (see Chapter VI). In selecting sizes the designer has to weigh the reuse of forms against the cost of added concrete and building height and the saving in reinforcing steel, as described more thoroughly in Chapter XXII.

The balance of the roof girder computations parallels that of the roof beams. Note that the width of RG6 along the stairwell was increased to 12 in. as an 8-in. beam was totally inadequate for the same reason as RB8.

17–6H. Roof Columns. (See Computation Sheet BG6.) Loads are accumulated as described in Art. 17–6C and Ex. 13–4 as the reactions of the supported beams and girders. The allowance for the dead weight of column includes the weight of column shaft from third-floor line through the roof and to the top of the parapet increased by any tributary slab area that is not included in the beam reactions. Aside from the interior columns the sizes are established by architectural considerations and are extremely large, but even so a minimum of about two-thirds of one per cent of reinforcing steel is used. The interior columns were made 9⅝ by 9⅝ in. and at least 1 per cent of vertical steel is called for regardless of load. For column B5 the concrete size is made 11½ by 11½ in. and the reinforcing steel is increased to obtain the capacity required.

17–6I. Special Conditions. The use of deep beams around the elevator hatch results in special conditions that deserve further comment. The intersection of RB3 with RB8 is shown in Fig. 17–4*a*. The interior

BEAM AND GIRDER BUILDING — *Sheet BG5*

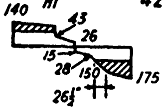
RG3

RB3=16.41 | RB2=9.90
(6.91) Elev=25.0
1.02*
617 2.83 30
2.1 3.9
18.6
9.60

5.65	1.26
10.44	5.97
4.79	5.11
7.26	17.74
5.64	12.11
33.78	42.79

Wall =0.72*
Slab =0.48
Beam =0.30*
1.50

$+\frac{wL}{10}$ 33.78×9.6=324.29
−6.91×6.215=−42.95
−16.41×2.83=−46.44
−4.245×1.415=−6.01
228.89×9.6
=2140



$11\frac{1}{2}$×27 d=24 R=124
T=30 t/d=0.15 A_s=5.10 2-$1\frac{1}{8}$"S+2-$1\frac{1}{8}$"B=5.08
A_{vL}=3.82 10-$\frac{1}{2}$ϕ3,7,7,7,8,8,8,9,9,10 2 Layers
u=224 SA
$A_{vR}=\frac{55.2+13.7}{16}$=4.30 11-$\frac{1}{2}$ϕ2,5,5,5,6,6,6,6,6,7,8 u_1=158 SA

RG2-RG3

−M=2140 d=24 R=323 d'/d=0.08 p=0.0185 A_s=5.33 2-$1\frac{1}{8}$"+2-1ϕ+2-$\frac{7}{8}$ϕTop=5.32
p'=0.0088 $\frac{8.8}{16}$ A_s'=1.39 2-$1\frac{1}{8}$"=2.54
d=18 R=574 d'/d=0.11 p=0.035 A_s=7.56 2-$1\frac{1}{8}$"+2-1ϕ+2-$1\frac{1}{4}$"Top=7.24
p'=0.04 $\frac{9.6}{16}$ A_s'=5.18 2-$1\frac{1}{8}$"+2-$\frac{7}{8}$ϕ+1-$1\frac{1}{4}$"Add=5.30

RG4

(2.84) RB1=9.63
7.47 8.66
16.13

5.17	4.46
3.07	3.07
8.24	7.53

Parapet=0.23
Beam =0.15
0.38

8×18 d=14.6
v_L=81 o_L=68 A_{vL}=0.36 6 ~~4~~-$\frac{1}{4}$ϕ4,~~10~~,~~12~~,~~18~~,8,8,8,8,8

$+\frac{wL}{10}$=489
R=96 A_s=1.91
2-$\frac{3}{4}$ϕS+2-$\frac{5}{8}$ϕB=2.08
2 Layers u=138

RG5

RB1=9.63
0.38
8.66 8.66
17.32

4.82
3.29
8.11

v =79
o =61
A_v=0.29
5 ~~3~~-$\frac{1}{4}$ϕ5,~~13~~,~~18~~,8,8,8,8

$+\frac{wL}{12}$=448
R=✓ A_s=1.75
2-$\frac{3}{4}$ϕS+2-$\frac{3}{4}$ϕB=1.76
2 Layers u_i=113

RG4-RG5

−M=489 d=15 d'/d=0.18 p'=0.0005 A_s'=✓
d'=$2\frac{3}{4}$ R=241 p=0.014 A_s=1.68 2-$\frac{7}{8}$ϕ+2-$\frac{3}{4}$ϕ=2.08

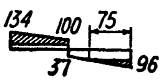
RG6

(5.48) RB4=22.38
5.83 10.30
16.13

14.29	8.09
7.58	7.58
21.87	15.67

Wall =0.72
Beam =0.22
0.94

$+\frac{wL}{10}$ 1071 R=367
p=0.0215 A_s=4.02
4-1"=4.00
p'=0.0163 $\frac{9.2}{16}$ A_s'=1.76
2-1"=2.00



12×18 d=15.6 d'/d=0.13
u=72
u_i=✓

A_{vL}=2.99 ~~8-$\frac{1}{2}$ϕ3,7,8,8,9,9,10,11~~
A_{vR}=1.01 Use $\frac{1}{2}$ϕ 8"c/c for 8 Spaces then @ 16"c/c

SHEET BG5

beam hangs below the bottom of the spandrel. The spandrel cannot be deepened because of sash opening. Beam RB3 is permitted to run underneath RB8 until close to the sash. A stirrup hanger is shown, the individual capacity of which is equal to the end reaction. In addition a check of the intensity of shear on that part of RB3 which is in contact with RB8 indicates that this is within reason. On this basis there is excess strength in the joint as either provision is sufficient to carry the load, but this indicates how the designer can insure a safe, adequate structure with a minimum of expense and without elaborate computations.

The intersection of RB4 with RG6 is shown in Fig. 17–4*b*. Again the stirrup hanger is adequate for the total reaction but the unit shear on that portion of RB4 that is in contact with RG4 is high and one of

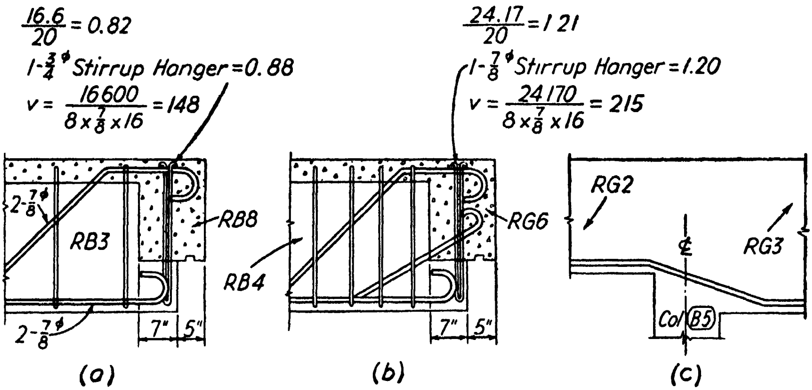


FIG. 17–4

the straight bottom bars is bent up across the reentrant corner because this corner is the point at which diagonal tension cracks would undoubtedly start. If it were not for the fact that RB4 extends several inches underneath RG6, additional diagonal tension reinforcement would have been used, but this bent bar coupled with the closely spaced stirrups and the hanger are more than adequate in this case.

The junction of RG2 and RG3 over column B5 is illustrated in Fig. 17–4*c*. Girder RG3 has been deepened for the elevator loads. The negative moment at the end of RG3 (except for the effect of the width of the column) must be the same as on the end of RG2, although its magnitude may be considerably different from the $WL/10$ moment taken from the heavier span. The design computations show that RG3 is adequate for this moment. RG2, however, is much shallower. If it were not for the cost of special formwork, a haunch or bracket on

BEAM AND GIRDER BUILDING			Sheet BG6
$-M=1071$ 12×18 $R=367$ $A_s=4.02$ $2\text{-}1^{\square}+2\text{-}\frac{3}{4}^{\phi}+1\text{-}1\frac{1}{4}^{\square}$ Top$=4.44$ See RG6 $A'_s=1.76$ 8×18 $R=550$ $p'=0.043$ $\frac{9.4}{16}A'_s=2.40$ $2\text{-}\frac{3}{4}^{\phi}+2\text{-}1^{\square}=2.88$ $d=15\frac{1}{2}$ $d'/d=0.16$ $p=0.034$ $A_s=4.22$ $2\text{-}1^{\square}+2\text{-}\frac{3}{4}^{\phi}+1\text{-}1\frac{1}{4}^{\square}$ $d'=2\frac{1}{2}$ Top$=4.44$			RG5-RG6
A1 D1 D6 RB6 = 6.21 RG4 = 8.24 Col Etc. = 10.6 25.05 $30\frac{7}{16}\times30\frac{7}{16}\times12=316$ $8\text{-}\frac{3}{4}^{\phi}$ = 45 (0.60%) 361	A2 D2 A3 D3 A4 D4 D5 2xRB7 = 13.34 RG1 = 12.92 Col Etc = 7.3 33.56 $32\frac{1}{8}\times12=208$ $6\text{-}\frac{3}{4}^{\phi}$ = 34 (0.68%) 242 Ex 13-6	B1 C1 C6 RG4 = 7.53 RG5 = 8.11 RB1 = 9.63 Col Etc = 6.6 31.87 $\left.\begin{matrix}32\frac{1}{8}\times12\\6\text{-}\frac{3}{4}^{\phi}\end{matrix}\right\}=242$	Roof Cols
B2 C2 B3 C3 B4 C4 C5 2xRB2 = 19.80 RG1 = 12.28 RG2 = 12.66 Col Etc. = 1.3 46.04 $9\frac{5}{8}\times9\frac{5}{8}=50.0$ $4\text{-}\frac{5}{8}^{\phi}$ = 15.4 (1.29%) 65.4 $p'=65.4\left[1.3-0.03\frac{11\times12}{9\frac{5}{8}}\right]=58.2$ Ex. 13-4	B5 Above = 19.83 RG3 = 33.78 RG2 = 12.66 RB2 = 9.90 RB5 = 16.67 Col Etc = 2.0 95.84 $11\frac{1}{2}\times11\frac{1}{2}=71.2$ $4\text{-}\frac{7}{8}^{\phi}$ = 30.7 (1.82%) 101.9	A5 Above = 21.49 RG3 = 42.19 RB7 = 6.67 RB8 = 16.39 Col Etc = 7.0 93.74 $\left.\begin{matrix}32\frac{1}{8}\times12\\6\text{-}\frac{3}{4}^{\phi}\end{matrix}\right\}=242$	
A6 Above = 24.95 RB8 = 22.24 RG6 = 15.67 Col Etc = 9.5 72.36 $\left.\begin{matrix}30\frac{7}{16}\times30\frac{7}{16}\times12\\8\text{-}\frac{3}{4}^{\phi}\end{matrix}\right\}=361$	B6 Above = 21.40 RG6 = 21.87 RB5 = 16.67 RG5 = 8.11 Col Etc = 6.2 74.25 $\left.\begin{matrix}32\frac{1}{8}\times12\\6\text{-}\frac{3}{4}^{\phi}\end{matrix}\right\}=242$		
Live Load = 125 $L_v=8.60$ $+M=0.080\times63\times\overline{9.33}^2=439$ $-M=0.110\times63\times\overline{9.33}^2=-603$ 1 in. Fin = 13 $L_m=9.33$ $0.105\times125\times\overline{9.33}^2=1143$ $0.120\times125\times\overline{9.33}^2=-1306$ 4 in. Slab = 50 $d=3$ 1582 -1909 188 $+73\times3.2$ = $+234$ $R=176$ $A_s=0.030$ $R=186$ -1675 $V=0.808$ $v=26$ $u=234$ SA $\left.\begin{matrix}\frac{1}{2}^{\square}@18\text{ S}\\\frac{5}{8}^{\phi}@18\text{ B}\end{matrix}\right\}$ alt$=0.0311$ $A_s=0.032$ $\frac{5}{8}^{\phi}@9=0.0344$			FS1
$w=188$ $L_v=9.20$ $+M=0.046\times63\times10^2=290$ $-M=0.080\times63\times10^2=-504$ $L_m=10.00$ $0.085\times125\times10^2=1060$ $0.115\times125\times10^2=-1438$ $d=3$ 1350 -1942 $+72\times3.2$ = $+231$ $R=150$ $A_s=0.0257$ -1711 $\left.\begin{matrix}\frac{1}{2}^{\phi}@18\text{ S}\\\frac{5}{8}^{\phi}@18\text{ B}\end{matrix}\right\}$ alt$=0.0278$ $R=191$ $A_s=0.0326$ $V=0.865$ $v=27$ $u_i=188$ SA $u=124$ $\frac{5}{8}^{\phi}@9=0.0344$			FS2 Ex 13-1

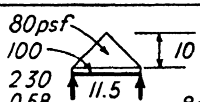
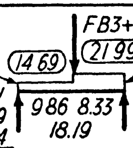
the end of RG2 could be brought down as deep as RG3. The designer, however, felt that extending all the bottom steel of RG2 into RG3 and of RG3 into RG2 would provide for this condition. In the design computations of negative moment at the junction of RG2 and RG3, figures were made on both the deep section and the shallow one. The deep section is within the allowable. The shallow section has an extremely high R value and both p and p' are above normal limits, but aside from these arbitrary limitations, there appears to be ample compressive reinforcement and the tensile strength is only a moderate amount above the usual limits. This detail is purposely introduced to illustrate how the designer attacks such a problem. It would be decidedly uneconomical to increase the size of all the roof girders to meet this peak condition and even haunches and brackets would considerably increase the cost. In such a case as this a more accurate analysis of conditions by the methods of Chapter XVIII may be justified or the designer may use the results of his previous studies to temper his judgment.

A similar condition is encountered at the junction of RB7 and RB8 over column A5 and an analysis is made on the computation sheets along the lines just discussed.

The student should turn to the detailed design drawings of this building in Chapter XXI to see how the detailer incorporates these requirements on the drawings.

17–6J. Floor Slabs. (See Computation Sheet BG6 and Ex. 13–1.) The detailed design of a typical floor slab was made in Ex. 13–1 so comments here can be of a general nature. The slab thickness is determined by the negative moment in FS2 and the same thickness is maintained throughout the floor for uniformity. It almost appears that a slab ½ in. thinner, except for this peak condition, could be used, but compressive reinforcement over the supports is almost useless in a thin slab as the protective covering locates the compression steel close to the neutral axis where it is of no value and haunched ends result in expensive formwork. The reinforcing steel for the semi-continuous span, which has the largest positive bending moment, is kept at the same spacing as the rest of the floor for ease in placing and the bar size is increased as required. Because this design applies on three levels it should be carefully studied for maximum economy.

17–6K. Floor Beams. (See Computation Sheet BG7 and Ex. 13–2.) Since the detailed design of a typical floor beam was discussed in Ex. 13–2, including the reasons for selecting the sizes used, and since many suggestions have already been considered in Articles 17–6F and G, only certain points of general interest need be considered here. Note that

BEAM AND GIRDER BUILDING	Sheet BG7
w=188 $L_v'=6.44$ $L_m=7.17$ d=3 $+M=0.080\times63\times\overline{7.17}^2=259$ $0.105\times125\times\overline{7.17}^2=\underline{675}$ 934 $-M=0.110\times63\times7.17^2=-356$ $0.120\times125\times7.17^2=\underline{-771}$ -1127 V=0.605 v=19 u=✓ R=104 $A_s=0.0178$ $\frac{1}{2}^\phi$@18 S, $\frac{5}{8}^\phi$@18 B } alt=0.0278 +56 x 3.2 $\underline{+179}$ -948 R=105 $A_s=0.0181$ $\frac{5}{8}^\phi$@9=0.0344 Extend Bent Rods of FS2 across for −M	FS3
Use $\frac{3}{8}^\phi$@12"c/c crosswise $p=\frac{0.11}{4\times12}=0.0023$	Temp
Slab·10@188=1.88 Beam $=\underline{0.15}$ 2.03 L=18.52 $9\frac{5}{8}\times18$ $T=55\frac{1}{2}$ d=15 V = 18.80 v = 149 $a=66\frac{1}{2}$ $A_v=1.78$ $8-\frac{3}{8}^\phi$2,5,5,5, 6,7,8,8, $+\frac{wL}{10}=835$ R=67 $A_s=3.18$ $\left.\begin{array}{l}2\text{-}1^\phi S\\2\text{-}1^\phi B\end{array}\right\}$ 3.16 2 Layers u=228 SA $u_1=137$	FB1 Ex 13 2
w=2.03 L=19.04 $9\frac{5}{8}\times18$ T=57 d=15 V = 19.33 v = 159 $a=69\frac{1}{2}$ $A_v=1.95$ $9-\frac{3}{8}^\phi$2,4,4,5, 5,6,6,8,8 $+\frac{wL}{12}=736$ R=✓ $A_s=2.80$ $\left.\begin{array}{l}2\text{-}\frac{7}{8}^\phi S\\2\text{-}1^\phi B\end{array}\right\}=2.78$ 2 Layers $u_1=161$ SA	FB2
−M=835 $d=15\frac{1}{2}$ R=361 p'=0.019 $\frac{9.2}{16}$ $A_s'=1.63$· $2\text{-}\frac{7}{8}^\phi+2\text{-}1^\phi=2.78$ SA d'/d=0.16 p=0.022 $A_s=3.28$ $2\text{-}1^\phi+2\text{-}1^\phi=3.16$ u=114	FB1-FB2
−M=736 $d=15\frac{1}{2}$ R=318 p'=0.0125 $\frac{9.2}{16}$ $A_s'=1.07$ $2\text{-}\frac{7}{8}^\phi=1.20$ SA d'/d=0.16 p=0.019 $A_s=2.83$ $2\text{-}1^\phi+2\text{-}1^\phi=3.16$	FB2-FB2
 2.30 $\underline{0.58}$ 2.88 8x12 d=10 T=8 V=2.88 v=41 $+\frac{wL}{8}=(8.82+1.67)12=126$ R=158 $A_s=0.72$ u = 82 $\left.\begin{array}{l}2\text{-}\frac{1}{2}^\circ S\\1\text{-}\frac{1}{2}^\circ B\end{array}\right\}=0.75$	FB3
 1.49* 10.71 2.79 $\underline{5.04}$ 18.54 3.98 3.31 $\underline{16.95}$ 24.24 FS3=3.2x188 = 0.61* Stair (BG3) = 1.15 Wall $0.7\times11\frac{1}{2}\times80=0.64$* Beam Etc. = $\underline{0.24}$* 2.64 $+\frac{wL}{8}=1325$ $A_s=4.99$ 2 Layers $v_L=\cancel{147}\ 138$ $a_L=85$ $A_{vL}=2.00$ $10\cancel{9}\text{-}\frac{3}{8}^\phi$2,5, 5,6,7,8,~~10~~, ~~15~~,8,8,8 R=210 $\left.\begin{array}{l}2\text{-}1\frac{1}{4}^\circ S\\2\text{-}1^\circ B\end{array}\right\}=5.12$ u=185 SA $v_R=\cancel{192}\ 180$ (d 16) $a_R=63\frac{1}{2}$ $A_{vR}=2.29$ $11\cancel{10}\text{-}\frac{3}{8}^\phi$2,3,4,4, 4,5,5,5,7,~~11~~,8,8 $9\frac{5}{8}\times18$ d=15 T=28 t/d=0.267 R=215 j=0.885	FB4
8.6x188=1.62 Beam $=\underline{0.15}$ 1.77 L=18.52 $9\frac{5}{8}\times18$ $T=55\frac{1}{2}$ d=15 V = 16.39 v = 130 a = 60 $A_v=1.26$ $7\cancel{6}\text{-}\frac{3}{8}^\phi$2,5,6,7,~~9~~,~~13~~,8, 8,8 $+\frac{wL}{10}=729$ $A_s=2.78$ 2 Layers u=228 SA R=✓ $\left.\begin{array}{l}2\text{-}\frac{7}{8}^\phi S\\2\text{-}1^\phi B\end{array}\right\}=2.78$ $u_1=137$	FB5

the one stem size, selected for the typical condition, is kept throughout the floor except for an occasional stair header. Check again the method of obtaining maximum bending moments by direct slide-rule computation, often without writing a single detailed figure, usually by breaking the loads down into combinations of couples — a method which applies at zero shear points but should be watched carefully in computing intermediate moments away from this only point at which the upward and downward forces are in balance.

In checking negative moments, as for example at the junction of FB1–FB2, the bottom rods from *one* side may furnish sufficient compressive reinforcement without including all the bottom bars from both sides. This is noted, because the detailer, in determining bar lengths (Chapter XXI) assumes that this steel is working in compression at a stress not exceeding 16,000 psi and therefore does not extend the bottom bars of FB1 well into FB2 and vice versa, but only laps these bottom bars an amount of $16{,}000/(4 \times 150)$ or 27 bar diameters ($27 \times \frac{7}{8} = 24$ in. in this case) at the center of columns. Should the whole bar combination be required for compressive reinforcement the bars would have to be extended through for development.

FB3 carries a solid brick wall between the elevator hatch and stairwell which, if sufficiently high in proportion to its length, would arch across the span and be self-supporting above a triangle of masonry which rests directly on the beam. Designers differ in their assumptions, some taking a 45° isosceles triangle of load and others one at 60°, the real conditions depending upon the type of masonry units used and the properties of the mortar joints, but in any case there must be sufficient unbroken wall surface above the crown of this arch to carry the thrust. The presence of window or other openings breaks into this arch action, creating small arches over each individual opening to which must be added the weight of the piers between openings, with another spreading of load below the openings. Although this action reduces the load carried by FB3, its supporting beam FB4 must still carry the weight of a full half of the wall which is brought to it either as the end reaction of FB3 or as the reaction of the arch previously described. FB4 also carries the front walls of the elevator and stair shafts which are pierced with openings that prevent arch action but that also reduce the total load. A brief computation allowing for the probable size of openings led the designer to take 70 per cent of the solid wall as load on FB4, clearly shown on the computation sheets for checking after full details have been made for these openings.

The load computations for FB6 and FB7 should be studied as they show the accumulating of weight of spandrel wall, sash, and sills, as

BEAM AND GIRDER BUILDING	Sheet BG8
$-M=736$ $d=15\frac{1}{2}$ $R=318$ $p'=0.0125$ $\frac{9.2}{16}A'_s=1.07$ $2\text{-}\frac{7}{8}^{\phi}=1.20$ SA $d'/d=0.16$ $p=0.019$ $A_s=2.83$ $2\text{-}1^{\phi}+2\text{-}1^{\phi}=3.16$ u ✓	FB2-FB5
Slab = 4.3 x 188 = 0.81 $L=16.13$ $V=10.48$ $+\frac{wL}{10}=406$ $R=70$ Sash = 8.5 x 10 = 0.09 8 x 18 $v=96$ Wall = 2.4 x 80 = 0.19 $d=15.6$ $a=36$ $A_s=1.49$ Sill = 0.5 x 130 = 0.06 $T=24$ $A_v=0.33$ $\left.\begin{matrix}1\text{-}\frac{3}{4}^{\phi}+1\text{-}\frac{5}{8}^{\phi}S\\1\text{-}1^{\phi}B\end{matrix}\right\}=1.52$ ($2\frac{1}{4}$" Spcg) Beam = $\underline{0.15}$ Use $\frac{3}{8}^{\phi}$ ⊔12" c/c 1.30 Full Lgth $u=178$ SA $u_1=$✓	FB6
$w=130$ $L=17.32$ $V=11.26$ $+\frac{wL}{12}=390$ $R=$✓ $A_s=1.43$ 8 x 18 $v=103$ $d=15.6$ $a=28$ $A_v=0.30$ $\left.\begin{matrix}2\text{-}\frac{5}{8}^{\phi}S\\1\text{-}1^{\phi}B\end{matrix}\right\}=1.41$ $u_1=$✓ Use $\frac{3}{8}^{\phi}$-⊔12" c/c Full Lgth	FB7
$-M=406$ $R=209$ ✓ $A_s=1.49$ $1^{\phi}+1^{\phi}=1.58$ $u=131$ [FB7-FB7 o k]	FB6-FB7
FB3+Wall=6.1 0.85* (7.65) (13.12) Stair (BG3) = 0.99 $+\frac{wL}{10}=630$ $R=324$ Sash = 0.04* 5.52 9.0 7.13 2.13 Wall $7\frac{1}{2}$ x 80 = 0.60* $p=0.011$ $\frac{9.2}{16}A'_s=0.79$ ~~$2\text{-}\frac{3}{4}^{\phi}$ 0.88~~ 2.70 16.13 3.40 Sill = 0.06* $p=0.019$ $A_s=2.37$ $2\text{-}1\frac{1}{8}^{\circ}=2.52$ $\underline{2.90}$ $\underline{10.22}$ Beam = $\underline{0.15*}$ $-\frac{wL}{24}=263$ $A'_s=0.94$ $2\text{-}\frac{7}{8}^{\phi}=1.20$ 11.12 15.75 1.84 $u=128$ $v_L=102$ $a_L=65$ $v_R=144$ $a_R=60$ 8 x 18 $d=15.6$ $A_{v_L}=0.68$ $A_{v_R}=1.26$ $T=8$ $d'/d=0.128$ $\frac{3}{8}^{\phi}$ 3,6, Bal 12" to ℄ $\frac{3}{8}^{\phi}$-3,6,6,6,8,8, Bal 12" to ℄	FB8 Ex 13-3
$-M=630$ $R=324$ $d'/d=0.125$ $p=0.019$ $A_s=2.37$ $2\text{-}\frac{7}{8}^{\phi}+1\text{-}1^{\phi}+1\text{-}\frac{3}{4}^{\phi}$ Top = 2.43 $p'=0.011$ $\frac{9.3}{16}A'_s=0.82$ $1\text{-}1\frac{1}{8}^{\circ}=1.27$	FB7-FB8
(3.78) FB2=38.66 LL = 0.125 $11\frac{1}{2}$ x 24 $+\frac{wL}{10}=1888$ $R=74$ 3.89 9.0 9.5 3.89 Bm Etc = $\underline{0.295}$ $d=21.4$ $\underline{19.85}$ 18.5 $\underline{18.81}$ 0.420 $T=55\frac{1}{2}$ $A_s=5.04$ 23.74 22.70 $\left.\begin{matrix}2\text{-}1\frac{1}{8}^{\circ}S\\2\text{-}1\frac{1}{8}^{\circ}B\end{matrix}\right\}=5.08$ $u=141$ Spcg = $3\frac{3}{4}$" $\left.\begin{matrix}v_L=110 & v_p=93 & A_{v_L}=3.22\\v_R=150 & v_p=87 & A_{v_R}=2.95\end{matrix}\right\}$ Use $15\text{-}\frac{3}{8}^{\phi}$ ⊓ 3,6,6,6,6,7,7,7,7,8,8,8,8,9,9	FG1
0.42 FB2=38.66 $11\frac{1}{2}$ x 24 $+\frac{wL}{12}=1621$ $R=$✓ 3.99 19.0 $d=21.4$ $A_s=4.33$ $\underline{19.33}$ $T=57$ $\left.\begin{matrix}2\text{-}1^{\circ}S\\2\text{-}1\frac{1}{8}^{\circ}B\end{matrix}\right\}=4.54$ 23.32 $u_1=145$ $v_L=108$ $v_p=90$ $A_v=3.20$ $15\text{-}\frac{3}{8}^{\phi}$ 3,6,6,6,6,7,7,7,8,8,8,9,9,9,10	FG2
$-M=1888$ $R=372$ $p'=0.0182$ $\frac{10.3}{16}A'_s=2.83$ $2\text{-}1^{\circ}+2\text{-}1\frac{1}{8}^{\circ}=4.54$ $d=21.0$ $d'/d=0.14$ $p=0.0216$ $A_s=5.22$ $2\text{-}1\frac{1}{8}^{\circ}+2\text{-}1\frac{1}{8}^{\circ}=5.04$	FG1-FG2

well as the slab reaction which is here taken as a full half panel not allowing for any reduction in end reaction by the continuity of the slab.

17–6L. Floor Girders. (See Computation Sheets BG8, 9.) Girder computations parallel closely those for the beams. A little experimenting established 11½ by 24 in. as a suitable size, the width being selected to use 2 by 12 in. soffit planks (which are eventually moved up for the roof as well), and the depth to satisfy diagonal tension requirements. A check of stem widths is made for bar spacing and coverage. As much steel as possible is kept in the bottom layer to increase the effective depth. For the condition pictured in FG1 the two center bars would be straight and the bent bars would be used on either side.

For spacing stirrups, the shear curves for girders are usually trapezoidal and ordinarily it is sufficiently accurate to compute spacings at either end of the trapezoid and one or two intermediate points, making up the schedule and checking the total to see that the space is properly filled. A quick way to do this is to find out what one stirrup is good for per inch breadth of beam stem: $A_v \times 16{,}000/b$, and this is constant throughout the girder. Dividing by the intensity of excess shear at any point $(v - v_c)$ gives the spacing. For more exact methods of spacing stirrups in trapezoidal prisms see page 255.

Wherever load sketches, shear or moment curves are necessary they are drawn but wherever possible time is saved by picturing them mentally and recording only the results. Some computers work with scratch paper at hand for rough work and record only the results on the design sheets. This practice is dangerous, though it saves a little mental wear and tear, because too often something of importance is lost on the scratch paper, so, where possible, it is decidedly quicker to picture the functions mentally than draw them all out.

FG6 is increased in width over the rest of the spandrels. Experiments indicated that an 8-in. width was insufficient even with maximum compressive reinforcement so 12 in. was used, the stairs being located clear of this beam. In checking negative moment at the junction of FG5–FG6 the moment at the end of FG5, except for the effect of the wide column, is the same as at the end of FG6, but the stem width is less. If it were not for the heavy column section more study would have to be given this joint, but as it is the computations show a safe detail. All these points at which special conditions develop illustrate how the designer has to consider the safety and comfort of occupants, the maintaining of shaft sizes, window openings, and other clearances, the adherence to self-imposed sizes of members, simplicity in formwork, compliance with codes and regulations, and ease in construction while keeping the stresses within specified limits.

BEAM AND GIRDER BUILDING	Sheet BG9
FB4=18.54 (280) *0.42 2.29 11.86 9.15 3.40 26.70; FB1=18.80 (13.96) 667 785 900 185 124 111 25 10 77 127; 0.51 6.68 9.65 10.56 27.40; Wall=0.880 Bm=0.295 1.18; $11\frac{1}{2}\times 24$ d=21.4 T=30; $+\frac{wL}{10}=1909$ R=141 $A_s=5.10$ $\left.\begin{matrix}2\text{-}1^{\square}S\\ 2\text{-}1\frac{1}{8}{}^{\square}B\end{matrix}\right\}=5.08$ u=162 SA $u_1=149$; $A_{V_L}=3.31$ $9\text{-}\frac{1}{2}{}^{\phi}$ 4,8,8,8,8,9,9,9,9; $A_{V_R}=3.26$ ~~9~~ 8-$\frac{1}{2}{}^{\phi}$ 4,8,8,10,10,~~12~~,~~16~~,~~18~~,11,11,11,11; *See FG1	FG3
$-M=1909$ R=362 $p'=0.0155$ $\frac{10.3}{16}$ $A'_s=2.46$ $2\text{-}1\frac{1}{8}{}^{\square}=2.54$; p=0.0212 $A_s=5.22$ $2\text{-}1\frac{1}{8}{}^{\square}+2\text{-}1\frac{1}{8}{}^{\square}=5.08$	FG2-FG3
FB1=18.80; 10.09 3.95 14.04; 747 866 1613; 8.71 3.91 12.66; Sash=0.09 Wall=0.19 Sill=0.06 Beam=0.15 =0.49; 8 x 18 d=15.6 T=24 t/d=0.256 j=0.895; $+\frac{wL}{10}=876$ R=150 $A_s=3.14$ $\left.\begin{matrix}1\text{-}1\frac{1}{4}{}^{\square}S\\ 1\text{-}1\frac{1}{4}{}^{\square}B\end{matrix}\right\}=3.12$ u=206 SA $u_1=162$ SA; $v_L=129$ $v_p=95$ $A_{V_L}=2.33$ $11\text{-}\frac{3}{8}{}^{\phi}$ 3,6,6,8,8,8,8,8,8,8,8; $v_R=116$ $v_p=77$ $A_{V_R}=1.90$ 12~~8~~-$\frac{3}{8}{}^{\phi}$ 4, 11 @ 8	FG4
w=0.49 FB1=18.80; 9.40 4.24 13.64; 1732; 8 x 18 d=15.6 T=25; $+\frac{wL}{12}=798$ R=✓ A=2.92 $\left.\begin{matrix}1\text{-}1\frac{1}{4}{}^{\square}S\\ 1\text{-}1\frac{1}{4}{}^{\square}B\end{matrix}\right\}=3.12$ $u_1=175$ SA Spcg=4"; $v_L=125$, $v_p=86$ $A_{V_L}=2.36$ 13 ~~14~~-$\frac{3}{8}{}^{\phi}$ 3,6,6,7, 9 @ 8	FG5
$-M=876$ R=428 $p'=0.0235$ $\frac{9.3}{16}$ $A'_s=1.75$ $1\text{-}1\frac{1}{4}{}^{\square}+1\text{-}1\frac{1}{4}{}^{\square}=3.12$; d=16 d'/d=0.125 p=0.0248 $A_s=3.18$ $1\text{-}1\frac{1}{4}{}^{\square}+1\text{-}1\frac{1}{4}{}^{\square}=3.12$	FG4-FG5
FB4=24.24 (3.26); 4.52 15.48 20.00; 583 1030 16.10; 4.52 8.76 13.28; From FG4 =0.49 Extra Width=0.07 0.56; 12 x 18 d=15.6 d'/d=0.13; $+\frac{wL}{10}=1028$ R=352 p=0.0205 $A_s=3.84$ $\left\{\begin{matrix}2\text{-}1\frac{1}{8}{}^{\square}S\\ 1\text{-}1\frac{1}{8}{}^{\square}B\end{matrix}\right\}=3.81$ $p'=0.0148$ $\frac{9.3}{16}$ $A'_s=1.61$ $2\text{-}1^{\square}=2.00$ u=108 $u_1=148$; $v_L=122$ $v_p=102$ $A_{V_L}=2.73$ ~~13~~-$\frac{3}{8}{}^{\phi}$ 2,5,5,5,5,5,5,5,5,6,6, 6,7,16 to ℄; $v_R=81$ a=74 $A_{V_R}=0.58$ ~~3~~-$\frac{3}{8}{}^{\phi}$ ~~6,15,22~~, 4,8,8,8,8,8, 16 to ℄	FG6
$-M=1028$ d=16 d'/d=0.125; 8 x 18 $\left\{\begin{matrix}R=512 & p'=0.035 & \frac{9.3}{16}A'_s=2.60 & 2\text{-}1\frac{1}{8}{}^{\square}=2.54\\ & p=0.03 & A'_s=3.84 & 1\text{-}1\frac{1}{4}{}^{\square}+2\text{-}1^{\square}+1\text{-}1\frac{1}{8}{}^{\square}=4.83\end{matrix}\right\}$; 12 x 18 $\left\{\begin{matrix}R=341 & p'=0.0131 & \frac{9.3}{16}A'_s=1.46 & 2\text{-}1\frac{1}{8}{}^{\square}=2.54\\ & p=0.0197 & A_s=3.15 & 1\text{-}1\frac{1}{4}{}^{\square}+2\text{-}1^{\square}+1\text{-}1\frac{1}{8}{}^{\square}=4.83\end{matrix}\right\}$	FG5-FG6

SHEET BG9

Comments are here made only on points of unusual interest but the student should check with care all the design computations even though they appear of a routine nature, as something of interest turns up in the design of each individual member.

17–6M. Columns. (See Computation Sheets BG10, 11.) Loads are accumulated as the sum of the reaction from the column shaft above, plus the reactions of supported beams and girders on this floor, plus the dead weight of the column shaft itself and any tributary slab area not included in the beam reactions. Live loads only are reduced 10, 20, and 30 per cent as provided for in the specifications, utilizing the supported floor area recorded at each stack of columns for that purpose. The size of exterior columns is determined by architectural requirements and in all cases is more than ample, though a minimum of ⅔ per cent of vertical steel is called for as general reinforcement. For interior columns the sizes are determined by the loads to be carried (giving consideration to standardization and reuse of forms), the shaft size, amount and percentage of reinforcement increasing progressively from third floor down to the footings. No computations have yet been made for bending in the columns as no direct method is practicable for including this effect in the design. The exterior columns (and interior columns with greatly unbalanced load conditions) are designed at conservative working stresses and check computations are made, as in Ex. 13–6, to show that the combined stresses are within the allowable or to suggest an increase in section.

17–6N. Foundation Walls. (See Computation Sheet BG11.) Foundation walls for this building were designed in Examples 15–1 and 15–2 with an alternative scheme to eliminate the column footings along exterior walls, making the structure wall-bearing, supported on a continuous spread footing (Ex. 15–3). Horizontal reinforcement is desirable in foundation walls to tie the building together and to develop the walls as beams to resist unequal settlements. A fair allowance is 2 or 3 rods, ⅝ or ¾ in. diameter at top and at bottom of walls with ½ in. rounds 12 in. c to c between. Rods should lap 40 diameters at splices and hook 1 or 1½ ft around corners.

17–6O. Footings. (See Computation Sheets BG11 to BG13.) A typical interior footing for this building was designed in Art. 15–8 and a combined exterior footing in Art. 15–10. The other footings follow the same procedure. Concrete stresses are reduced in this example; 3000 psi is used in the superstructure and 2000 psi in the footings because the footings are poured early before all the facilities are available on the job and are poured in earth pits without forms, making it more difficult to obtain accurate control.

BEAM AND GIRDER BUILDING — Sheet BG10

A1, D1, D6

Above 25.05 (81 sf)
3B6 = 10.48
3G4 = 14.04
Col Etc = 9.2
33.72
−10% LL = −1.01 32.71 { $30\frac{7}{16} \times 30\frac{7}{16} \times 12 = 316$
57.76 { $8\text{-}\frac{3}{4}\phi = 45$
2nd Fl = 33.72 { (0.60%) 361
−20% = −2.02 31.70
89.46 do.
1st Fl = 33.72
−30% = −3.03 30.69 Part of continuous wall.
120.15 Steel as above

A2, A3, A4, D2, D3, D4, D5

Above = 33.56 (181 sf)
2x3B7 = 22.52
3G1 = 23.74
Col Etc = 6.6
52.86
−10% LL = −2.26 50.60 { $32\frac{1}{8} \times 12 = 208$
84.16 { $6\text{-}\frac{7}{8}\phi = 46$
2nd Fl = 52.86 { (0.95%) 254
−20% = −4.52 48.34
132.50 do.
1st Fl = 52.86
−30% = −6.78 46.08 Part of continuous wall.
178.58 Steel as above

Columns 1st, 2nd, 3rd — Ex 13-6

B1, C1, C6

Above 31.87 (181 sf)
3G4 = 12.66
3G5 = 13.64
3B1 = 18.80
Col Etc = 7.6
52.70
−10% LL = −2.26 50.44 { $32\frac{1}{8} \times 12 = 208$
82.31 { $6\text{-}\frac{3}{4}\phi = 34$
2nd Fl = 52.70 { (0.68%) 242
−20% = −4.52 48.18
130.49 do.
1st Fl = 52.70
−30% = −6.78 45.92 Part of continuous wall.
176.41 Steel as above

B2, B3, B4, C2, C3, C4, C5

Above = 46.04 (400 sf)
3G1 = 22.70
3G2 = 23.32
2x3B2 38.66
Col Etc 2.1
86.78
−10% LL = −5.00 81.78 { $13\frac{1}{2} \times 13\frac{1}{2} = 98.5$
127.82 { $4\text{-}\frac{7}{8}\phi = 30.7$
2nd Fl = 86.78 { (1.32%) 129.2
Extra Wt 1.2
−20% = −10.00 77.98 { 17 x 17 = 156.0
205.80 { 4-1□ = 51.2
1st Fl = 86.78 { (1.38%) 207.2
Extra Wt = 2.4
−30% = −15.00 74.18 19 x 19 = 194.9
279.98 10-1φ = 100.5
(2.19%) 295.4

Ex 13-4

B5

Above = 95.84 (300 sf)
3G3 = 26.70
3G2 = 23.32
3B2 = 19.33
3B5 = 16.39
Col Etc = 2.0
87.74
−10% LL = −3.75 83.99 { 16 x 16 = 138.2
179.83 { $6\text{-}\frac{7}{8}\phi = 45.1$
2nd Fl = 87.74 { (1.41%) 183.3
Extra Wt = 1.2
−20% = −7.50 81.44 { 19 x 19 = 194.9
261.27 { 6-1□ = 76.8
1st Fl = 87.74 { (1.67%) 271.7
Extra Wt = 2.4
−30% = −11.25 78.89 { 22 x 22 = 261.4
340.16 { 8-1φ = 79.9
{ (1.31%) 341.3

A5

Above = 93.74 (100 sf)
3B7 = 11.26
3B8 = 11.12
3G3 = 27.40
Col Etc = 8.3
58.08
−10% LL = −1.25 56.83 { $32\frac{1}{8} \times 12 = 208$
150.57 { $6\text{-}\frac{3}{4}\phi = 34$
2nd Fl = 58.08 { (0.68%) 242
−20% = −2.50 55.58
206.15 do.
1st Fl = 58.08
−30% = −3.75 54.33 { $32\frac{1}{8} \times 12 = 208$
260.48 { $8\text{-}\frac{7}{8}\phi = 61.5$
{ (1.25%) 268.5

Part of continuous wall, but use steel as above

Loads are taken from the column computations increased by the assumed dead weight of the footing for interior columns and for the weight of any superimposed wall as well for exterior columns.

17–6P. General Summary. Computation Sheets BG1 to BG13 inclusive are now in shape to turn over to the detailers and few if any questions should arise. In drawing scale sections (Chapter XXI) minor changes in the designer's provisions may be desirable. In such cases the original computations are best left untouched and corrections and additions made in colored pencil or else the whole computation is voided with a page reference to the revision. Should the detailer need some additional data for anchoring or bending bars such computations are best made in colored pencil on the original design notes, available for future reference. For a building with a greater variety of members on each floor, and particularly if several computers work on the same structure, the advantages of the marking system, index, and entire head sheet will be more apparent. This is best made on transparent paper and prints given to each person interested who completes his part of the work and marks on the print any data he may have to contribute.

A building must not only be safe and adequate for the owner's needs, but it must also be as economical as possible to construct. Chapter XXII gives some suggestions for economy in framing. Frequently designs are roughed out for various column spacings and beam depths and for different types of construction and comparative cost estimates are prepared to aid in selecting the final schemes. In Chapter XVI typical panels of this building have been designed as clay-tile and joist slabs, ribbed slabs, two-way slabs, and flat slabs. It seems unnecessary to carry those designs any further at this time. Comparative costs of several of these schemes will be worked out in Chapter XXII.

17–7. Summary. In this chapter the practical application of the previously developed theories to a design problem has been illustrated. The arrangement of computations for maximum simplicity and ease of reference deserves the student's careful attention. The use of a skeleton key plan is a great aid to all who use the designer's notes and the index (especially on a arge project) is a great time saver and also serves as a progress record, showing just what members have been designed. The abbreviated form of computation permits a maximum of information to be contained in a small space. A logical procedure from penthouse to footings insures that all required accumulated loads will be available as the designer proceeds through the structure. Wind stresses and other horizontal loads as well as bending in the columns and the effect of rigid connections are all best studied independently of the preliminary design. These considerations form the basis of Chapters

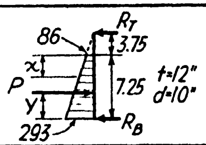
BEAM AND GIRDER BUILDING — Sheet BG11

Columns 1st, 2nd, 3rd

(A6) Above = 72.36 (0)
3B8 = 15.75
3G6 = 13.28
Col Etc = 7.8 — 36.83 — { $30\frac{7}{16} \times 30\frac{7}{16} \times 12$; $8\text{-}\frac{3}{4}^{\phi}$ P = 361 }
109.19
2nd Fl — 36.83
146.02 — do.
1st Fl — 36.83
182.85 — Part of continuous wall. Steel as above

(B6) Above = 74.25 (118 sf)
3G6 = 20.00
3G5 = 13.64
3B5 = 16.39
Col Etc = 7.6
57.63
−10% LL = −1.48 — 56.15 — { $32\frac{1}{8} \times 12$; $6\text{-}\frac{3}{4}^{\phi}$ P = 242 }
130.40
2nd Fl 57.63
−20% LL = −2.96 — 54.67
185.07 — do
1st Fl = 57.63
−30% LL = −4.44 — 53.19
238.26 — Part of continuous wall. Steel as above

Side and End Walls — Ex 15-1



t = 12" d = 10"

Earth ⇔ 28.6 pcf liquid
P = 1370 y = 2.96 R_T = 370 R_B = 1000
$x(86 + 14.3x) = 370$; $x^2 + 6x = 25.9$, $x = 2.82$
M = 23.6 A_s = 0.135 $\frac{1}{2}^{\phi}$ @ 16" c/c = 0.15 u = 97

Wall as Vertical Dist. Beam — $2\text{-}\frac{5}{8}^{\phi}$ Top, $2\text{-}\frac{5}{8}^{\phi}$ Bott

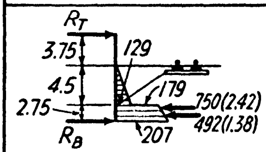
Vert Load — L = 17.3
Sash = 0.09
Span = 0.19
Sill = 0.06
Floor = 0.81
Wall = 1.80
2.95

$\frac{wL}{12}$ M = 883 k in.

$A_s = \frac{M}{f_s jd} = \frac{883}{17.5 \times 140} = 0.36^{\square''}$

Use $2\text{-}\frac{5}{8}^{\phi}$ Top − $2\text{-}\frac{5}{8}^{\phi}$ Bott $= 0.62^{\square''}$

Track Walls — Ex 15-2



$\frac{50000}{5 \times 16} = 625$ 625 × 0.286 = 179
R_T = 227 R_B = 1015
$14.3x^2 = 227$ x = 3.98 M = 17.4 A_s = 0.099
$\frac{1}{2}^{\phi}$ - 16" c/c = 0.15

Column Footings

Column Nos.	Tributary Area	Column Load	Wall	Footing	Dead + Red. LL	Red. LL	Dead L.	½ Live L	Dead + ½ Live L	Req'd Area for p = 4.04 ksf	Size	Bearing Under D + Red L
A1	81	120.1	7.4	7.2	134.7	27.5	107.2	16.8	124.0	30.7	5'-7" sq	4.34
A2 A3 A4	181	178.6	14.8	10.7	204.1	61.5	142.6	37.6	180.1	44.6	5'4" × 8'-4"	4.54
A5	100	260.5	20.6	15.6	306.7	34.0	272.7	20.8	293.5	72.6	6'-7" × 11'-0"	4.24
A6	0	182.9	10.3	11.0	204.2	0	204.2	0	204.2	50.4	7'-1" sq	4.07
B1	181	176.4	11.8	10.6	198.8	61.5	137.3	37.6	174.8	43.2	5'4" × 8'-2"	4.56
B2 B3 B4	400	280.0	0	16.0	296.0	136.0	160.0	83.0	243.0	*	7'-9" sq	4.93
B5	300	340.0	0	20.4	360.4	102.0	258.4	62.2	320.6	79.3	8'-11" sq	4.54
B6	118	238.3	23.6	14.3	276.2	40.0	236.2	24.4	260.6	63.5	6'0" × 10'-6"	4.38

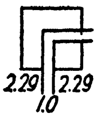
* 296 ÷ 5 = 59.2 sf 7'-9" sq = 60.1 243 ÷ 60.1 = 4.04 ksf
Note that design base pressures equal above values less weight of footing

Continued on Sheet BG12

BEAM AND GIRDER BUILDING Sheet BG12

Column Footings (Continued)

(A1)



t=18 d=14 p=4.11

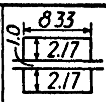
$v=\frac{104\times4.1}{12\times\frac{7}{8}\times14}=29$

$M=0.85\times5.58\times\overline{2.54}^2\times4.1\times\frac{12}{2}=751^*$

$A_s=3.09$ $13\text{-}\frac{1}{2}^{\circ}=3.25$

R=32 u=75

(B1)(A2)(A3)(A4)



t=18 $d=14\frac{1}{2}$ p=4.31

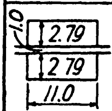
$v=\frac{0.96\times4.31}{12\times\frac{7}{8}\times14.5}=27.2$

$M=4.31\times\overline{2.42}^2\times\frac{12}{2}=151$

R=60 $A_s=4.96$ $16\text{-}\frac{5}{8}^{\phi}=4.96$ Cross

u=87 Use $6\text{-}\frac{1}{2}^{\phi}$ Long

(A5)



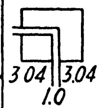
t=19 v=39

$d=15\frac{1}{2}$ M=222 R=81

p=4.00 $A_s=9.0$ $21\text{-}\frac{3}{4}^{\phi}$ Cross

u=104 Use $7\text{-}\frac{1}{2}^{\phi}$ Long

(A6)



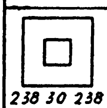
t=23 v=37

d=19 M=1260 R=41

p=3.78 $A_s=3.79$ $13\text{-}\frac{5}{8}^{\phi}=4.03$

u=122

(B2)(B3)(B4)



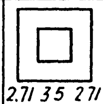
t=20 v=38 (Art 15·8)

d=16 M=1050

p=4.68 $A_s=3.74$

Cap=3'-0"sq x 8" $15\text{-}\frac{1}{2}^{\circ}=3.75$

(B5)



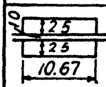
t=22 v=32

d=18 M=1425 u=118

p=4.26 R=74 $A_s=4.53$

Cap=3'-6"sq x 8" $15\text{-}\frac{5}{8}^{\phi}=4.65$

(B6)



t=18 v=35

$d=14\frac{1}{2}$ M=188 R=74

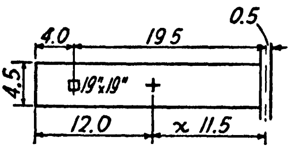
p=4.15 $A_s=8.65$ $20\text{-}\frac{3}{4}^{\phi}=8.80$

u=95 Use $7\text{-}\frac{1}{2}^{\phi}$ Long

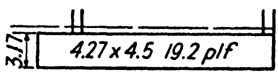
* By J C 865(a) design moment equals 85 per cent of the moment of the base pressure on one side of the critical section See J.C. 863-871

Footings

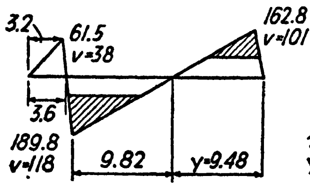
(C2-D2) (C3-D3) (C4-D4) (C5-D5)



	Int. Col	Ext. Col
Full LL	=166.0	75.2
Red LL	=136.0	61.6
Half LL	= 83.0	37.6
D. L	=144.0	120.4
DL+½ LL	=227.0 +	158.0 = 385.0
DL+Red. LL	=280.0 +	182.0 = 462.0



DL+½ LL = 385.0
Footing = 53.0
4.04) 438.0
108 sf



DL+Red LL = 462.0
Footing = 53.0
515.0

x=11.5 L=24.0 B=4.5 DL+Red. LL p=4.77

y=9.48 −M=9260

R=157 $\begin{cases} d_m=33 \\ d_v=33\frac{1}{2}''+4\frac{1}{2}''=38'' \end{cases}$

$A_s=15.8$

$13\text{-}1\frac{1}{8}^{\circ}=16.51$ Top $\begin{cases} 10\ Str \\ 3\ Bt \end{cases}$ u=142 SA

Art 15·10

Left of Int Col v=38 $A_v=0$ +M=1250 $A_s=2.11$ Bott $3\text{-}1\frac{1}{8}^{\circ}=3.78$

Right " " " v=118 a=82 u=152 SA

$A_v=10.7$ $7\text{-}\frac{1}{2}^{\circ}$ stirrups $=10.5$ Use $6\text{-}\frac{1}{2}^{\circ}$ Cross

@ 7,7,7,7,8,10,15

Ext Col v=101; a=69 $A_v=7.1$ $5\text{-}\frac{1}{2}^{\circ}$ stirrups $=7.55$ @ 6,7,9,12,17

BEAM AND GIRDER BUILDING — Sheet BG13

C1-D1
C6-D6

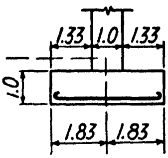
Scheme #1 = Combined footing similar to C2-D2 of same length and thickness but having width, main steel and stirrups reduced to $\frac{350.1}{438.0} = 80\%$

Scheme #2 - Using foundation walls for stiffness as below is more economical, BUT WALLS MUST NEVER BE REMOVED.

	Int. Col.	Ext. Col.
Full LL	= 72.2	33.6
$\frac{1}{2}$ LL	= 37.6	16.8
Red LL	= 59.3	27.5
DL	= 117.1	92.6
DL + $\frac{1}{2}$ LL	= 154.7 + 109.4 = 264.1	
DL + Red. LL	= 176.4 + 120.1 = 296.5	

D + $\frac{1}{2}$ LL = 264.1
Wall = 71.0
Footing = 15.0
4.04)350.1
86.6
x = 11.4 L = 24.0
B = 3.67

D + Red LL = 296.5
Wall + Ftg = 86.0
382.5
p = 4.34
v = 29 M = 62.8
A_s = 0.422
$\frac{3}{8}$ φ 3" c/c = 0.44
Spac'g for bond = 28"



Found Wall as Dist Beam (See C2-D2 above)
Active pressure = 12.35 klf
y = 9.73 from ℄
V_R = 107.7   M = 6 28 V_L = 126.9
12/132 d = 128 A_s = 2.80   3-1" Top = 3.00
v = 94
a = 43
A_{v_L} = 0.55
3-$\frac{1}{2}$ φ Wall rods @ 16" c/c = 0.60

SHEET BG13

XVIII and XIX. No design is complete until the adequate, intelligible information is given to the builder; the detailing of this building is considered in Chapter XXI. Finally, before starting on the final computations the designer makes economic studies to determine column spacings, types of construction, etc., as discussed in Chapter XXII.

CHAPTER XVIII

CHECK BUILDING DESIGN BY RIGID FRAME ANALYSIS

18–1. Arbitrary moment coefficients in building design are very useful for preliminary studies, especially for very regular types of framing, but their use is rapidly being displaced by more precise attacks. Such coefficients could apply to only one set of ratios of moments of inertia and span lengths and one ratio of dead to live load. Negative moment may exist all across an interior span where the coefficients would indicate positive moment. Many methods have long existed for the solution of these indeterminate problems, such as equation of elastic line, moment-areas, elastic weights, three moment equation, least work, slope deflections, moment distribution, column analogy, and also certain graphical methods including Fidler's* characteristic points supplemented by Ostenfeld's** diagrams or Ritter's*** fixed points. In addition to these, certain approximations are possible, such as the averaging of beam stiffnesses either side of a joint as suggested by the P.C.A. in 1937† and the fixing of the far ends of columns and using only two cycles of moment distribution as recommended by the 1940 J.C. and developed by the P.C.A. in 1941‡. Three reasons account for their not having been employed more often: Codes permitted the use of arbitrary coefficients; accurate methods for indeterminate structures required excessive computation; and computers have been unfamiliar with satisfactory abridgments suitable for the design office.

The building worked out in Chapter XVII will be reviewed in part by several methods and conclusions will be drawn as to the suitability of arbitrary coefficients and the amount of work in various other methods. This building has practically equal spans, constant moments of inertia, with live load double the dead load. The arbitrary coefficients will approximate this case much more closely than for the average building. The labor of making computations in this case is considerably less than

* Minutes of Proceedings, Inst. C. E., Vol. LXXIV, 1883, p. 196, and " A Practical Treatise on Bridge Construction," by Thomas Claxton Fidler, London, 1909.

** " Technish Statik " by A. Ostenfeld, Vol. II, Copenhagen, 1913.

*** " Der Kontinuierliche Balken," W. Ritter, 1900.

† " Continuity in Concrete Building Frames," Portland Cement Association, second edition, October, 1937.

‡ " Continuity in Concrete Building Frames," Portland Cement Association, third edition, 1941.

for a more irregular building. The savings of the approximate methods are not fully realized in this particular structure.

For any indeterminate structure it is necessary to approximate the sizes of the members of a frame and check the capacity. It is not possible to design each member directly, because the stiffness of the member affects the distribution of stress. Accordingly, rapid approximate methods have great importance.

Before studying this chapter the reader should review and thoroughly understand the theories developed in Chapter XII.

18–2. Precision of Analysis. The designer wishes a reasonably close approximation to the actual distribution of moments. Although the so-called " exact " methods are interesting mathematical problems, the use of such hair-splitting refinements is not justified for several reasons:

1. E_c. The modulus of elasticity varies from, say, 2,000,000 to 4,000,000 psi. It will vary considerably in the same structure with different conditions of age and moisture.

2. I_c. Most textbook solutions have been made for planar frames with assumed values for moments of inertia. Concrete structures that are poured in place are monolithic. It is uncertain how much of the adjoining structure is to be taken as flange in determining stiffnesses of beams and girders. A much wider portion of slab acts than that limited amount available for tee in stress computations. In making load tests, the engineer is often annoyed by the plate action of the slab carrying loads outside of the area he wishes to test.

3. A_s. It is uncertain whether the moment of inertia of beams should be determined for the transformed area, or whether the gross section of concrete should be used and the steel forgotten. There is a growing tendency toward the latter practice. In designing we assume the concrete has cracked in tension before the steel is brought into play. Such cracks occur at relatively distant spacings, and the moment of inertia used as a measure of stiffness may well be taken for the gross section.

4. *Variable I.* Most concrete beams are tee-beams. Methods have been developed for beams of variable sections and with haunches. They are applicable when the contours of the section are known. Concrete beams act as tee-beams to resist positive bending in the center portion, and as rectangular beams to resist negative bending outside of the points of inflection. As the points of inflection vary under different loading conditions, an exact determination of the deflection of an ordinary concrete beam is a complicated affair. (Inside of the column faces the depth is large and I very great.)

5. *Joints.* Practical considerations make the division of a structure into " pours " inevitable. Consequently joints usually occur at the

top of each column just below the beam soffits, and just above the top of the rough floor slab. If the building is very long the floor will be subdivided by cross joints at mid-span of the beams. With steel across the joints, the concrete can take the compression and the rods the tension. Some yielding is possible. The exact effect of such joints is uncertain and needs further study.

6. *Connections.* Allowance is made for angle changes in the tangents of all the members that meet at a joint, but no allowance is made for possible changes in the angles at which the members meet. The computations are all for " spring " in the members, and disregard possible " give " in the joints.

7. *Loads.* Somewhat compensating these effects is the fact that in computations loads are customarily placed in those positions that will produce absolute maximum values, regardless of the fact that such loading conditions may never be realized in the life of the structure.

8. *Discrepancies.* Minor discrepancies will occur in construction. Column centers will vary a small fraction of an inch and cross sections of members may vary quite a bit from the theoretical sizes. Such variations usually do not amount to anything with statically determinate structures, but the effect in continuous frames may be to change considerably the assumed ratios of member stiffness.

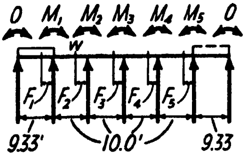
On the other hand it should be noted that tests of models and full-sized structures have demonstrated that reinforced concrete frames are elastic in their behavior to a surprising degree, and that designs made by the principles of continuity have proved capable of severe overloading and satisfactory in all respects. The above argument is merely to demonstrate that no hair-splitting is practicable and that a reasonably accurate approximation is all that is necessary or desired.

18–3. Floor Slabs. The floor slabs of the building shown on Fig. 17–2 are examples of rectangular reinforced concrete beams of approximately equal spans, lying on equally yielding supports, and here treated as though free-ended. This is about the simplest possible example of continuous beams. Several methods will be illustrated for this case:

Three Moment Equation. On Computation Sheet TM1 the necessary governing data have been assembled. The unknowns here sought are the negative moments over the interior supports, numbered M_1 to M_5 inclusive. Moments at exterior supports are taken as zero since the torsional resistance of the spandrels is small. (See Ex. 14–7.) The moment of inertia is constant throughout span: lengths are taken from center to center of beams.

By loading one span at a time, the effect of each loaded span on all five support moments was determined. To permit using both live and dead load coefficients results were expressed in terms of unit load w.

BEAM & GIRDER BUILDING BY THREE MOMENTS — Sheet TM1



Load in First or Sixth Span

$$\begin{cases} 0+38.66M_1+10M_2=-\frac{w\times 9.33^3\times 12}{4} \\ 10M_1+40M_2+10M_3=0 \longrightarrow & M_1=+201.50M_5 \\ 10M_2+40M_3+10M_4=0 & M_2=-53.99M_5 \\ 10M_3+40M_4+10M_5=0 & M_3=+14.464M_5 \\ 10M_4+38.66M_5+0=0 & M_4=-3.866M_5 \\ M_5[38.66\times 201.50-539.9]=-2436.5w;\ M_5=0.3361w \end{cases}$$

Table I	M_1	M_2	M_3	M_4	M_5
	−67.72w	+18.15w	−4.86w	+1.30w	−0.34w
D.L.=63	−4.27	+1.14	−0.31	+0.08	−0.02
L.L.=125	−8.47	+2.27	−0.61	+0.16	−0.04
T.L.=188	−12.74	+3.41	−0.92	+0.24	−0.06

Load in Second or Fifth Span

$$\begin{cases} 0+38.66M_1+10M_2=-\frac{w\times 10^3\times 12}{4} \\ 10M_1+40M_2+10M_3=-\frac{w\times 10^3\times 12}{4} \\ 10M_2+40M_3+10M_4=0 \longrightarrow & M_2=-53.99M_5 \\ 10M_3+40M_4+10M_5=0 & M_3=+14.464M_5 \\ 10M_4+38.66M_5+0=0 & M_4=-3.866M_5 \end{cases}$$

$$\left.\begin{aligned} 38.66M_1-539.9M_5&=-3000w \\ 10M_1-2015.0M_5&=-3000w \end{aligned}\right\} \begin{aligned} M_1&=\frac{-1475.1}{28.66}M_5=-51.47M_5 \\ M_5&=\frac{-3000w}{-2529.7}=1.186w \end{aligned}$$

Table II	M_1	M_2	M_3	M_4	M_5
	−61.04w	−64.03w	+17.15w	−4.59w	+1.19w
D.L.=63	−3.85	−4.03	+1.08	−0.29	+0.07
L.L.=125	−7.63	−8.00	+2.14	−0.57	+0.15
T.L.=188	−11.48	−12.03	+3.22	−0.86	+0.22

Load in Third or Fourth Span

$$\begin{cases} 0+38.66M_1+10M_2=0 & M_1=-0.2587M_2 \\ 10M_1+40M_2+10M_3=-w\times 10^3\times 12/4 \\ 10M_2+40M_3+10M_4=-w\times 10^3\times 12/4 \\ 10M_3+40M_4+10M_5=0 \longrightarrow & M_3=+14.464M_5 \\ 10M_4+38.66M_5+0=0 & M_4=-3.866M_5 \end{cases}$$

$$\left.\begin{aligned} 37.41M_2+144.64M_5&=-3000w \\ 10M_2+539.9M_5&=-3000w \end{aligned}\right\} M_2=\frac{395.26}{27.41}M_5=+14.42M_5$$

$$M_1=-3.730M_5$$

$$M_5=-\frac{3000w}{684.1}=-4.385w$$

Table III	M_1	M_2	M_3	M_4	M_5
	+16.36w	−63.23w	−63.42w	+16.95w	−4.39w
D.L.=63	+1.03	−3.98	−4.00	+1.07	−0.28
L.L.=125	+2.05	−7.90	−7.93	+2.12	−0.55
T.L.=188	+3.08	−11.88	−11.93	+3.19	−0.83

Check Location of Fixed Points:

$$F_5=\left(\frac{M_5}{M_4+M_5}\right)L_5=\frac{1.00}{4.866}\times 10=2.06 \qquad F_3=\left(\frac{M_3}{M_2+M_3}\right)L_3=\frac{14.46}{68.45}\times 10=2.11$$

$$F_4=\left(\frac{M_4}{M_3+M_4}\right)L_4=\frac{3.866}{18.33}\times 10=2.11 \qquad F_2=\left(\frac{M_2}{M_1+M_2}\right)L_2=\frac{53.99}{255.49}\times 10=2.11$$

Sheet TM1

BEAM AND GIRDER BUILDING BY THREE MOMENTS — Sheet TM2

Table IV — Dead Load Only

Load in	M_1	M_2	M_3	M_4	M_5
1	-4.27	+1.14	-0.31	+0 08	-0 02
2	-3 85	-4 03	+1.08	-0 29	+0.07
3	+1.03	-3.98	-4 00	+1 07	-0 28
4	-0.28	+1 07	-4 00	-3 98	+1 03
5	+0.07	-0 29	+1 08	-4 03	-3 85
6	-0 02	+0.08	-0.31	+1 14	-4 27
Σ	-7.32	-6.01	-6.46	-6.01	-7 32

Table V — D L.+ L L. in Spans 1-3-5

	M_1	M_2	M_3	M_4	M_5
DL	-7 32	-6 01	-6 46	-6 01	-7 32
LL1	-8 47	+2 27	-0 61	+0.16	-0 04
L.L3	+2 05	-7 90	-7 93	+2 12	-0 55
LL5	+0 15	-0 57	+2 14	-8 00	-7 63
Σ	-13 59	-12 21	-12 86	-11 73	-15 54

Table VI — D L +L.L in Spans 1-2-4-6

	M_1	M_2	M_3	M_4	M_5
DL	-7 32	-6 01	-6.46	-6 01	-7 32
LL1	-8.47	+2 27	-0.61	+0 16	-0 04
L.L2	-7 63	-8 00	+2.14	-0 57	+0 15
LL4	-0 55	+2 12	-7 93	-7 90	+2 05
LL6	-0 04	+0 16	-0 61	+2 27	-8 47
Σ	-24.01	-9 46	-13 47	-12 05	-13 63

Table VII — D L + L L in Spans 2-4-6

	M_1	M_2	M_3	M_4	M_5
DL	-7 32	-6.01	-6.46	-6 01	-7 32
LL2	-7 63	-8 00	+2.14	-0 57	+0 15
LL4	-0.55	+2 12	-7.93	-7 90	+2 05
LL6	-0 04	+0 16	-0.61	+2.27	-8.47
Σ	-15 54	-11 73	-12 86	-12 21	-13 59

Table VIII — D L + L L in Spans 1-3-4-6

	M_1	M_2	M_3	M_4	M_5
DL	-7.32	-6 01	-6 46	-6 01	-7 32
L.L1	-8 47	+2 27	-0 61	+0 16	-0 04
LL3	+2.05	-7 90	-7 93	+2 12	-0 55
LL4	-0.55	+2 12	-7 93	-7 90	+2 05
L.L6	-0.04	+0.16	-0 61	+2 27	-8 47
Σ	-14 33	-9 36	-23 54	-9 36	-14 33

Table IX — D L + L L in Spans 2-3-5

	M_1	M_2	M_3	M_4	M_5
DL	-7 32	-6 01	-6.46	-6 01	-7.32
LL2	-7.63	-8.00	+2 14	-0 57	+0.15
LL3	+2 05	-7.90	-7.93	+2.12	-0 55
LL5	+0 15	-0.57	+2 14	-8.00	-7 63
Σ	-12 75	-22 48	-10 11	-12 46	-15 35

Max.+ M_A (First Span) L.L. 1-3-5

$$\frac{WL}{8} = 24.55;\ R_1 = \frac{4 \times 24\ 55 - 13\ 59}{112} = 0.7554$$

$$x = \frac{0\ 7554}{0.188} = 4.018;\ +M_a = 4\ 018 \times 0.7554 \times \frac{12}{2} = +18.21$$

Max + M_B (Second Span) L.L 2-4-6

$$\frac{WL}{8} = 28.20;\ +M_b = 28\ 20 - \frac{15.54 + 11.73}{2} = +14.57$$

Max + M_C (Third Span) L.L 1-3-5

$$\frac{WL}{8} = 28.20;\ +M_c = 28.20 - \frac{12.21 + 12.86}{2} = +15.67$$

BEAM AND GIRDER BUILDING BY THREE MOMENTS Sheet TM3

TABLE X- D.L.+Max L.L. or D.L.+Min L.L. Every Tenth Point of Span

		Simple Span	Load 1st Span	Load 2nd Span	Load 3rd Span	Load 4th Span	Load 5th Span	Load 6th Span	Σ all	Σ (+)	Σ (−)	D.L. w=63	Max. L.L. w=125	Min. L.L. w=125	D.L.+Max.L.L.	D.L.+Min.L.L.
M_3	1.0	0	-4.86	+17 15	-63 42	-63.42	+17.15	-4.86	-102 26	+34 30	-136 56	-6 44	+4.29	-17 07	-2 15	-23.51
Third Span	0.9	+54.00	-2.56	+9 03	-63 40	-55 38	+14 98	-4.24	-47 57	+78 01	-125 58	-3 00	+3 00	-8 95	0	-11 95
	0.8	+96 00	-0 26	+0 91	-63 38	-47 35	+12 80	-3.63	-4 91	+109 71	-114 62	-0 31	+5 79	-6 41	+5 48	-6 72
	0.7	+126.00	+2 04	-7 20	-63 36	-39 31	+10 63	-3.01	+25 79	+138.67	-112 88	+1 62	+9.41	-6 19	+11.03	-4.57
	0.6	+144 00	+4.34	-15 32	-63 34	-31 27	+8.45	-2.40	+44 46	+156 79	-112 33	+2 80	+11 74	-6 12	+14 54	-3.32
	0.5	+150 00	+6 65	-23 44	-63.33	-23 24	+6 28	-1 78	+51 14	+162 93	-111 79	+3 22	+12 45	-6 06	+15 67	-2 84
	0.4	+144 00	+8 95	-31 56	-63 31	-15.20	+4 11	-1 16	+45 83	+157 06	-111 23	+2 89	+11 72	-5 99	+14 61	-3 10
	0.3	+126 00	+11 25	-39 68	-63.29	-7 16	+1 93	-0 55	+28 50	+139 18	-110 68	+1 80	+9 49	-5 92	+11 29	-4 12
	0.2	+96 00	+13.55	-47 79	-63 27	+0 88	-0 24	+0 07	-0.80	+110 50	-111 30	+0 05	+5 90	-6 00	+5 95	-5.95
	0.1	+54 00	+15.85	-55 91	-63.25	+8 91	-2 42	+0.68	-42.14	+79 44	-121 58	-2 65	+3 18	-8 45	+0 53	-11.10
M_2	1.0	0	+18 15	-64 03	-63 23	+16.95	-4 59	+1 30	-95 45	+36 40	-131 85	-6 01	+4 55	-16 48	-1 46	-22.49
Second Span	0.9	+54 00	+9.56	-63 73	-55 27	+14.82	-4 01	+1 14	-43 49	+79 52	-123 01	-2 74	+3 19	-8 63	+0 45	-11 37
	0.8	+96 00	+0 98	-63 43	-47 31	+12.68	-3 43	+0 97	-3.54	+110.63	-114 17	-0 22	+5 90	-6 34	+5 68	-6 56
	0.7	+126 00	-7 61	-63 13	-39 35	+10 55	-2.86	+0.81	+24 41	+137 36	-112 95	+1 54	+9 28	-6 23	+10 82	-4.69
	0.6	+144 00	-16 20	-62 83	-31 40	+8 41	-2 28	+0 64	+40 34	+153.05	-112 71	+2 54	+11 28	-6 24	+13.82	-3 70
	0.5	+150.00	-24 79	-62 54	-23 44	+6 28	-1 70	+0.48	+44.29	+156 76	-112 47	+2 79	+11 78	-6 24	+14 57	-3 45
	0.4	+144 00	-33 37	-62 24	-15.48	+4 15	-1.12	+0.32	+36 26	+148.47	-112 21	+2 28	+10 78	-6 25	+13 06	-3 97
	0.3	+126 00	-41 96	-61 94	-7.52	+2 01	-0.54	+0 15	+16 20	+128 16	-111 96	+1 02	+8 78	-6 25	+9 80	-5 23
	0.2	+96 00	-50 55	-61 64	+0.44	-0 12	+0 04	-0 01	-15.84	+96 48	-112 32	-1 00	+4 36	-6.34	+3 36	-7.34
	0.1	+54 00	-59 13	-61 34	+8 40	-2 26	+0.62	-0 18	-59 89	+63 02	-122 91	-3 77	+1 13	-8 61	-2 64	-12 38
M_1	1.0	0	-67 72	-61.04	+16 36	-4.39	+1.19	-0.34	-115 94	+17.55	-133 49	-7.30	+2.19	-16.68	-5.11	-23.98
First Span	0.9	+47.05	-60.95	-54 93	+14.72	-3 95	+1.07	-0.31	-57.30	+62 84	-120 14	-3 61	+1.97	-9 14	-1 64	-12.75
	0.8	+83 64	-54 18	-48 83	+13.09	-3 51	+0 95	-0 27	-9.11	+97 68	-106 79	-0 57	+5.44	-6.58	+4.87	-7.15
	0.7	+109.78	-47 40	-42 73	+11 45	-3.07	+0.83	-0 24	+28 62	+122.06	-93.44	+1 80	+9 33	-5 76	+11 13	-3.96
	0.6	+125 46	-40 63	-36 62	+9.82	-2 63	+0 71	-0.20	+55 91	+135.99	-80 08	+3.52	+11 92	-4 93	+15 44	+1.41
	0.5	+130.69	-33.86	-30 52	+8.18	-2 20	+0 60	-0 17	+72 72	+139 47	-66 75	+4.58	+13 20	-4 11	+17 78	+0.47
	0.4	+125 46	-27.09	-24 42	+6 54	-1.76	+0.48	-0.14	+79 07	+132 48	-53 41	+4 98	+13.17	-3.29	+18.15	+1.69
	0.3	+109 78	-20 32	-18 31	+4.91	-1.32	+0.36	-0.10	+75.00	+115.05	-40 05	+4 73	+11 84	-2 47	+16.57	+2 28
	0.2	+83 64	-13 54	-12 21	+3 27	-0 88	+0 24	-0 07	+60 45	+87 15	-26 70	+3 81	+9 20	-1 65	+13.01	+2.16
	0.1	+47 05	-6 77	-6 10	+1 64	-0 44	+0.12	-0 03	+35.47	+48 81	-13.34	+2.23	+5.26	-0.82	+7.49	+1.41
		Moments for Load W = 1										Moments in kf				

Sheet TM3

Table I gives values of each moment in terms of w and numerical values in inch-kips for dead load, live load, and total load on the first span. A load in the sixth span may be read from this same table by reversing the subscripts of the moments. Table II covers the effects of a load in the second or fifth spans; Table III, load in the third or fourth spans.

It is evident from the foregoing computations that, with a series of unloaded spans with a loaded span outside of the series, the ratio of the moment at the left end of any span to the moment at the right end is constant. Hence the moment line crosses the axis at a fixed point whose location is a function merely of the span lengths and their stiffnesses. The computations below these tables locate these so-called "fixed points." In beams of equal spans and constant moments of inertia, the fixed points are very close to the fifth point. Some designers use these fixed points quite extensively.* They are an integral part of the graphical method described below.

In Table IV are accumulated all dead load effects to obtain the dead load moment curve. Partial live loading produces maximum conditions. Several combinations are possible, each producing a maximum at some point. Five tables sum up the effects for live loads on the various combinations of spans as listed below.

Table	*Loaded Spans*	*Maximum Moments Produced*
V	1–3–5	Positive in spans 1–3–5
VII	2–4–6	Positive in spans 2–4–6
VI	1–2–4–6	Negative over 1st interior support
IX	2–3–5	Negative over 2nd interior support
VIII	1–3–4–6	Negative over 3rd interior support

Beneath the tables maximum positive moments are computed, placing live load on the span and on alternate spans beyond. In the first span, moment was obtained by computing reaction, zero shear point, and taking moments about it. The other positive moments were obtained by subtracting from the $WL/8$ moment parabola the average of the two negative end moments. This is slightly approximate† but well within the limits of accuracy required.

The three moment equation is an "exact" one but, as here presented, it assumes that the slabs are freely resting on equally yielding knife-edge supports at each beam and does not take into consideration any restraint due to the torsional stiffness of the beams. For even a simple

* Notably Taylor, Thompson and Smulski, Concrete, Plain and Reinforced, John Wiley & Sons, 1925, Vol. II, p. 153.

† For an exact determination see p. 247.

problem the work is rather extended and becomes more complicated with either an increasing number of spans or variations in the length of the spans and their moments of inertia. This method is presented first to establish values against which to check the other methods, and so Table X has been prepared to show the values at the tenth-points of each span.

*Graphical Analysis.** In the hands of the skilled operator the graphical method will produce results quicker than the analytical three moment method; will furnish the entire maximum and minimum moment curves which are of help in bending up reinforcing steel; will eliminate irregular errors (as the values must fall on the curve); and should be used every once in a while to give the designer a mental picture of the effect of each individual load and its contribution to the whole.

The first step consists in locating graphically the " fixed points " just described. The procedure is simple as is illustrated in Fig. 18–1*a*, where the spans are drawn to convenient scale, which need not be very large for satisfactory results, say ⅛ in. or ¼ in. per ft. Vertical lines are drawn through the third points of each span, i.e., through the centroids of triangular M/EI diagrams whose bases are one span length. A vertical line is also drawn through a point near each reaction, obtained by setting back a distance of one-third of the left span from the near third-point of the right span or, conversely, by laying off one-third of the right span from the right third-point of the left span. This line passes through the centroid of an M/EI diagram whose base is the sum of the right and left spans and whose vertex lies on the reaction line. A simple graphical construction is applied to these elastic weights: from a zero moment point of the original beam, in this case the left reaction, a random inclined line is drawn, cutting the right " third-point vertical " of the first span and the " inverted vertical " of the first two spans. A line from the first intersection through the right reaction of the first span locates a point on the left " third-point vertical " of the second span; the line connecting this with the second intersection point originally described cuts the beam axis at the " left fixed point of the second span." The next random line starts from this fixed point and the construction proceeds to the right-hand end when all the left fixed points have been established. The process is repeated, beginning at a zero moment point at the right end, and all the right fixed points

* Very little has been written in English on this subject. A paper by L. H. Nishkian and D. B. Steinman on " Moments in Restrained and Continuous Beams by the Method of Conjugate Points " in Transactions, A.S.C.E., Vol. 90, 1927, and particularly the thorough discussion this received should be carefully read. The method in this text differs somewhat from that paper, but the discussions brought out several alternative variations, all of which are of interest.

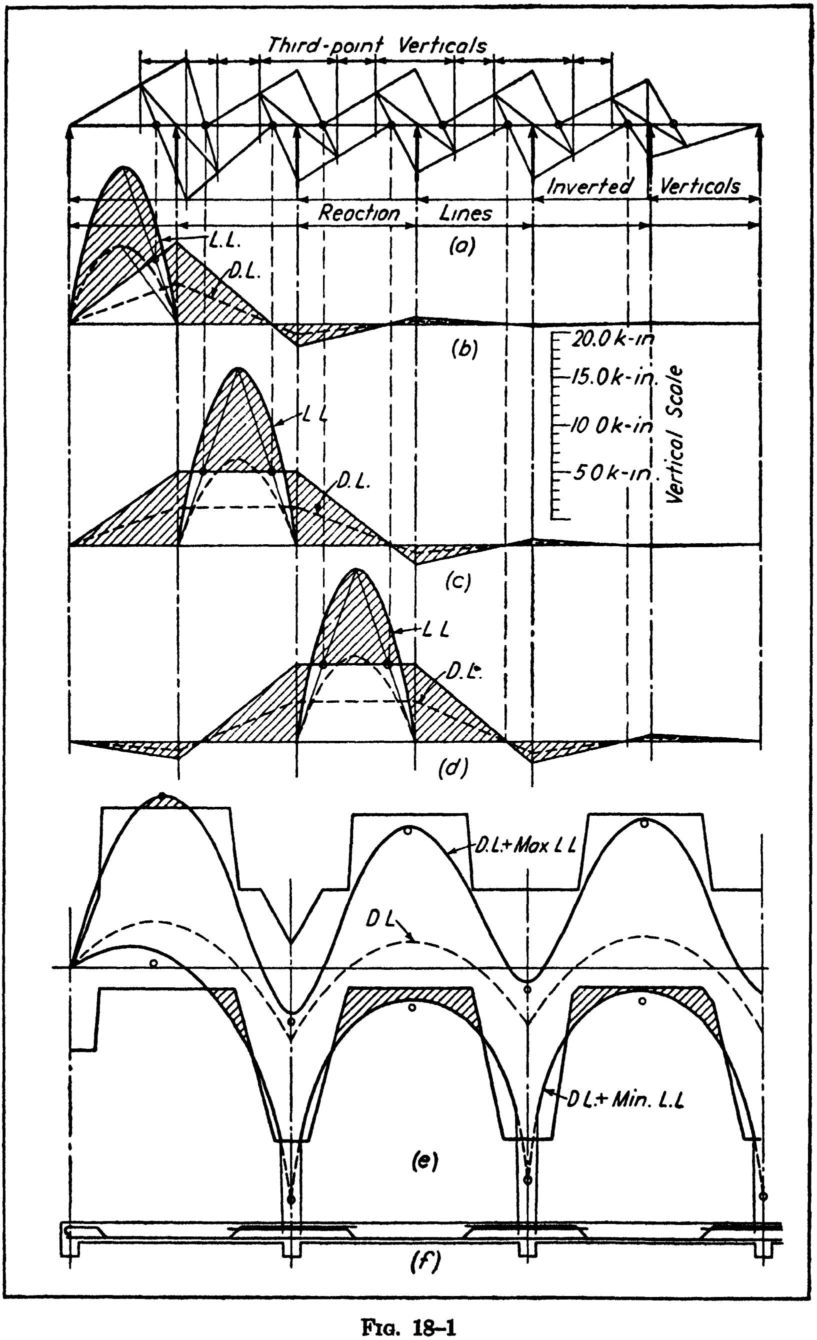

Third-point Verticals
Inverted Verticals
Reaction Lines
(a)
L.L.
D.L.
(b)
20.0 k-in
15.0 k-in.
10 0 k-in
5 0 k-in.
Vertical Scale
L L
D.L.
(c)
L L
D.L.
(d)
D.L.+ Max L L
D L
D L.+ Min. L.L
(e)
(f)

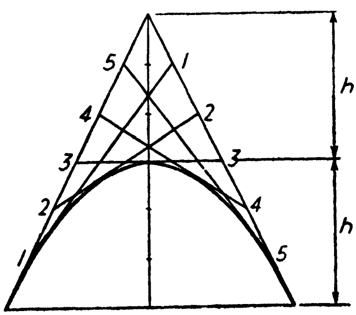
FIG. 18-1

are located. Though rather tedious, it can be shown by similar triangles that the location of each fixed point obtained graphically is identically the same as that obtained analytically at the bottom of Computation Sheet TM1.

For simplicity the spans are then loaded one at a time with their M/EI diagrams as shown in Figs. 18–1*b*, *c*, and *d* respectively, paralleling exactly the computations on Sheet TM1, the last three spans being the reverse of the first three. The moment parabolas are easily constructed* and in this case separate parabolas are drawn for live and dead loads, though in practice it is usually simpler to draw one parabola and obtain the values for the other by direct proportion. Moment closing lines are then drawn as explained below and the correct moment at any point is measured along a vertical ordinate through that point between the boundary lines, the areas for live load only being shaded in the figure. Values *above* the moment closing line are positive; those *below*, negative. This moment closing line for a uniformly loaded span is drawn through the intersections of the fixed-point verticals with the legs of an isosceles triangle inscribed within the parabola, and is extended to intersect the reaction verticals. From these intersections the moment closing line in any unloaded span goes through the far fixed point. The reader should scale sufficient points to ascertain that these graphical values agree exactly with Tables I, II, and III on Sheet TM1. Ordinarily all three diagrams of Figs. 18–1*b*, *c*, and *d* are superimposed and have a common base line to save time and space in drawing. The moment closing lines are distinguished by numbering, by various types of lines, or by color.

Finally, Fig. 18–1*e* accumulates all these partial loadings into one complete picture, just as Table X did for the analytical solution. First,

* The simplest graphical construction of a parabola of this type is by inscribing it in an isosceles triangle whose altitude is double the mid-ordinate of the parabola.

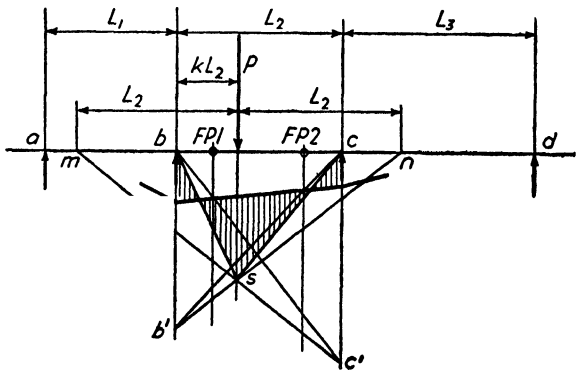


As shown in the figure, both legs are divided into the same number of equal parts and the division points are connected beginning at the top on one side and the bottom on the other. The resulting lines are tangent to the parabola.

a dead load curve is drawn by taking a series of vertical ordinates across all the diagrams and on each vertical ordinate plotting the algebraic sum of the intercepts on the individual diagrams, accumulated with a pair of dividers, having in mind the necessity of using the symmetrical position in diagrams b, c, and d to allow for each of the three right-hand spans. The result is the dotted curve marked "dead load only." Accumulate similarly the live load effects due to the possibility of partial live loading; *positive* values only are added algebraically and plotted *above* the dead load curve; the accumulated *negative* values are plotted *below* the dead load curve. The result is a pair of envelope curves such that the area between them contains all possible moment values for various combinations of the loads chosen and no moment can lie outside of these envelopes unless the original conditions are changed. At critical points the vertical ordinates can be taken closer together if desired.*

The use of these maximum and minimum bending moment curves is illustrated in Figs. 18-1e and f. Part f shows a section through this

* An extension of this graphical method permits the drawing of influence lines for moments. It is first necessary to understand the procedure for a single concentrated load. The fixed points are located as already described. For a single concentrated load P, distant kL_2 from the left reaction of span L_2, draw the simple beam moment triangle bsc as shown in the figure. Lay off the span length L_2 each way from the point of application of the load, thus giving points m and n, and draw lines msc' and msb' through the vertex of the moment triangle. The intersection of the fixed point verticals with lines bc' and cb' establish two points on the moment closing line thus permitting the construction of the moment curve.



To draw an influence line as, for example, negative moment at point $b(M_b)$, it is only necessary to place unit loads at successive points throughout the spans and to indicate the above construction sufficiently to establish the moments at b (noting that it is not necessary to draw the moment triangle; simply indicate point s on the load line) and to project this value of M_b back on the load line, where it becomes one point on the influence line. Since very little of the actual construction need be shown, it will be found that an influence line is obtained quite rapidly.

floor slab with vertical dimensions doubled for clarity. On *e* are shown the resisting moments at each section obtained from $M_s = A_s f_s jd$. Where bars bend up the positive moment decreases from the bend point and the value of that bar drops to zero where it crosses the neutral axis. Where top bars bend down their value drops from 100 per cent at the bend point to zero where they cross the neutral axis. At the ends of bars 40 diameters of embedment are required to develop the bars in bond along an inclined line as shown. Bars are bent up at the quarter-points of the center-to-center span for continuous ends and at the seventh-point for non-continuous ends. It is customary to extend top bars to the quarter-point of each adjacent span. The cross-hatched areas indicate deficiencies where the resisting moment is insufficient to take care of the absolute maximum moment under all possible partial loadings. This diagram shows that negative moment is possible all the way across interior slab spans for this building. The horizontal line is drawn for 90 psi tension on the unreinforced center position of the span. ($M_c = 12 \times 4^2 \times 90/6 = 2880$ lb-in.) At the supports no attention is paid to the portion inside of the beam widths because almost the entire reaction will be concentrated close to the face of the beam, so that the moment at the beam face will be practically the maximum value; and again, as soon as the beam side is encountered, the effective depth of slab increases as the compression works down into the beam stem.

Approximate Moment Distribution. On Computation Sheet MJ1 is a solution of this same slab by an abbreviated approximate method of moment distribution.* This method seems rather rough on first examination but, when used, as here, to obtain values for a single span loaded at a time, summing up for various combinations of loaded spans, the results obtained are within the degree of precision consistent with the various factors of the problem.

No explanation of the set-up of the problem is required beyond noting that the stiffness of each member is taken as proportional to $1/L$, I being constant for all spans. This ignores the reduced stiffness of AB at B due to the hinged support at A. The fixed end moments (FEM) are those for all spans loaded; $-wL^2/12$ in all cases except $-wL^2/8$ for M_{BA}, A being hinged. The results desired are, however, first, the moments at all supports for a single span loaded and, second, the maximum and minimum end moments for the necessary load combinations. Accordingly, the work proceeds considering only one span loaded at a time. On the line marked " 1st Distribution " are recorded the carry-over moments resulting from the unlocking of the far end of that

* This method is discussed at some length in " Continuity in Concrete Building Frames," a pamphlet published by the Portland Cement Association.

BEAM AND GIRDER BUILDING BY ABBREVIATED MOMENT DISTRIBUTION & 1940 J.C. CODE — **Sheet MJ1**

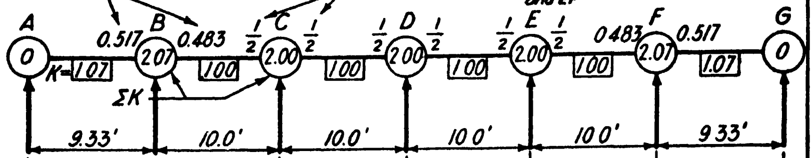
Abbreviated Moment Distribution

L.L. = 125 psf
D.L. = 63 psf

F.E.M Span AB & FG = $-\dfrac{125 \times 9.33^2 \times 12}{8} = -16.33$

F.E.M. Span BC, CD, DE = and EF $-\dfrac{125 \times 10.0^2 \times 12}{12} = -12.5$

Distribution Ratios



	A–B	B–C	C–D	D–E	E–F	F–G
F.E.M.	0 −16 33	−12.5 −12 5	−12 5 −12 5	−12 5 −12 5	−12 5 −12 5	−16.33 0
1st Dist.		−3 13 −3 02	−3 13 −3 13	−3.13 −3 13	−3 02 −3.13	
Σ	−16 33	−15 63 −15 52	−15.63 −15.63	−15 63 −15 63	−15 52 −15 63	−16.33
(a) Load AB		−7 89 +1 97	−0 49	+0 12	−0 03	
(b) " BC		−8 08 −7 76	+1.94	−0 49	+0 12	(See Fig. 18-2b)
(c) " CD		+1 96 −7 82	−7 82	+1 96	−0 49	
(d) " DE		−0 49 +1 96	−7 82	−7 82	+1.96	
(e) " EF		+0 12 −0 49	+1.94	−7 76	−8 08	
(f) " FG		−0 03 +0 12	−0 49	+1 97	−7 89	
(g) Max L.L.	+13 42 −16	49 +11 61 −16	07 +12 39 −16	62 Σ (+)	Lines (a) to (f)	
(h) Min L.L.	−4 31 + 2	08 − 6 10 + 4	05 − 6 03 − 3	88 Σ (−)	Lines (a) to (f)	
(i) D L	+4 67 − 7	26 + 2 56 − 6	06 + 2 98 − 6	42 63/125 Σ	(a) to (f)	
(j) Max L.L.+D.L.	+18 10 −23	77 +14 16 −22	13 −15 37 −23	04 g + i		
(k) Min L.L.+D.L.	+0 36 − 5	18 − 3 54 − 2	01 − 3 05 − 2	54 h + i		

Moments by Coefficients of 1940 J.C. Code

	Left Support	Center 1st Span	1st Int Support	Center 2nd Span	2nd Int. Support	Center 3rd Span	3rd Int. Support
D.L. Coef of WL	−0 040	+ 0.080	− 0 110	+ 0 046	− 0 080	+ 0 046	− 0.080
D.L. Moment	−2 56	+ 5 12	− 7.04	+ 2.94	− 5 12	+ 2 94	− 5.12
Max L L Coef.	−0.040	+ 0.105	− 0.120	+ 0.085	− 0.115	+ 0.085	− 0.115
Max. L.L. Mom	−5.08	+13.33	−15.24	+10.80	−14.61	+10.80	−14.61
Min. L.L. Coef	−0.040	− 0.020	+ 0.015	− 0.045	+ 0 036	− 0 045	+ 0.036
Min L L. Mom.	−5.08	− 2.54	− 1.91	− 5.72	+ 4.57	− 5.72	+ 4.57
D L + Max. L.L.	−7.64	+18.45	−22.23	+13.74	−19.73	+13.74	−19.73
D.L. + Min. L.L.	−7.64	+ 2.58	− 5.13	− 2.78	− 0.55	− 2.78	− 0.55

For L.L. : WL = 125 x 9.20 x 9.20 x 12 = 127 k-in.
For D.L. : WL = 125 x 9.20 x 9.20 x 12 = 64 k-in.

SHEET MJ1

span, all other spans being unloaded. This moment added to the original FEM is a new, unbalanced moment, given in the summation line following; the distributed value on the far side of the support is the result of two cycles of distribution for this span loaded. These results are given on lines *a* to *f*, with the distribution extended to all the other supports.

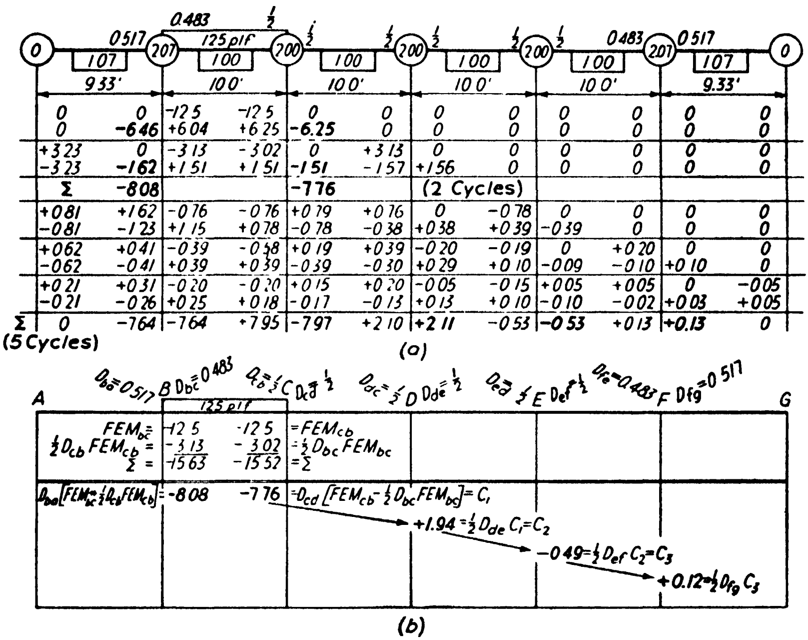


FIG. 18–2

The correctness of this method will be apparent upon inspection of Fig. 18–2, in part *a* of which five cycles of distribution are shown in complete detail for span *BC* loaded, a summation line being inserted after two cycles. A suggestion as to the accuracy of two cycles is thus shown.

In Fig. 18–2*b* the approximate distribution procedure of Computation Sheet MJ1 is repeated with definite designation as to how each figure was obtained in terms of the FEM and the distribution factors.

Note that on account of symmetry the moment values obtained by loading successively *AB*, *BC*, and *CD* are those required for loads on *GF*, *FE*, and *ED*, positions being reversed.

Lines *a* to *f* of Sheet MJ1 give all support moments for each one of the six spans loaded individually. Summing up separately the positive and negative values gives maximum and minimum live load support moments; summing up both positive and negative values and multiplying

by 63/125 (the ratio of dead to live load) evaluates the dead load support moments. The positive moments near mid-span were obtained by deciding which spans must be loaded for a maximum or minimum, summing up the support moments for those conditions only, and deducting the mean of the support moments from the simple beam moments. Finally, the dead and live effects were combined for absolute maximum and minimum values. Comparison with Table X, Sheet TM3, shows how well the values here obtained agree with those of the exact method.

The simplicity of this method of approximate moment distribution is readily apparent, and experiments with problems involving variations in span lengths, moments of inertia, and loading conditions show an excellent agreement with more precise methods — fully as close as the other variables in the problem justify. In the hands of the skilled operator this is probably the quickest way to approximate those critical values ordinarily needed by the designer.

1940 Joint Committee. The 1940 J.C. Code specifies in Appendix 3 certain coefficients of *WL* "for the special case of equal spans and uniformly distributed loads. They are based on the assumption of continuity over the supports with negligible restraints at end and intermediate supports." On Computation Sheet MJ1 the applicable coefficients are recorded, the values of *WL* are computed on the basis of the clear span as called for by the Code, and the maximum and minimum combinations of dead and live load are tabulated for comparison with the results of the other analyses.

General Discussion. Each of the methods just described for continuous slabs has its use and value. The three moment equation is mathematically precise — far more so than the conditions require. Although a good many figures are set down in reaching usable results they are simple to handle since, with a clear mental picture of each step, the computer has control of each operation and quickly spots erratic results. This is one of the older methods and is universally understood. This is helpful when computations must be submitted for approval to various agencies. The advantages of the graphical method are: the small likelihood of serious errors such as might occur with a misplaced decimal point in computations; the fact that all points on the envelope curves can readily be obtained as an aid in cutting and bending reinforcing steel in any unusual problem where the ordinary rules of thumb fail to work; the relative ease with which variable moments of inertia, end haunches, or brackets can be handled; and the very clear mental picture that a few applications of this method engenders. Approximate moment distribution is easy to remember and apply and, ordinarily, obtains results within the required precision. Few figures need be recorded, and those are clearly pictured and controlled. It gives only

critical points on the moment curves and those only approximately. It is one of the best methods for daily use supplemented by recourse to one of the others for checking and more complete delineation. The coefficients of the 1940 J.C. Code are subject to the same limitation as any coefficients, viz., that they can only apply with precision to the particular combination of dead and live load ratios, span length ratios, and other factors for which they were derived. Unless the designer possesses an unusual memory, the tabulation of coefficients must be always available. For the more serious and complicated problems it is well to use two of these methods, checking one against the other.

Shears. A little thought shows that the reaction and shear at the outer supports, especially when torsional restraint is neglected as here, are less than $wL/2$ and at the first interior support are considerably greater than $wL/2$. For this slab shear is not a factor in the design and no analysis will be made. Should critical shear values be desired they can be obtained as explained on page 247.

Strictly, the load transmitted to the first interior row of beams actually is greater than the sum of the two simple beam reactions from the slabs on either side. However, no live load reduction was taken and no allowance was made for restraint of the slab at the spandrels so the load on the first interior row of beams will be taken the same as for the remaining interior rows.

18-4. Typical Girder-Column Bent. The slab problem just completed is representative of a prismatic beam continuous over a series of equally yielding knife-edge supports and without end restraint. Here some methods of attack will be considered for the more complicated problem of a " rigid frame " where the stiffnesses of the columns must be taken into account.* This becomes a complicated matter for any

* Chapter XII outlines the theory of moment distribution and the following summary may be of help to the reader. The moment curve for any beam with continuous ends is made up of three parts:

1. The simple beam moment curve for the given loads and spans.
2. A triangular moment curve from the restraint over the left support.
3. A triangular moment curve from the restraint over the right support.

Items 2 and 3 in turn are each made up of three parts: (*a*) the fixed end moment curve for the given load and span which assumes the tangent to the elastic curve to remain horizontal over each support; (*b*) a change in this end moment to allow for the fact that the stiffness of all the frame to the left of the left support will never balance exactly the moment in the girder and there will be an angle change in the tangent to the elastic curve owing to the stiffness of the left part of the frame; and (*c*) a change in this end moment to allow for the fact that the frame to the right of the right support will not balance exactly the girder moment at that point and any change in the conditions at the right support will be reflected back to cause a change of lesser amount at the left support. Since the major part of the girder end moments consists of the fixed end moment and the modifications for variations in stiffness at the supports is usually considerably smaller, the best accuracy is obtained by starting with the fixed end moments.

but the simplest cases and workable approximations are desirable, especially those where it is not necessary to take into account the effect of members very far removed from the member in question. In this article the arrangement will be the reverse of that used for slabs, the approximate methods being presented first and the more accurate check later.

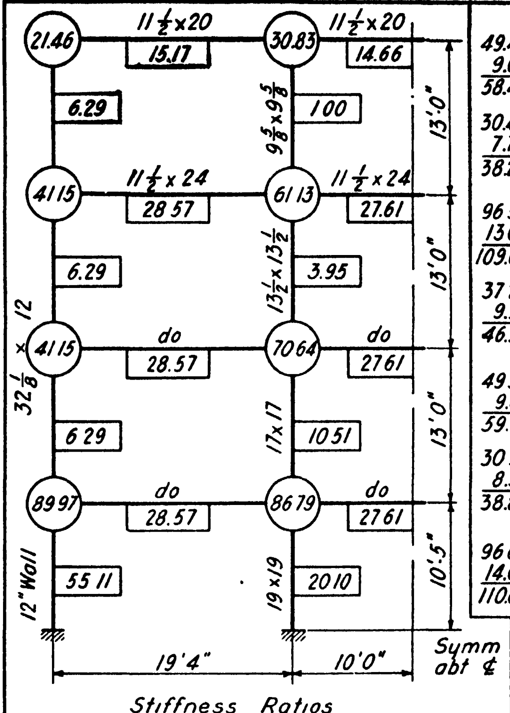
Note that throughout the various solutions in this article the uniform practice is adopted of putting the $K = I/L$ value of each member in a box alongside the member, the ΣK values at each joint in a circle at the joint, and the distribution factors D in general on the diagonal at the end of the member or at the top of each column when moment distribution is used.

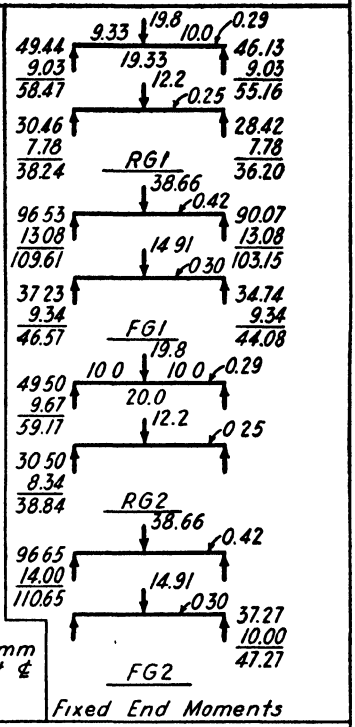
1937 P.C.A. Approximate Moment Distribution. Rarely, if ever, in a framed bent will the stiffnesses of the horizontal and vertical members be identical, so that the simplification, used in the previous article, of taking the relative stiffnesses inversely proportional to the lengths will not be applicable. Instead it will be necessary to run through a rough preliminary design (somewhat along the lines used in Chapter XVII) with arbitrary moment coefficients to establish rough sizes for analyses, or else from experience with similar structures assume the relative stiffnesses directly. The sizes determined in that chapter are used on Computation Sheet RF1, where all center-to-center distances are recorded, the fixed end moments and simple beam moments are computed for quick use in later studies, and the relative stiffnesses are figured, using for columns the gross concrete section plus $(n - 1)$ times the steel area, and for beams the gross concrete area, neglecting reinforcing steel and using the 10 ft of adjoining slab as tee. These relative stiffnesses are recorded in boxes on the frame diagram of the half-width of building which, because of symmetry, is all that is required.

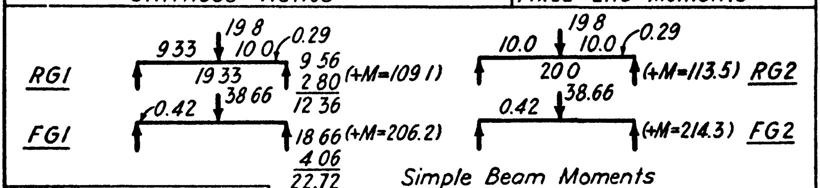
The approximation to be used now is the result of experimenting with a number of possibilities whose object was the summing up of the effects of the entire surrounding structure in the D values which, in this method, are obtained by taking the joint stiffness (ΣK) as the sum of the columns above and below the joint in question, plus the *average* stiffness of beams each side of the joint instead of their sum as in the regular method. There is no theoretical justification for this except that experience and trial show that the results thus obtained are good. The advantage is that any desired beam may be isolated for study without going through the entire frame, as is done farther on in this article. Here the joint values and the relative participations will differ from those of the accurate moment distribution below.

The first applications are on Computation Sheet RF2, on which, in

BEAM AND GIRDER BUILDING P.C.A. MOM. DIST Sheet RF1



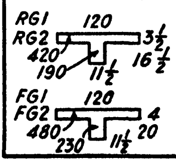




K Values:

Columns

	I	L	$K=\frac{I}{L}$
Roof $(9\frac{5}{8})^4/12 + 9\times4\times0.31\times\overline{2.6}^2 = 715+75 =$	790	13.0	60.77 — 1.00
3RD $(13\frac{1}{2})^4/12 + 9\times4\times0.60\times\overline{4}^2 = 2770+350 =$	3120	13.0	240.0 — 3.95
2ND $(17)^4/12 + 9\times4\times1.00\times\overline{6.1}^2 = 6960+1340 =$	8300	13.0	638.5 — 10.51
1ST $(19)^4/12 + 9\times(10-2)\times0.79\times\overline{7.1}^2 = 10860+2870 =$	12730	10.42	1221.7 — 20.10
Ext. $(32.13\times12^3)/12 + 9\times6\times0.44\times\overline{3.8}^2 = 4630+340 =$	4970	13.0	382.3 — 6.29
Wall $(240\times12^3)/12 + 9\times6\times0.44\times\overline{3.8}^2 = 34560+340 =$	34900	10.42	3349.3 — 55.11



RG1, RG2: $e = 1.75 + \frac{190}{610}\times10 = 4.86$

$I = 420(3\frac{1}{2}^2/12 + 3.11^2) + 190(\frac{16\frac{1}{2}^2}{12} + 6.89^2) = 17820$ — L: 19.33, 20.00; K: 921.9 — 15.17, 891.0 — 14.66

FG1, FG2: $e = 2 + \frac{230}{710}\times12 = 5.89"$

$I = 480(4^2/12 + 5.89^2) + 230(\frac{20^2}{12} + 6.11^2) = 33550$ — L: 19.33, 20.00; K: 1736 — 28.57, 1678 — 27.61

SHEET RF1

the upper left corner, roof girder RG1 is isolated. In the boxes are the K values; in the joint circles, the modified ΣK values; and above these circles, the modified D values. Two cases are considered: (1) live and dead load on this span with dead load only on the adjacent span; and (2) live and dead load on both spans. Since this is an exterior girder there is no span on the left end to consider. The FEM's are set down and the D percentage is computed; after which the carry-over value is recorded and the sum taken to determine the approximate negative end moments in the girder. To make this easy to follow each operation is marked with a reference letter in parentheses to correspond to the following tabulation.

At the left end:

(*a*) FEM with proper sign.

(*b*) D_L percentage of the left end unbalanced moment $\times -1$.

(*c*) D_R percentage of the right end unbalanced moment $\times +\frac{1}{2}$.

(*d*) The algebraic sum is the approximate left end moment in RG1.

At the right end:

(*e*) FEM with proper sign.

(*f*) D_L percentage of left end unbalanced moment $\times +\frac{1}{2}$.

(*g*) D_R percentage of right end unbalanced moment $\times -1$.

(*h*) The algebraic sum is the approximate right end moment in RG1.

Between these computations the positive moment is evaluated thus:

(*k*) Maximum simple beam positive moment from Sheet RF1.

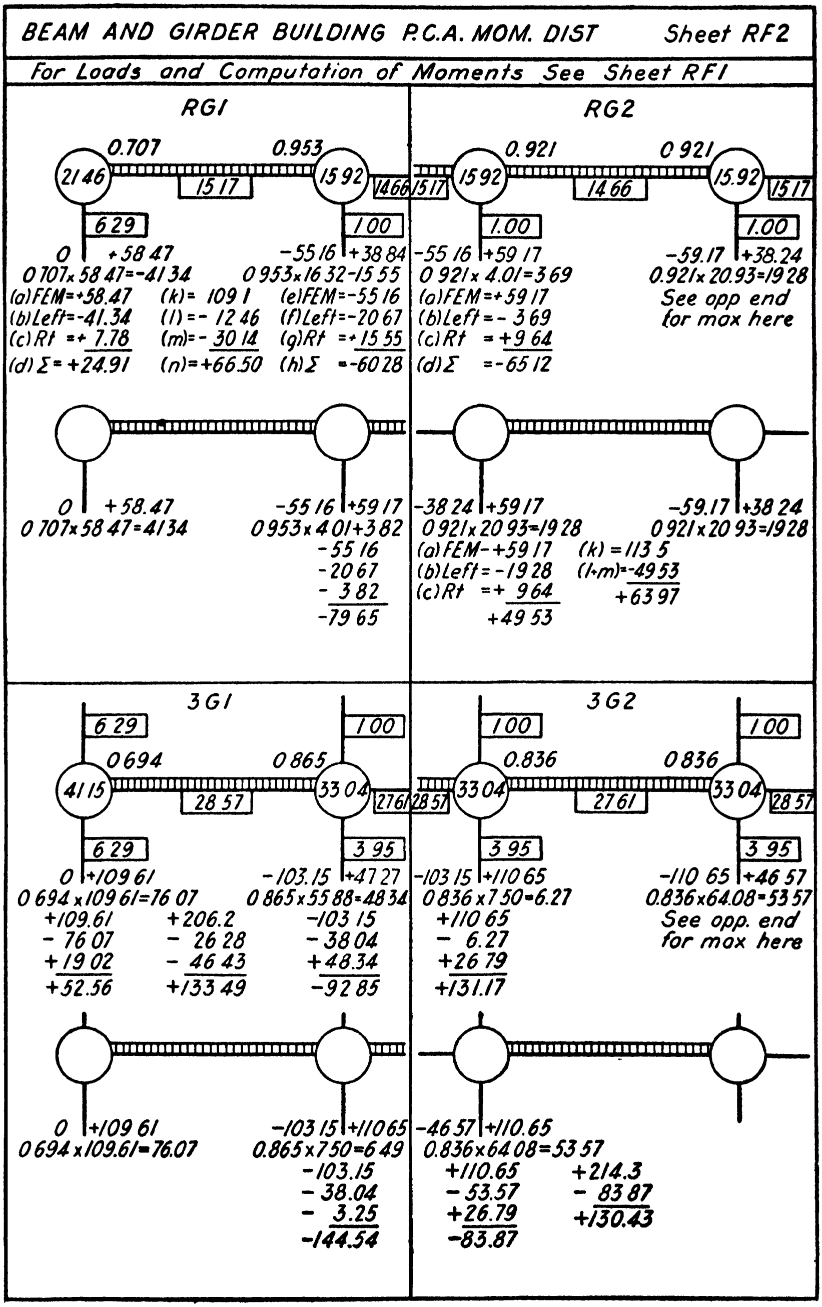
(*l*) One-half of left end moment in RG1 just found.

(*m*) One-half of right end moment in RG1 just found.

(*n*) The algebraic sum is the approximate maximum positive moment in RG1.

This first arrangement of live load produces maximum negative moment at the left end and maximum positive moment near mid-span. For maximum negative moment at the right end the second loading diagram applies. It is unnecessary to repeat the stiffnesses and other values, nor is it necessary to carry through the computations on the left end farther than to get the joint participation for use in determining the carry-over. The procedure e–f–g–h above will give the maximum negative moment at the right end of RG1. We now have all the critical values for the design of RG1.

The same procedure is followed for RG2 in the upper right corner of Sheet RF2 where two loading cases are considered: (1) live and dead load on this span and the one to the left, with dead load only on the right span (this gives maximum negative moment at the left end and, by symmetry, the corresponding maximum on the right end); and (2) live and dead load on this span and dead load only on the two adjoining


BEAM AND GIRDER BUILDING P.C.A. MOM. DIST Sheet RF2
For Loads and Computation of Moments See Sheet RF1
RG1
0.707
0.953
21 46
15 17
15 92
14 66
6 29
1 00
0 +58 47
0 707x 58 47=-41 34
(a)FEM=+58.47
(b)Left=-41.34
(c)Rt =+ 7.78
(d)Σ= +24.91
(k)= 109 1
(l)=- 12 46
(m)=- 30 14
(n)=+66.50
-55 16 +38 84
0 953x 16 32-15 55
(e)FEM=-55 16
(f)Left=-20 67
(g)Rt =+15 55
(h)Σ =-60 28
RG2
0.921
0 921
15 17
15 92
14 66
15.92
15 17
1.00
1.00
-55 16 +59 17
0 921x 4.01=3 69
(a)FEM=+59 17
(b)Left=- 3 69
(c)Rt =+9 64
(d)Σ =-65 12
-59.17 +38.24
0.921x 20.93=19 28
See opp end
for max here
0 +58.47
0 707x58 47=41 34
-55 16 +59 17
0 953x 4 01+3 82
-55 16
-20 67
- 3 82
-79 65
-38 24 +59 17
0 921x20 93=19 28
(a) FEM-+59 17
(b)Left=-19 28
(c)Rt =+ 9 64
+49 53
(k) =113 5
(l+m)=-49 53
+63 97
-59.17 +38 24
0 921x20 93=19 28
3G1
6 29
1 00
0 694
0 865
41 15
28 57
33 04
27 61
6 29
3 95
0 +109 61
0 694 x109 61=76 07
+109.61
- 76 07
+19 02
+52.56
+206.2
- 26 28
- 46 43
+133 49
-103.15 +47 27
0 865x55 88=48 34
-103 15
-38 04
+48.34
-92 85
3G2
1 00
1 00
0.836
0 836
28 57
33 04
27 61
33 04
28 57
3 95
3 95
-103 15 +110 65
0 836 x7 50=6.27
+110 65
- 6.27
+26 79
+131.17
-110 65 +46 57
0.836x64.08=53 57
See opp. end
for max here
0 +109 61
0 694 x109.61=76.07
-103 15 +110 65
0.865x7 50=6 49
-103.15
- 38.04
- 3.25
-144.54
-46 57 +110.65
0.836x64 08=53 57
+110.65
- 53.57
+26.79
-83.87
+214.3
- 83 87
+130.43

Sheet RF2

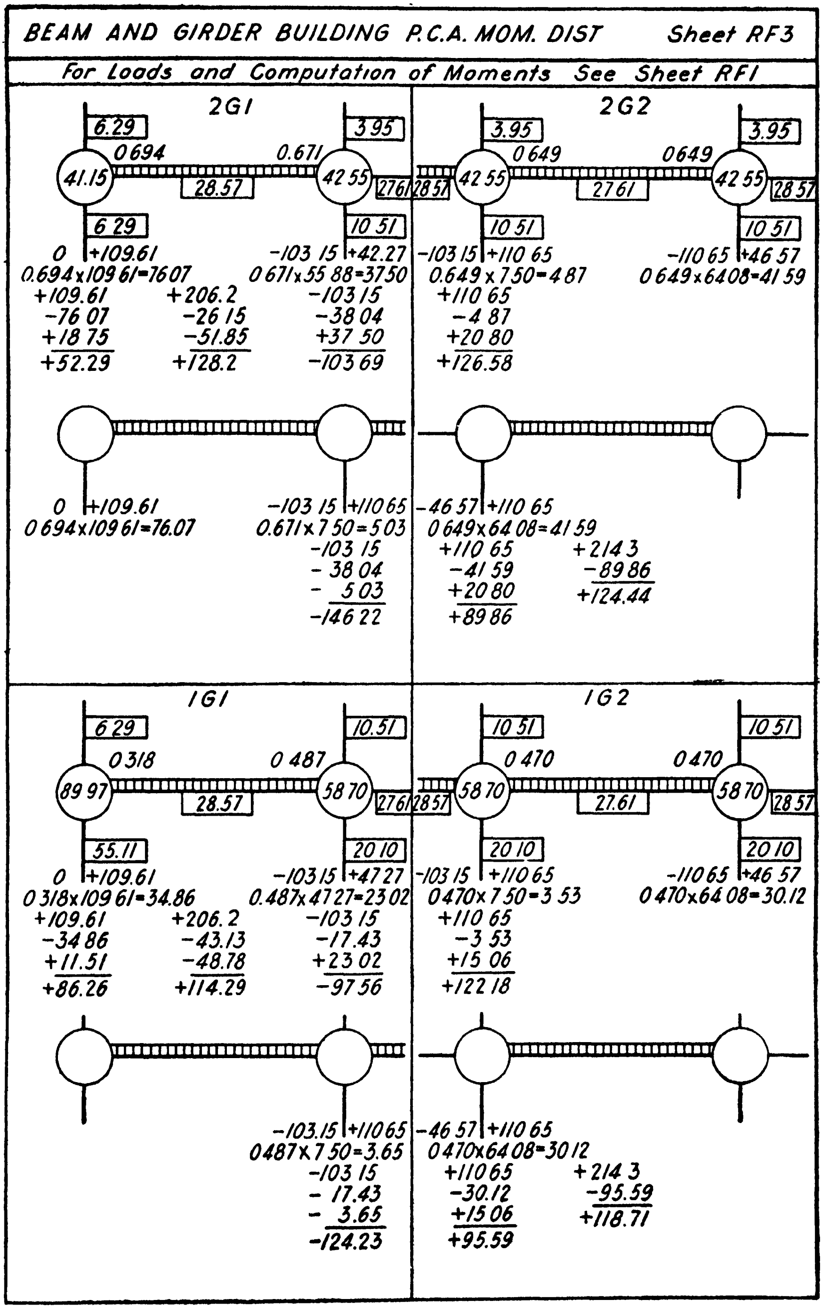
spans (this gives maximum positive moment in this span). Similar computations are made for the girders on the third, second, and first floors. Because of the varying column stiffnesses the moments in the floor girders will vary somewhat even though the loads and spans are similar.

The ease and simplicity of this routine speak for themselves as results are obtained without recourse to the complicated methods usually associated with highly indeterminate structures. Any member may be analyzed with reference only to its own end conditions and without having to solve an entire bent. As far as accuracy and reliability are concerned, comparison with later and more exact analyses is the only test. This is frankly an approximation and one very simple to use, resulting from cut-and-try experiments with a number of frames, where the results obtained were reasonably accurate.

*1940 J.C. Approximate Moment Distribution.** The 1940 J.C. Code in Appendix 2 recommends an abbreviated method of moment distribution " that is applicable to the fixed end moments for any type of loading in the four spans adjacent to the three supports or joints under consideration " and " is approximate to the extent that it includes only two cycles of distribution for fixed end moments at three successive supports or joints." In J.C. Appendix 2 formulas are derived for the solution of the moment distributions, but these are rather complicated for use, and the student who is familiar with Chapter XII and the earlier part of this chapter should have no difficulty in following the computations on Sheets RF4 and RF5, which use the data originally established on RF1. Ordinarily it is unnecessary and undesirable to write out the details of the individual terms. In the previous method each separate girder is solved independently whereas under this approximation several spans are considered at the same time.

It will be noted that at the footings the columns are treated as " free-ended," i.e., the carry-overs from points K and L are taken as 0. This requires comment. The actual condition is intermediate between fully fixed and fully free, but careful studies indicate that in a structure of this sort the footings are considerably closer to a free-ended condition, especially when lateral forces are taken into account (Chapter XIX) and small end restraints would result. The correct solution requires knowledge of the elastic properties of the subsoil and involves considerations too complex for this text. The reader is referred to current books on

* Some excellent suggestions for simplifying the application of this method are contained in " Continuity in Concrete Building Frames," Third Edition, 1941, by The Portland Cement Association, a copy of which should be obtained by every designer of concrete building frames.


BEAM AND GIRDER BUILDING P.C.A. MOM. DIST
Sheet RF3
For Loads and Computation of Moments See Sheet RF1
2G1
6.29
0.694
0.671
3.95
41.15
28.57
42.55
27.61
6.29
10.51
0 +109.61
0.694×109.61=76.07
+109.61
-76.07
+18.75
+52.29
+206.2
-26.15
-51.85
+128.2
-103.15 +42.27
0.671×55.88=37.50
-103.15
-38.04
+37.50
-103.69
2G2
3.95
0.649
0.649
3.95
28.57
42.55
27.61
42.55
28.57
10.51
10.51
-103.15 +110.65
0.649×7.50=4.87
+110.65
-4.87
+20.80
+126.58
-110.65 +46.57
0.649×64.08=41.59
0 +109.61
0.694×109.61=76.07
-103.15 +110.65
0.671×7.50=5.03
-103.15
-38.04
-5.03
-146.22
-46.57 +110.65
0.649×64.08=41.59
+110.65
-41.59
+20.80
+89.86
+214.3
-89.86
+124.44
1G1
6.29
0.318
0.487
10.51
89.97
28.57
58.70
27.61
55.11
20.10
0 +109.61
0.318×109.61=34.86
+109.61
-34.86
+11.51
+86.26
+206.2
-43.13
-48.78
+114.29
-103.15 +47.27
0.487×47.27=23.02
-103.15
-17.43
+23.02
-97.56
1G2
10.51
0.470
0.470
10.51
28.57
58.70
27.61
58.70
28.57
20.10
20.10
-103.15 +110.65
0.470×7.50=3.53
+110.65
-3.53
+15.06
+122.18
-110.65 +46.57
0.470×64.08=30.12
-103.15 +110.65
0.487×7.50=3.65
-103.15
-17.43
-3.65
-124.23
-46.57 +110.65
0.470×64.08=30.12
+110.65
-30.12
+15.06
+95.59
+214.3
-95.59
+118.71

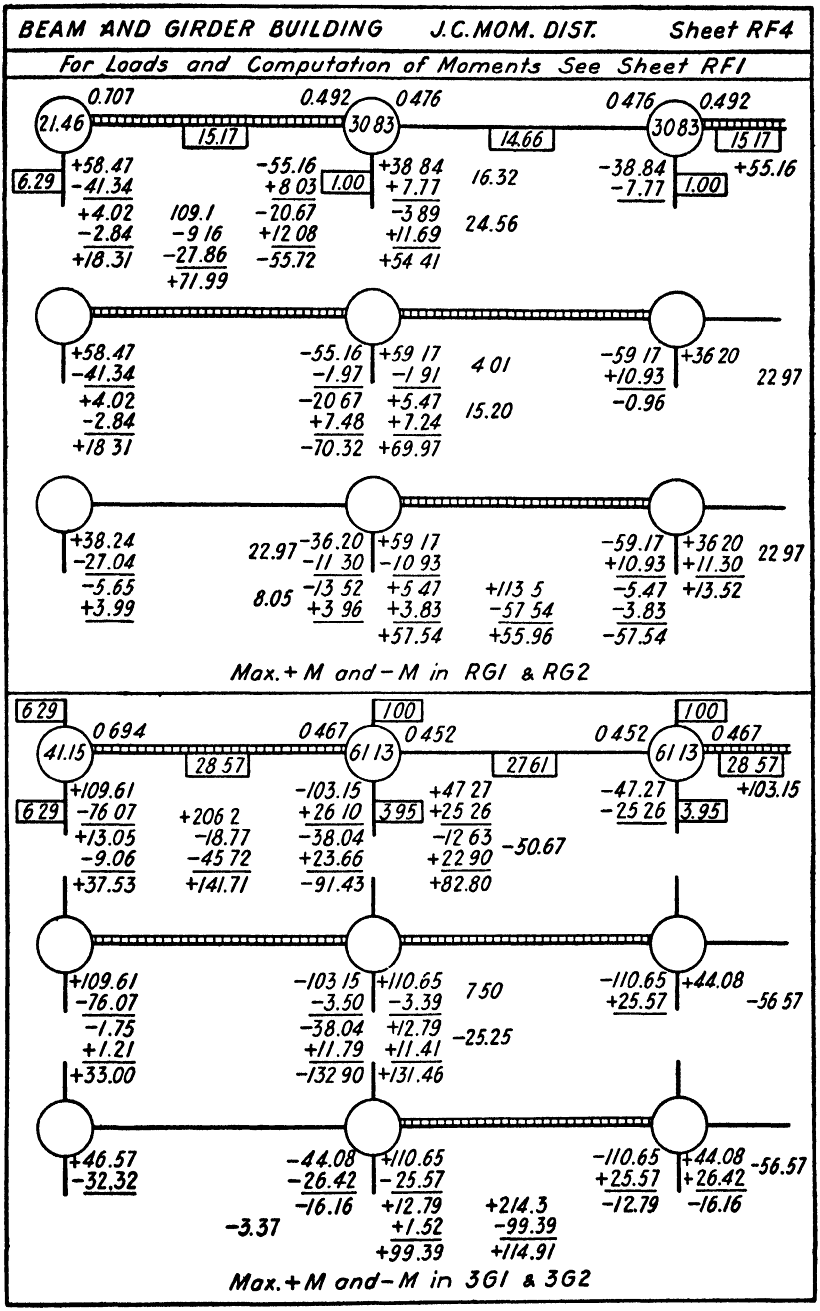
Sheet RF3

soil mechanics and is cautioned against assuming full fixity except in those cases where the moment can be transmitted to the subsoil with absolutely no rotation — an almost impossible condition because even rock is yielding and in some cases the tension on one side developed by the moment exceeds the compression on that side generated by the vertical loading.

Although J.C. Appendix 2 would seem to suggest the possibility of solving such problems by substitution of numerical values in a type formula, it will be found in practice both quicker and less open to careless and algebraic errors to perform directly the steps of the moment distribution as illustrated on the computation sheets. Only a relatively few numbers need be dealt with and the algebraic signs are easily watched. The problem requiring the most thought is the combination of positions of the live load to produce critical values for designing. Advantage is taken of symmetry wherever possible.

This method is slightly more elaborate than the one previously discussed, in that three consecutive supports are considered simultaneously so that, instead of solving each beam independently, the critical values for several spans are obtained at the same time. The amount of numerical work is slightly greater in this method. Both are approximations, the former frankly averaging beam values on a purely empirical basis, the latter equally frankly neglecting entirely all effects beyond the next adjacent support. Each is reasonably precise and far better than the use of arbitrary coefficients or those approximations which neglect entirely the effects of column stiffnesses. Their relative precision depends upon the type of problem and can be judged only by experience and comparison with more exact solutions.

Moment Distribution. For a check on the foregoing, a solution is made on Computation Sheets RF6 to RF8 by a more elaborate method of moment distribution. In a regular structure such as this, it frequently happens that a complete solution of the bent by moment distribution is not over difficult although longer than the approximations. It should be remarked that the solution here given has not been corrected for side-sway or list under unsymmetrical loading, and hence, although accurate for the symmetrical conditions, is not so exact for the unsymmetrical ones. Sheet RF6 has live load applied to alternate panels and staggered on successive floors; RF7 loads the reverse. These two cases cover all possible conditions for maximum positive and for exterior negative moments. For the interior negative moments a different loading pattern is required for each moment desired as the two adjacent panels must be loaded. Since the effect of distant members falls off rapidly a scheme of abbreviation is used: four cycles


BEAM AND GIRDER BUILDING
J.C. MOM. DIST.
Sheet RF4
For Loads and Computation of Moments See Sheet RF1
0.707
21.46
15.17
0.492
0.476
30.83
14.66
0.476
0.492
30.83
15.17
6.29
+58.47
-41.34
+4.02
-2.84
+18.31
109.1
-9.16
-27.86
+71.99
-55.16
+8.03
1.00
-20.67
+12.08
-55.72
+38.84
+7.77
-3.89
+11.69
+54.41
16.32
24.56
-38.84
-7.77
1.00
+55.16
+58.47
-41.34
+4.02
-2.84
+18.31
-55.16
-1.97
-20.67
+7.48
-70.32
+59.17
-1.91
+5.47
+7.24
+69.97
4.01
15.20
-59.17
+10.93
-0.96
+36.20
22.97
+38.24
-27.04
-5.65
+3.99
22.97
8.05
-36.20
-11.30
-13.52
+3.96
+59.17
-10.93
+5.47
+3.83
+57.54
+113.5
-57.54
+55.96
-59.17
+10.93
-5.47
-3.83
-57.54
+36.20
+11.30
+13.52
22.97
Max. + M and - M in RG1 & RG2
6.29
1.00
1.00
0.694
41.15
28.57
0.467
0.452
61.13
27.61
0.452
0.467
61.13
28.57
+109.61
6.29
-76.07
+13.05
-9.06
+37.53
+206.2
-18.77
-45.72
+141.71
-103.15
+26.10
3.95
-38.04
+23.66
-91.43
+47.27
+25.26
-12.63
+22.90
+82.80
-50.67
-47.27
-25.26
3.95
+103.15
+109.61
-76.07
-1.75
+1.21
+33.00
-103.15
-3.50
-38.04
+11.79
-132.90
+110.65
-3.39
+12.79
+11.41
+131.46
7.50
-25.25
-110.65
+25.57
+44.08
-56.57
+46.57
-32.32
-3.37
-44.08
-26.42
-16.16
+110.65
-25.57
+12.79
+1.52
+99.39
+214.3
-99.39
+114.91
-110.65
+25.57
-12.79
+44.08
+26.42
-16.16
-56.57
Max. + M and - M in 3G1 & 3G2

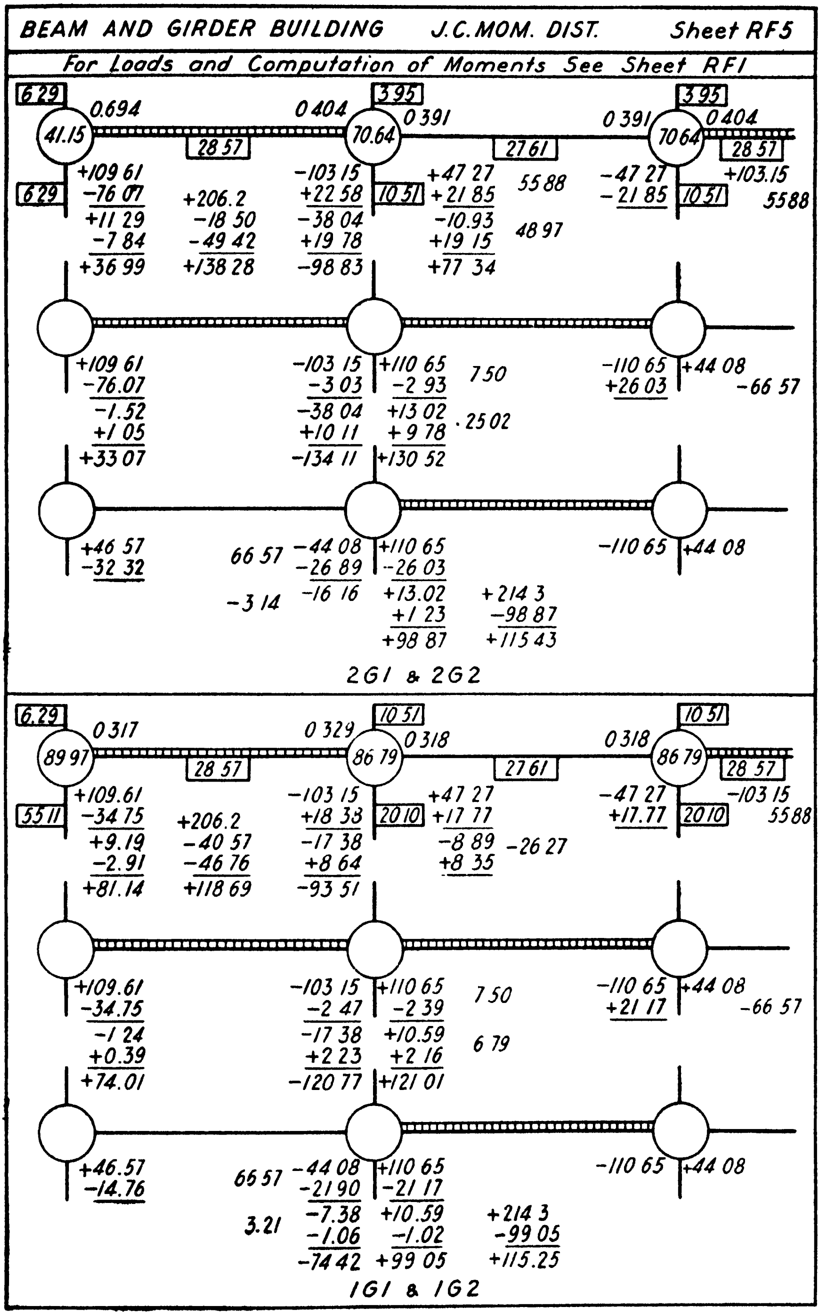
Sheet RF4

carried through for the point in question; three cycles for those immediately adjacent; two cycles for those once more removed; and one cycle for the joints next after these. Beyond those points the effects are too small to consider.

On RF6 the symmetry of the structure is utilized and only one half of the bent is shown. The arrangement should be clear enough. At each joint separate columns are provided for each member, the arrangement, being from left to right, the left girder, column above, column below, and right girder. The line above the frame line for the girders is used to record the percentage participation of each member (D); below, each pair of horizontal lines takes care of one distribution and one carry-over. A little attention is required in making the carry-overs as the girders are carried across horizontally but the columns carry up and down. For convenience in checking, the unbalanced moments are recorded to the right of the other computations and under the tabulations the sums are set down for the final moments in the ends of the members.

The positive moment values computed at mid-span of each girder are the extremes: maximum in alternate spans, minimum in the others, and reversed on the second sheet. They were obtained by deducting the mean of the end moments from the simple beam moments, *all three being taken for the same loading condition.* The method illustrated on sheet RF8 can be applied successively to each of the interior joints to determine its absolute maximum negative moment.

Several points about this method are at once apparent. First, and most important, the entire bent must be considered as a whole; this means not only more figures to handle, but it also requires that the preliminary design of rough sizes to determine approximate stiffnesses must be done completely before starting work. It prevents the designing of a few members roughly, checking them by a more accurate approximate analysis and the changing of the preliminary design, if desirable, for the next group. Here the entire bent must be designed, checked by more accurate analysis, revised, and possibly the whole operation may have to be repeated. Second, the computations are rather extensive, even though not particularly difficult. Here, only one absolute maximum negative moment value is obtained, and at least three more sheets similar to RF8 would be required. Third, distant effects show up in this method. Here, the stiffness of the entire basement wall is used for illustration of the stiffness of the basement column, and its effect is felt in diminishing degree all the way to the roof; the approximate methods dispose of it within the next couple of joints. This method is most likely to be held in reserve for unusual problems


BEAM AND GIRDER BUILDING J.C.MOM. DIST. Sheet RF5
For Loads and Computation of Moments See Sheet RF1
6.29 0.694 0.404 3.95 0.391 0.391 3.95 0.404
41.15 28.57 70.64 27.61 70.64 28.57
+109.61 -103.15 +47.27 55.88 -47.27 +103.15
6.29 -76.07 +206.2 +22.58 10.51 +21.85 -21.85 10.51 55.88
+11.29 -18.50 -38.04 -10.93
-7.84 -49.42 +19.78 +19.15 48.97
+36.99 +138.28 -98.83 +77.34
+109.61 -103.15 +110.65 7.50 -110.65 +44.08
-76.07 -3.03 -2.93 +26.03 -66.57
-1.52 -38.04 +13.02 .25.02
+1.05 +10.11 +9.78
+33.07 -134.11 +130.52
+46.57 66.57 -44.08 +110.65 -110.65 +44.08
-32.32 -26.89 -26.03
-3.14 -16.16 +13.02 +214.3
+1.23 -98.87
+98.87 +115.43
2G1 & 2G2
6.29 0.317 0.329 10.51 0.318 0.318 10.51
89.97 28.57 86.79 27.61 86.79 28.57
+109.61 -103.15 +47.27 -47.27 -103.15
55.11 -34.75 +206.2 +18.38 20.10 +17.77 +17.77 20.10 55.88
+9.19 -40.57 -17.38 -8.89 -26.27
-2.91 -46.76 +8.64 +8.35
+81.14 +118.69 -93.51
+109.61 -103.15 +110.65 7.50 -110.65 +44.08
-34.75 -2.47 -2.39 +21.17 -66.57
-1.24 -17.38 +10.59 6.79
+0.39 +2.23 +2.16
+74.01 -120.77 +121.01
+46.57 66.57 -44.08 +110.65 -110.65 +44.08
-14.76 -21.90 -21.17
3.21 -7.38 +10.59 +214.3
-1.06 -1.02 -99.05
-74.42 +99.05 +115.25
1G1 & 1G2

Sheet RF5

and to check against while one of the approximate methods is used in daily design, possibly checked by another approximation. Finally, all that the designer needs is critical values, and if various methods do not produce identical results, only a few pounds of added reinforcing steel are, in general, necessary to be safe under any and all values obtained.

Slope Deflections. The applications of the method of slope deflections as illustrated on Computation Sheets RF9, RF10, and RF11 should be clear to anyone who has studied Chapter XII. Only a few comments are necessary. The required data are taken from Computation Sheet RF1. On Sheet RF9 the eight equations for joints A to H inclusive are set up. Each line represents the slope deflection equation applied to an individual member. The summation states that the sum of the moments at any one joint must be equal to zero. As soon as the equilibrium at a joint is established, all coefficients are divided so that the coefficient of the first unknown in that particular equation will be unity.

On Computation Sheet RF10 these eight simultaneous equations are solved by the method of detached coefficients. The lines are numbered for ready reference and each operation is described in detail. Although the computations are rather lengthy it should be noted that any additional condition of symmetrical loading can be solved with very little extra effort by carrying a separate column of constant terms. For that reason case 1, corresponding to the loading on Computation Sheet RF6, and case 2, corresponding to sheet RF7, are both carried through and values of $2E\theta_h$ are obtained for each case.

On Computation Sheet RF11 values of all the angles are established by successive substitution in appropriate equations on Sheet RF10. The similarity between case 1 and case 2 disappears at this point. Values have been completely carried through for case 1. The reader can very easily compute corresponding values for case 2.

Several things are immediately noticeable about this method. First, the amount of detailed computation work is very great, especially when it is realized that only one case has been completely evaluated. Second, to obtain even reasonable precision it is necessary to carry the individual coefficients to at least the four decimal places here used and even then perfect agreement is not obtained. Third, experience shows that considerably more than ordinary care is required to carry through such computations without error in the algebraic signs. Fourth, in considering the amount of labor involved note that both of these cases are for symmetrical loading conditions (R assumed equal to 0) and if side-sway is taken into account the computations become increasingly complicated. Fifth, by mere comparison of this solution of one case only with the other methods, it is readily apparent that slope

BEAM AND GIRDER BUILDING - MOMENT DISTRIBUTION — Sheet RF6

For Loads and Computation of Moments See Sheet RF1

Col Above	Col Below	Girder
0	0 293	0 707
a	0	+58 47
	-17 13	-41 34
	-3 57	+4 02
	-0 13	-0 32
	+2 49	+6 18
	-2 54	-6 13
	+0 06	+1 52
	-0 46	-1 12
	-21 28	+21 28

+0 45 +109 1
-10 64
+8 67 -26 69
+71 77
+1 58

Girder	Col Above	Col Below	Girder
0 492		0 032	0 476
-55 16	0	b 0	+38 84
+8 03		+0 52	+7 77
-20 67		-0 54	-3 89
+12 35		+0 80	+11 95
-0 16		-0 01	-5 98
+3 03		+0 20	+2 92
-3 07		-0 11	-1 46
+2 28		+0 15	+2 21
-53 37		+1 01	+52 36

-16 32
-25 10
-6 15
-4 64

+113.5
-52 36
+61 14

Col Above	Col Below	Girder
0 153	0 153	0 694
0 c	0	+46 57
-7 13	-7 13	-32 32
-8 57	-8 39	-15 55
+4 97	+4 97	+22 56
-0 07	-0 47	-0 17
+0 11	+0 11	+0 49
-1 27	-1 10	-3 13
+0 84	+0 94	+3 82
-11 12	-11 17	+21 27

-32 51 +206 2
-10 64
-0 71 -44 08
+151 48
-5 50

Girder	Col Above	Col Below	Girder
0 467	0 016	0 065	0 452
-44 08	0	d 0	+110 65
-31 09	-1 07	-4 33	-30 09
-16 16	+0 26	+1 57	+15 05
-0 34	-0 01	-0 05	-0 33
+11 28	+0 40	+1 55	+0 17
-6 26	0 21	-0 87	-6 06
+0 25	+0 10	+0 38	+3 03
-1 76	-0 06	-0 24	-1 70
-88 16	-0 53	-1 99	+90 72

+66 57
+0 72
+13 40
+3 76

+214 3
-90 72
+123 58

Col Above	Col Below	Girder
0 153	0 153	0 694
0 e	0	+99 61
-16 77	-16 77	-76 07
-3 57	-1 63	+11 29
-0 93	-0 93	-4 23
+2 49	+0 63	+11 15
-2 19	-2 19	-9 94
+0 06	+0 06	+2 71
-0 43	-0 43	-1 96
-21 34	-21 21	+42 56

+6 09 +206 2
-21 28
+14 32 -47 28
+137 64
+2 83

Girder	Col Above	Col Below	Girder
0 404	0 056	0 149	0 391
-103 15	0	f 0	+47 27
+22 58	+3 13	+8 33	+21 85
-38 04	-2 17	-4 03	-10 93
+22 29	+3 09	+8 22	+21 57
-2 12	-0 03	-0 45	-10 79
+5 41	+0 75	+2 00	+5 23
-4 97	-0 44	-0 51	-2 62
+3 45	+0 48	+1 27	+3 33
-94 55	+4 81	+14 83	+74 91

-55 88
-55 17
-13 59
-8 54

+214 3
-74 91
+139 39

Col Above	Col Below	Girder
0 070	0 613	0 317
0 g	0	+46 57
-3 26	-28 55	-14 76
-8 39	0	-10 95
+1 35	+11 86	+6 13
-0 47	0	-1 22
+0 12	+1 04	+0 53
-1 10	0	-1 38
+0 17	+1 52	+0 79
-11 58	-14 13	+21 71

-19 34 +206 2
-10 86
-1 69 -38 03
+157 31
-2 48

Girder	Col Above	Col Below	Girder
0 329	0 121	0 232	0 318
-44 08	0	h 0	+110 65
-21 90	-8 05	-15 44	-21 17
-7 38	+4 17	0	+10 59
-2 43	-0 89	-1 71	-2 35
+3 07	+4 11	0	+1 18
-2 75	-1 01	-1 94	-2 66
+0 27	+1 00	0	+1 33
-0 86	-0 31	-0 60	-0 83
-76 06	-0 98	-19 69	+96 74

+66 57
+7 38
+8 36
+2 60

+214 3
-96 74
+117 56

Symm Abt. ℄

k

l

Max +M Alternate Spans

deflections can seldom be applied directly to an ordinary design problem. The computing labor is prohibitive. It is more a useful tool for checking other approximations.

A.C.I. Design Coefficients. A committee of the A.C.I., A. J. Boase, author-chairman, has prepared tables of design coefficients which permit direct determination of moments in advance of the design of the members. These tables appear in the " Reinforced Concrete Design Handbook " of the A.C.I.

18–5. General Summary. In this chapter practical applications of the methods of continuity have been illustrated and some comparisons made of their relative value. In many cases the methods of continuous *beams* will be satisfactory but in the majority of cases the structures should be analyzed as *rigid frames,* taking into account the stiffnesses of the columns as well as of the beams. For the latter case some approximate method of moment distribution that will permit of dealing with only a few connected members at a time is the easiest to apply, produces results that are much more accurate than the use of arbitrary coefficients, and, although never as precise as the more exact methods, is much quicker to use, is reasonably accurate and perhaps as precise as the numerous variables discussed at the outset of this chapter justify. In every case it is necessary first to arrive at approximate proportions by the use of arbitrary moment coefficients or some other preliminary analysis or, if experience is available, by the outright assumption of relative stiffnesses. It is necessary to analyze for several different loading conditions to obtain maximum values. It is then necessary to check the original design and make such alterations as are required to provide adequate strength at every point. Considerable ingenuity can be used in increasing the strength of a member without materially raising its moment of inertia, since an increase in this latter function would raise the participation of the member in the joint moment, thus requiring still further increase in strength. It does not seem necessary to carry through the detailed computations of the modifications of the original design required by these more exact methods. The reader is referred to Arts. 9–6 and 13–9 on direct stress and bending and to Art. 13–2*a* for the design of a beam with restrained ends. It will be noted that these more exact methods require some modifications of the original design, but in most cases the changes are not greater than can be best cared for by changes in the amount and bending of the reinforcing steel. This chapter has dealt only with vertical loading systems, dead and live, and has purposely left the consideration of horizontal forces to be dealt with in Chapter XIX.

BEAM AND GIRDER BUILDING - MOMENT DISTRIBUTION Sheet RF7

For Loads and Computation of Moments See Sheet RF1

Col. Above	Col Below	Girder
0	0 293	0.707
a	0	+38 24
	-11 20	-27 04
	-8 39	-5 65
	+4 11	+9 93
	-0 30	+1 87
	-0 46	-1 11
	-1 26	-0 88
	+0.63	+1 51
	-16 87	+16.87

-14 04 +1 57 -2 14

+109 1
-8 44
-27 42
+73 24

Girder	Col Above	Col Below	Girder
0 492		0 032	0 476
-36 20	0 b	0	+59 17
-11 30		-0 74	-10 93
-13 52		+0 45	+5 47
+3 74		+0 24	+3 62
+4 97		+0 42	-1 81
-1 76		-0 11	-1 71
-0 56		+0 11	+0 86
-0 20		-0 01	-0 20
-54 81		+0 34	+54 47

+22 97 +7.60 +3 58 +0 41

+113 5
-54 47
+59 03

Col Above	Col Below	Girder
0 153	0 153	0 694
0 c	0	+109 61
-16 77	-16 77	-76 07
-5 60	-3 57	+13 05
-0 59	-0 59	-2 69
+2 06	+1 97	+12 36
-2 51	-2 51	-11 37
-0 23	+0 07	+3 10
-0.45	-0 45	-2 04
-24.09	-21 85	+45 95

+3 88 +16.39 +2 94

+206 2
-22 98
-43 49
+139 73

Girder	Col Above	Col Below	Girder
0 467	0 016	0 065	0.452
-103 15	0 d	0	+47 27
+26 10	+0 89	+3 63	+25 26
-38 04	-0 37	-1 87	-12 63
+24 71	+0 85	+3 44	+23 92
-1 35	+0 12	-0 06	-11 96
+6 19	+0 21	+0 86	+5 99
-5 69	-0 06	-0 36	-3 00
+4 25	+0 15	+0 59	+4 12
-86 98	+1 79	+6 23	+78.97

+55 88 +52 91 -13 25 -9 11

+214.3
-78 97
+135 33

0.153	0 153	0 694
0 e	0	+46 57
-7 13	-7 13	-32 32
-8 39	-3 84	-13 45
+3 93	+3 93	+17 82
-0 30	-0 20	-0 42
+0 14	+0 14	+0 64
-1 26	-0 25	-2 61
+0 63	+0 63	+2 86
-12 38	-6.72	+19 09

+25 68 -0 92 -4 12

+206 2
-9 55
-42 71
+153 94

0 404	0 056	0 149	0 391
-44 08	0 f	0	+110 65
-26 89	-3 73	-9 92	-26 03
-16 16	+1 82	+3 38	+13 02
-0 83	-0 12	-0 31	-0 81
+8 91	+1 72	+1 89	+0 41
-5 22	-0 72	-1 93	-5 06
+0 32	+0 43	+0 37	+2 53
-1 47	-0 20	-0 54	-1 43
-85 42	-0 80	-7 06	+93 28

+66 57 +2 06 +12 93 +3 65

+214 3
-93 28
+121.02

0.070	0 613	0 317
0 g	0	+109 61
-7 67	-67 19	-34 75
-3 57	0	+9.19
-0 39	-3 45	-1 78
+1 97	0	+5 14
-0.50	-4 36	-2 25
+0 07	-0	+0 99
-0 07	-0 65	-0 34
-10.16	-75.65	+85.81

+5 62 +7 11 +1.06

+206 2
-42 91
-45 46
+117 83

0.329	0 121	0 232	0 318
-103 15	0 h	0	+47 27
+18 38	+6 76	+12 96	+17 77
-17 38	-4 96	0	-8 89
+10 27	+3 78	+7 25	+9 93
-0 89	-0 16	0	-4 97
+1 98	+0 73	+1 40	+1 91
-1 13	-0 97	0	-0 96
+1 01	+0 37	+0 71	+0 97
-90.91	+5 55	+22.32	+63.03

+55 88 +31 23 -6 02 -3 06

+214.3
-63 03
+151.27

Symm. Abt. ℄

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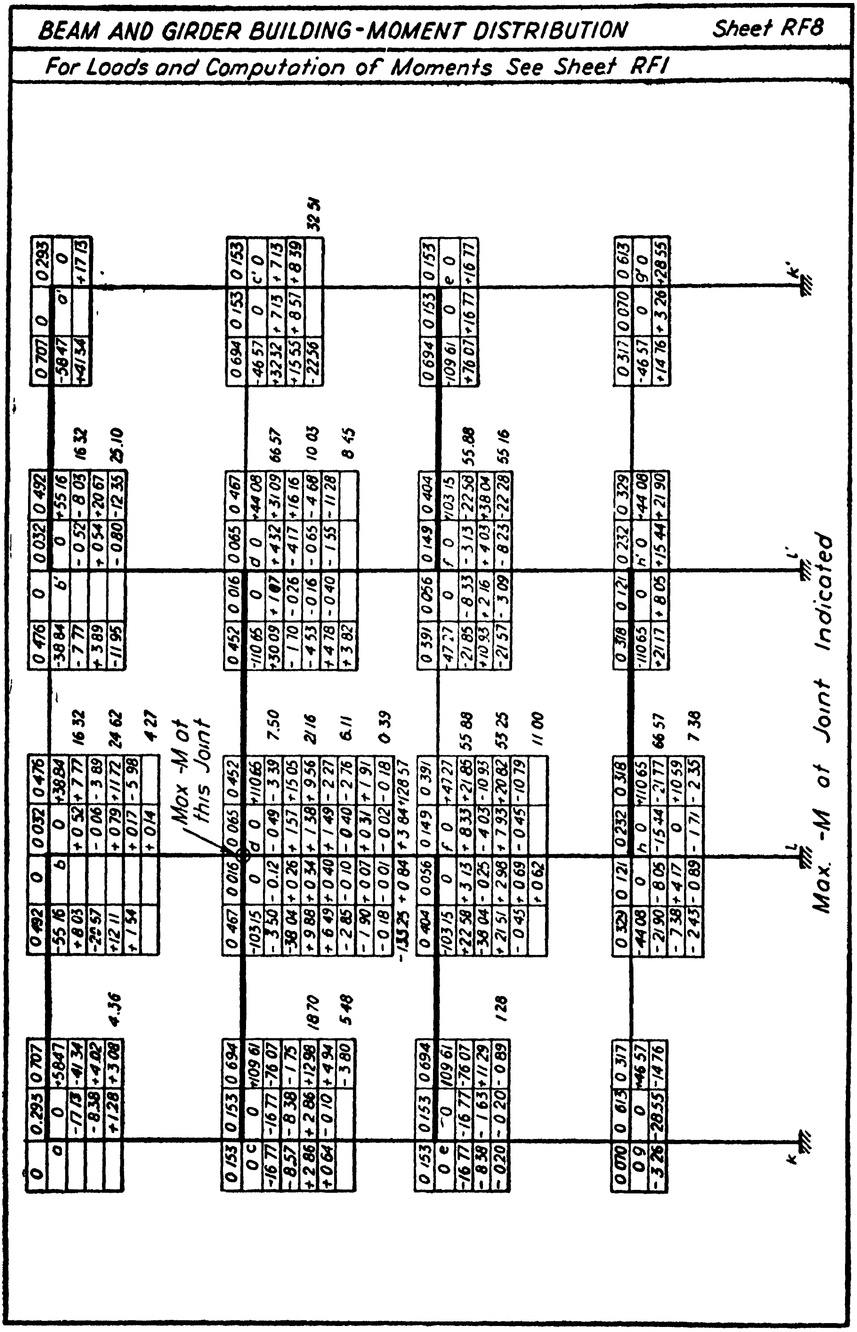
Max. + M Alternate Spans


BEAM AND GIRDER BUILDING-MOMENT DISTRIBUTION
Sheet RF8
For Loads and Computation of Moments See Sheet RF1
Max -M at this Joint
Max. -M at Joint Indicated

SHEET RF8

BEAM AND GIRDER BUILDING - SLOPE DEFLECTIONS — Sheet RF9

For Stiffness Ratios and Fixed End Moments See Sheet RF1

Equations	$2E\theta_a$	$2E\theta_b$	$2E\theta_c$	$2E\theta_d$	$2E\theta_e$	$2E\theta_f$	$2E\theta_g$	$2E\theta_h$	Const Case I	Const Case II
$M_{ab}=2EK_{ab}(2\theta_a+\theta_b)+M$	+30.34	+15.17							+58.47	+38.24
$M_{ac}=2EK_{ac}(2\theta_a+\theta_c-(3R'=0))$	+12.58		+6.29						0	0
(0.0232992) Joint A	+42.92	+15.17	+6.29						+58.47	+38.24
Equation 1	+1.0000	+0.3534	+0.1466						+1.3623	+0.8910
$M_{ba}=2EK_{ab}(2\theta_b+\theta_a)+M$	+15.17	+30.34							−55.16	−36.20
$M_{bb'}=2EK_{bb'}(2\theta_b-\theta_b)+M$		+14.66							+38.84	+59.17
$M_{bd}=2EK_{bd}(2\theta_b+\theta_d-(3R'=0))$		+2.00		+1.00					0	0
(0.0659196) Joint B	+15.17	+47.00		+1.00					−16.32	+22.97
Equation 2	+1.0000	+3.0982		+0.0660					−1.0758	+1.5142
$M_{ca}=2EK_{ca}(2\theta_c+\theta_a-(3R'=0))$	+6.29		+12.58						0	0
$M_{cd}=2EK_{cd}(2\theta_c+\theta_d)+M$			+57.14	+28.57					+46.57	+109.61
$M_{ce}=2EK_{ce}(2\theta_c+\theta_e-(3R''=0))$			+12.58		+6.29				0	0
(0.1589825) Joint C	+6.29		+82.30	+28.57	+6.29				+46.57	+109.61
Equation 3	+1.0000		+13.0843	+4.5421	+1.0000				+7.4038	+17.4261
$M_{dc}=2EK_{dc}(2\theta_d+\theta_c)+M$			+28.57	+57.14					−44.08	−103.15
$M_{db}=2EK_{bd}(2\theta_d+\theta_b-(3R'=0))$		+1.00		+2.00					0	0
$M_{dd'}=2EK_{dd'}(2\theta_d-\theta_d)+M$				+27.61					+110.65	+47.27
$M_{df}=2EK_{df}(2\theta_d+\theta_f-(3R''=0))$				+7.90		+3.95			0	0
Joint D Equation 4		+1.00	+28.57	+94.65		+3.95			+66.57	−55.88
$M_{ec}=2EK_{ec}(2\theta_e+\theta_c-(3R''=0))$			+6.29		+12.58				0	0
$M_{ef}=2EK_{ef}(2\theta_e+\theta_f)+M$					+57.14	+28.57			+109.61	+46.57
$M_{eg}=2EK_{eg}(2\theta_e+\theta_g-(3R'''=0))$					+12.58		+6.29		0	0
(0.1589825) Joint E			+6.29		+82.30	+28.57	+6.29		+109.61	+46.57
Equation 5			+1.0000		+13.0843	+4.5421	+1.0000		+17.4261	+7.4038
$M_{fe}=2EK_{ef}(2\theta_f+\theta_e)+M$					+28.57	+57.14			−103.51	−44.08
$M_{fd}=2EK_{df}(2\theta_f+\theta_d-(3R''=0))$			+3.95			+7.90			0	0
$M_{ff'}=2EK_{ff'}(2\theta_f-\theta_f)+M$						+27.61			+47.27	−110.65
$M_{fh}=2EK_{fh}(2\theta_f+\theta_h-(3R'''=0))$						+21.02		+10.51	0	0
(0.2531645) Joint F			+3.95		+28.57	+113.67		+10.51	−55.88	+66.57
Equation 6			+1.0000		+7.2329	+28.7772		+2.6608	−14.1468	+16.8532
$M_{ge}=2EK_{eg}(2\theta_g+\theta_e-(3R'''=0))$					+6.29		+12.58		0	0
$M_{gh}=2EK_{gh}(2\theta_g+\theta_h)+M$							+57.14	+28.57	+46.57	+109.61
$M_{gk}=2EK_{gk}(2\theta_g-\frac{1}{2}\theta_g-(3R^{IV}=0))$							+82.67		0	0
(0.1589825) Joint G					+6.29		+152.39	+28.57	+46.57	+109.61
Equation 7					+1.0000		+24.2273	+4.5421	+7.4038	+17.4261
$M_{hg}=2EK_{gh}(2\theta_h+\theta_g)+M$							+28.57	+57.14	−44.08	−103.15
$M_{hf}=2EK_{hf}(2\theta_h+\theta_f-(3R'''=0))$						+10.51		+21.02	0	0
$M_{hh'}=2EK_{hh'}(2\theta_h-\theta_h)+M$								+27.61	+110.65	+47.27
$M_{hl}=2EK_{hl}(2\theta_h-\frac{1}{2}\theta_h-(3R^{IV}=0))$								+30.15	0	0
(0.0951475) Joint H						+10.51	+28.57	+135.92	+66.57	−55.88
Equation 8						+1.0000	+2.7184	+12.9324	+6.3340	−5.3168

BEAM AND GIRDER BUILDING - SLOPE DEFLECTIONS — Sheet RF10

No.	Operation	$2E\theta_a$	$2E\theta_b$	$2E\theta_c$	$2E\theta_d$	$2E\theta_e$	$2E\theta_f$	$2E\theta_g$	$2E\theta_h$	Const Case I	Const Case II
1	Eq 1	+1.0000	+0.3534	+0.1466						+1.3623	+0.8910
2	Eq 2	+1.0000	+3.0982		+0.0660					−1.0758	+1.5142
3	Eq 3	+1.0000		+13.0843	+4.5421	+1.0000				+7.4038	+17.4261
4	Eq 4		+1.00	+28.57	+94.65		+3.95			+66.57	−55.88
5	Eq 5			+1.0000		+13.0843	+4.5421	+1.0000		+17.4261	+7.4038
6	Eq 6			+1.0000		+7.2329	+28.7772		+2.6608	−14.1468	+16.8532
7	Eq 7					+1.0000		+24.2273	+4.5421	+7.4038	+17.4261
8	Eq 8						+1.0000	+2.7184	+12.9324	+6.3340	−5.3168
9	Line 1 − Line 2		−2.7448	+0.1466	−0.0660					+2.4381	−0.6232
10	(0.3643253) x Line 9		−1.0000	+0.0534	−0.0240					+0.8883	−0.2270
11	Line 1 − Line 3		+0.3534	−12.9377	−4.5421	−1.0000				−6.0415	−16.5351
12	(2.829654) x Line 11		+1.0000	−36.6092	−12.8526	−2.8297				−17.0954	−46.7886
13	Line 10 + Line 12			−36.5558	−12.8766	−2.8297				−16.2071	−47.0156
14	(0.02735544) x Line 13			−1.0000	−0.3522	−0.0774				−0.4434	−1.2861
15	Line 4 − Line 12			+65.1792	+107.5266	+2.8297	+3.95			+83.6654	−9.0914
16	(0.0153423) x Line 15			+1.0000	+1.6497	+0.0434	+0.0606			+1.2836	−0.1395
17	Line 14 + Line 16				+1.2975	−0.0340	+0.0606			+0.8402	−1.4256
18	(0.7707129) x Line 17				+1.0000	−0.0262	+0.0467			+0.6476	−1.0987
19	Line 5 − Line 16				−1.6497	+13.0409	+4.4815	+1.0000		+16.1425	+7.5433
20	(0.6061708) x Line 19				−1.0000	+7.9050	+2.7166	+0.6062		+9.7851	+4.5725
21	Line 14 + Line 6				−0.3522	+7.1555	+28.7772		+2.6608	−14.5902	+15.5671
22	(2.8392958) x Line 21				−1.0000	+20.3166	+81.7070		+7.5548	−41.4259	+44.1996
23	Line 18 + Line 20					+7.8788	+2.7633	+0.6062		+10.4327	+3.4738
24	(0.12692288) x Line 23					+1.0000	+0.3507	+0.0769		+1.3241	+0.4409
25	Line 22 − Line 20					+12.4116	+78.9904	−0.6062	+7.5548	−51.2110	−39.6271
26	(0.08056979) x Line 25					+1.0000	+6.3642	−0.0488	+0.6087	−4.1261	−3.1927
27	Line 24 − Line 26						−6.0135	+0.1257	−0.6087	+5.4502	+3.6336
28	(0.16629248) x Line 27						−1.0000	+0.0209	−0.1012	+0.9063	+0.6042
29	Line 7 − Line 24						−0.3507	+24.1504	+4.5421	+6.0797	+16.9852
30	(2.8514399) x Line 29						−1.0000	+68.8634	+12.9515	+17.3359	+48.4323
31	Line 28 − Line 30							−68.8425	−13.0527	−16.4296	+47.8281
32	(0.01452591) x Line 31							−1.0000	−0.1896	−0.2387	−0.6947
33	Line 30 + Line 8							+71.5818	+25.8839	+23.6699	+43.1155
34	(0.013970031) x Line 33							+1.0000	+0.3616	+0.3307	+0.6023
35	Line 32 + Line 34								+0.1720	+0.0920	−0.0924
36	(5.81395) x Line 35								+1.0000	+0.5349	−0.5372

BEAM AND GIRDER BUILDING - SLOPE DEFLECTIONS Sheet RF11

Evaluate Case I:

From Line	36	$2E\theta_h = -0.5349$
" "	34	$2E\theta_g = -0.3307 + 0.1934 = -0.1373$
" "	30	$2E\theta_f = +17.3359 - 6.9278 - 9.4549 = +0.9532$
" "	24	$2E\theta_e = -1.3241 + 0.0106 - 0.3343 = -1.6478$
" "	20	$2E\theta_d = +9.7851 - 0.0832 + 2.5895 - 13.0259 = -0.7345$
" "	16	$2E\theta_c = -1.2836 - 0.0578 + 0.0715 + 1.2117 = -0.0582$
" "	10	$2E\theta_b = +0.8883 + 0.0176 - 0.0031 = +0.9028$
" "	3	$2E\theta_a = -7.4038 + 1.6478 + 3.3362 + 0.7484 = -1.6759$

$$\left.\begin{aligned} M_{ab} &= 15.17(-2\times1.6759+0.9028)+58.47 &&= +21.32\\ M_{ac} &= 6.29(-2\times1.6759-0.0582) &&= -21.44 \end{aligned}\right\} -0.12$$

$$\left.\begin{aligned} M_{ba} &= 15.17(2\times0.9028-1.6759)-55.16 &&= -53.19\\ M_{bb'} &= 14.66(0.9028)+38.84 &&= +52.08\\ M_{bd} &= 1.00(2\times0.9028-0.7345) &&= +1.07 \end{aligned}\right\} -0.04$$

$$\left.\begin{aligned} M_{ca} &= 6.29(-2\times0.0582-1.6759) &&= -11.27\\ M_{cd} &= 28.57(-2\times0.0582-0.7345)+46.57 &&= +22.26\\ M_{ce} &= 6.29(-2\times0.0582-1.6478) &&= -11.10 \end{aligned}\right\} -0.11$$

$$\left.\begin{aligned} M_{dc} &= 28.57(-2\times0.7345-0.0582)-44.08 &&= -87.71\\ M_{db} &= 1.00(-2\times0.7345+0.9028) &&= -0.57\\ M_{dd'} &= 27.61(-0.7345)+110.65 &&= +90.37\\ M_{df} &= 3.95(-2\times0.7345+0.9532) &&= -2.04 \end{aligned}\right\} +0.05$$

$$\left.\begin{aligned} M_{ec} &= 6.29(-2\times1.6478-0.0582) &&= -21.10\\ M_{ef} &= 28.57(-2\times1.6478+0.9532)+109.61 &&= +42.69\\ M_{eg} &= 6.29(-2\times1.6478-0.1373 &&= -21.59 \end{aligned}\right\} \pm 0$$

$$\left.\begin{aligned} M_{fe} &= 28.57(2\times0.9532-1.6478)-103.15 &&= -95.76\\ M_{fd} &= 3.95(2\times0.9532) &&= +7.53\\ M_{ff'} &= 27.61(0.9532)+47.27 &&= +73.59\\ M_{fh} &= 10.51(2\times0.9532-0.5349) &&= +14.41 \end{aligned}\right\} -0.23$$

$$\left.\begin{aligned} M_{ge} &= 6.29(-2\times0.1373-1.6478) &&= -12.09\\ M_{gh} &= 28.57(-2\times0.1373-0.5349)+46.57 &&= +23.44\\ M_{gk} &= 55.11(-\tfrac{3}{2}\times0.1373) &&= -11.35 \end{aligned}\right\} \pm 0$$

$$\left.\begin{aligned} M_{hg} &= 28.57(-2\times0.5349-0.1373)-44.08 &&= -78.57\\ M_{hf} &= 10.51(-2\times0.5349+0.9532) &&\doteq -1.23\\ M_{hh} &= 27.61(-0.5349)+110.65 &&= +95.88\\ M_{he} &= 20.10(-\tfrac{3}{2}\times0.5349) &&= -16.13 \end{aligned}\right\} -0.05$$

CHAPTER XIX

LATERAL LOADS ON FRAMES

19–1. Lateral forces on structures are common, chiefly wind load on all outdoor structures, tractive and centrifugal forces on bridges, and earthquake forces.

Wind Load. The effect of wind on structures is very complicated and it is common for design purposes to attempt a reasonable approximation of that effect by assuming a lateral uniform load varying usually from 15 to 30 psf of exposed surface. For large and important structures more exact determinations of wind effect may be necessary and the reader is referred for information to the American Civil Engineers' Handbook, Merriman-Wiggin (John Wiley & Sons, Inc., New York, 1930) and to a series of articles which appeared in the *Engineering News-Record* in 1934 and 1935, "Aerodynamics and the Civil Engineer," by W. Watters Pagon.

Earthquake Forces. The vibrations which radiate out through the ground from the origin of earthquake movement, a region of fresh earth crust adjustment, are exceedingly complex. This is obvious when it is remembered that the Tokyo earthquake of 1923 was caused by risings and fallings of several hundred feet in an area of sea bottom of some 360 square miles lying 70 miles to the south: the San Francisco earthquake of 1906 was caused by a slip on the San Andreas fault extending for nearly 300 miles. These vibrations and the response of structures subjected thereto defy mathematical treatment but our knowledge of such effects has advanced sufficiently within recent years so that it is a relatively simple matter to design structures which shall be secure against earthquake damage. The usual assumption is that safety is secured if the structure is designed to sustain on any horizontal plane a lateral force equal to a certain fraction of the weight of structure and load lying above that plane. The usual fraction assumed is one-tenth, which is equivalent to assuming that the lateral acceleration induced by an earthquake may approach one-tenth that of gravity. This static load method of designing for earthquakes has been used with success in Tokyo, which is a region of extremely severe earthquake effects. The rule must be applied with due caution, however, as there are situations where it will not suffice, chiefly with tall, slender structures, tall buildings, chimneys,

and elevated water tanks. A more elaborate method of design is indicated for these structures which takes account of their elastic and vibratory characteristics and their response to ground movement. The whole problem is summarized well in a paper, "Earthquakes and Structures," by L. M. Hoskins and J. D. Galloway, in Transactions of the A.S.C.E., 1940, and in the papers attached thereto in discussion. There are many references in the paper and the discussion which will give the student a competent grasp of the subject. An excellent brief treatment is that by H. D. Dewell, "Earthquake-Resistant Construction," *Engineering News-Record,* Vol. 100, 1928.

19–2. Frames under Lateral Loads. Portal and Cantilever Methods. The stress analysis of simple frames under lateral loading has already been illustrated (solution by slope deflection in Ex. 12–5 and by moment distribution in Ex. 12–8). As the number of joints increases the difficulty of solving the simultaneous equations of the slope deflection set-up increases tremendously and the problem soon becomes too unwieldy for practical office procedures. The increase of labor for a moment distribution solution with increased complexity of frame is very much less than for slope deflection and even for a tall building of many bays width the labor is by no means prohibitive. Usually, however, an approximate solution would be used in practice, just as in the design of steel structures. These approximate methods assume a definite elastic distortion of the loaded frame which will differ, often markedly, from the actual distortion. For ordinary structures the strength provided to meet the requirements set forth by the approximate methods will be adequate to the actual demands but for high and for irregular frames this may not hold true. Such structures require for their design an experienced engineer with a specialized knowledge in this field.

Portal Method. The most used method of analysis for lateral loads on building frames is to assume that the beams of any floor with the supporting columns below form a series of independent portals with points of inflection at mid-lengths of members. Each interior column, accordingly, acts as the leeward column of one portal and the windward column of the adjoining portal, the first carrying compression and the second tension. It is common to assume that all portals are equally loaded or, in other words, to assume that the shear in an exterior column is one-half that in an interior column. If the bays are of equal length the resultant direct stress in interior columns will be zero and the wall columns will carry all the direct stress; if the bays are unequal the column direct stress will be alternately tension and compression across the frame. Some designers desire to keep the direct stress at zero in the interior columns and accomplish this by making the shear in any

column proportional to the combined length of beams supported.by the column. So-called exact analysis shows, however, that this alternation of direct stress may occur.

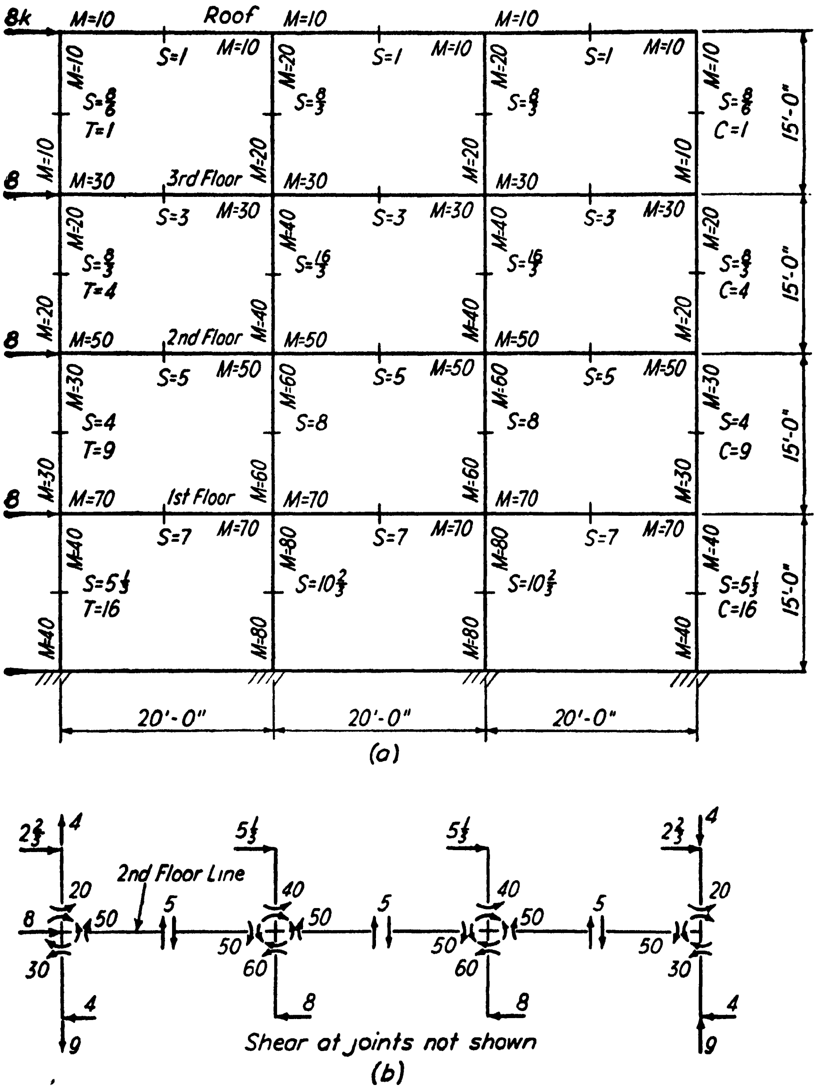


FIG. 19-1

Example 19-1. Compute the stresses due to wind load in the frame shown in Fig. 19-1, using the portal method.

Discussion. The record of moments, shears, and direct stresses on the

diagram was made directly without a setting down of the various operations. The student can check the values easily by noting the order of procedure.

By observing the assumptions relative to points of contraflexure the shears S in the columns were written throughout. The moment at the end of any member equals the shear in the member multiplied by the half-length; accordingly the moments (M) at top and bottom of each column were next recorded. The student should study carefully part b of the figure to get a clear picture of the directions of shears and of the moments at the several joints, and note that at any joint the combined beam moments equal the combined column moments. This observation made it possible to write next the beam end moments: the shear (S) in any beam equals the moment divided by the half-length. From the beam shears the column direct stress (T = tension, C = compression) was obtained to complete the computation. It was not thought necessary to record the direct stress in beams.

As a check to this work apply the condition $\Sigma M = 0$ to the free body consisting of the frame lying above the lowest line of inflection points.

Cantilever Method. Frames are frequently analyzed for lateral load stresses on the assumption that they act as vertical cantilever beams with the intensity of direct stress in the columns varying directly with the distance of the column from the centroid of the column areas forming the bent. Here also the assumption is made that there is a point of inflection at the mid-point of each beam and each column.

Example 19-2. Compute the wind stresses in the bent of Ex. 19-1 by the cantilever method. See Figs. 19-1 and 19-2.

Discussion. Considering the free body lying above the inflection points of any story we may write the equation $\Sigma M = 0$; taking V as the direct stress in an interior column, $3V$ as the direct stress in an exterior column, $M = 20V + 60\ (3V) = 200V$. This assumes columns of equal area. Solution of this relationship made it possible to write the column direct stresses throughout the frame. From the column direct stresses the beam shears were next found. Any beam end moment equals the shear times the half-length. Next the column end moments for the top story were written, followed by the column shears. Similarly the next lower story values followed, and so on.

The student will note several relationships here and in the portal method solution which are useful in checking The experienced designer often computes these stresses directly in a table instead of working on a sketch, his work being facilitated by these relationships.

19-3. Bowman's Method. In 1930 Professor H. L. Bowman suggested an approximate method based on exact analyses of a considerable number of steel frames.* Since his deductions were made with regard to the relative stiffness of the various parts, the rules may also be applied to reinforced concrete structures.

In Bowman's method the following assumptions are made:

(1) The points of contraflexure in exterior beams are taken as 0.55

* Sutherland and Bowman, Structural Theory, John Wiley & Sons, Inc., 1930.

of their length from their outer ends. The points of contraflexure are taken at the center of interior beams except (*a*) in the center bay where the number of bays is odd and (*b*) in the two bays next to the center where the number of bays is even. In these two cases (*a* and *b*) the inflection points are located as required by the conditions of symmetry

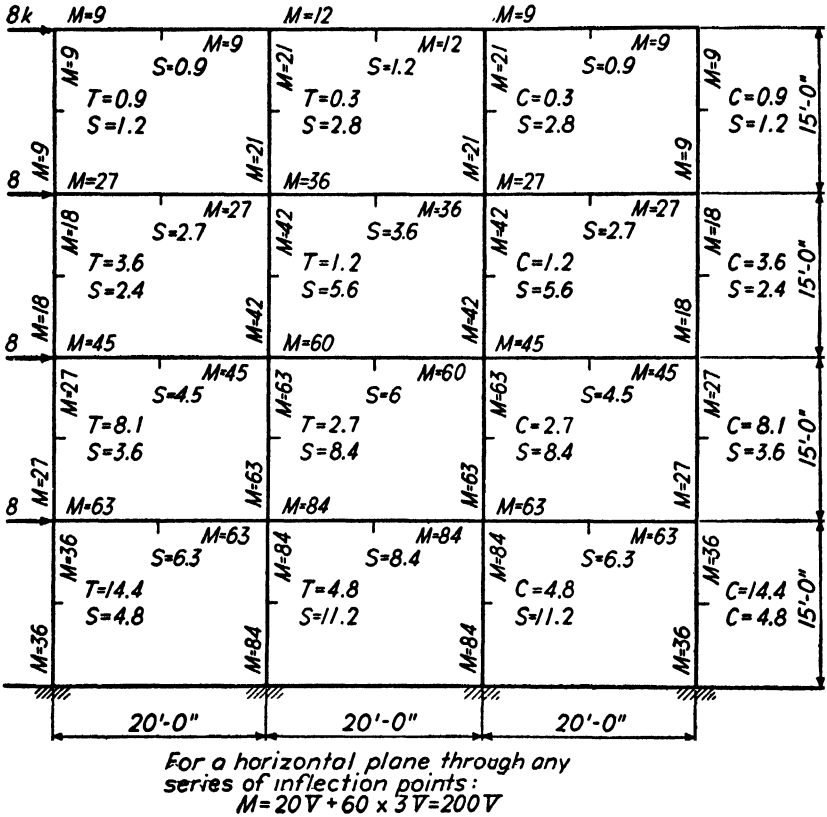


FIG. 19-2

and equilibrium; for example, if the bent is symmetrical a point of inflection must exist at the mid-point of the center beam.

(2) The points of contraflexure in columns are taken at the following fractional parts of their lengths, measured from the bottom: in all bottom-story columns, 0.60; in all top-story columns, bents of two or more stories, 0.35; in bents of three or more stories in columns of the story next to the top, 0.40; in bents of four or more stories, in columns of the story second from the top story, 0.45; in bents of five or more stories in columns not above specified, 0.50.

(3) The shear is divided among the columns of any story as follows.

Bottom story: A fractional part of the shear equal to (number of bays − ½) ÷ (number of columns) is divided among the columns in proportion to their moments of inertia, or divided equally if the columns have not yet been proportioned. The remaining part of the shear is divided among the bays directly as (moment of inertia of beam above bay ÷

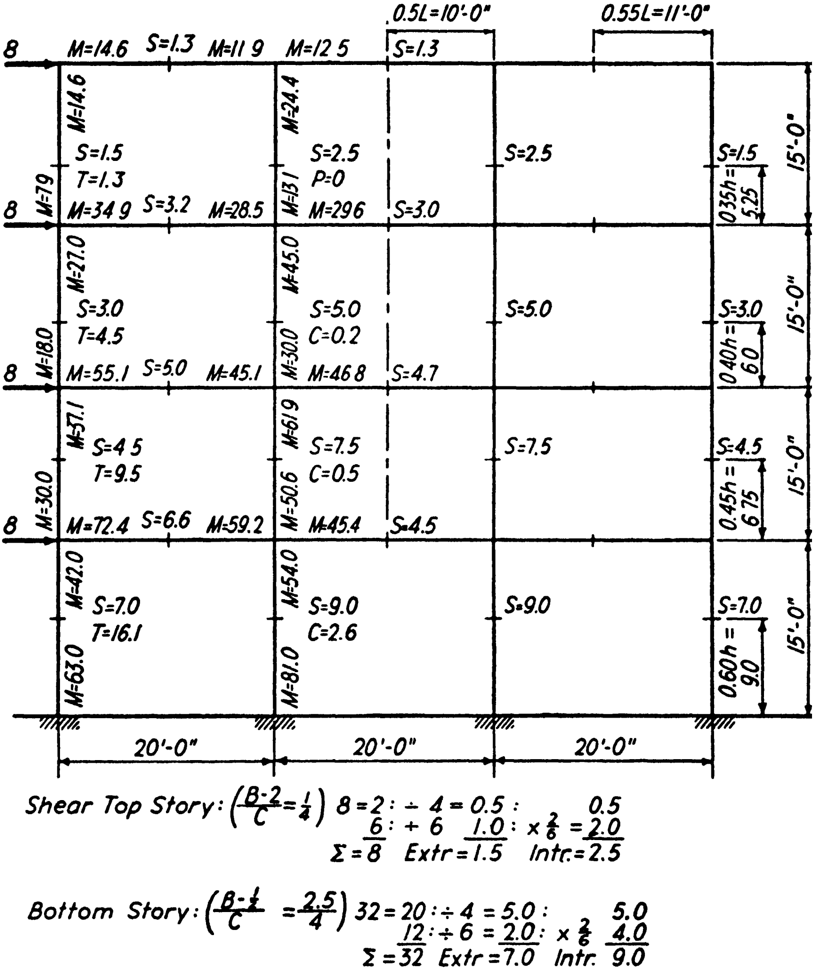


Fig. 19-3

length of bay) when the frame has been proportioned, inversely as their widths when the frame has not been proportioned; the shear in a bay is divided equally between the columns adjacent to the bay.

All stories above the bottom story: The division is the same as just

described except that the first fractional element is (number of bays − 2) ÷ number of columns.

Example 19-3. Compute the wind stresses in the bent of Ex. 19-1, using Bowman's method. See Figs. 19-1 and 19-3.

Discussion. First the points of inflection were located and then the shears distributed to the columns throughout the bent. The necessary computations appear on the figure.

Discussion. The order of procedure was: column shears, column end moments (unlike top and bottom), exterior beam moments at exterior columns, exterior beam moments at interior columns ($\frac{9}{11}$ of above), interior beam moments (from joint summation), beam shears, column direct stresses.

The student should check by applying $\Sigma M = 0$ to the free body above the bottom-story inflection points.

19-4. Frames under Lateral Loads. Method of Moment Distribution. The extension of the moment distribution procedure of Ex. 12-8 (page 222) to frames of two and more stories is cumbersome as it requires the setting up of simultaneous equations,* but a direct solution by moment distribution is possible, suggested by Prof. Clyde T. Morris† which is somewhat laborious but by no means prohibitively so. In order to master this procedure it is essential that the student visualize clearly the physical stages represented by the several steps of the process.

The first step is to allow the frame to lurch sideways under the action of the given loads, the joints being fixed without rotation. Each column will deflect with point of inflection at mid-length and will carry a portion of the shear in the story in proportion to I/L^3. (The basic slope deflection equation for this situation becomes $M = -6EKR = -6EID/L^2$, where D is the column deflection. Hence $V = 2M/L = -12EID/L^3$. For any story E and D are constant.) In moment distribution solution this step is represented by writing on the sketch of the frame, or in tabular form, end moments for all columns equal to the shear in the column (determined as indicated) multiplied by the half-length.

The second step is the unlocking of each joint of the frame in turn and the recording of the distribution of moments resulting, no further side movement being permitted. The third step is the recording of the carry-over moments accompanying this unlocking.

For equilibrium under the given lateral loading the end moments for the columns in any story must total in amount the product of the story shear times the story height. After the third step this condition will not be met; the total will be too small. Accordingly a correction must be made, the addition to each column moment of its share of the de-

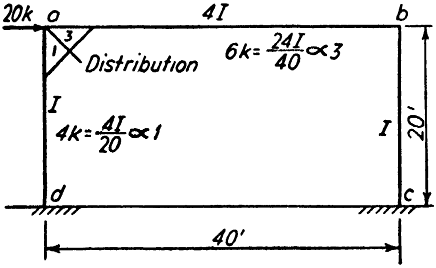
* Cross and Morgan, Continuous Frames of Reinforced Concrete, John Wiley & Sons, Inc., 1932, pp. 227–228.

† In Trans., A.S.C.E., Vol. 96, 1932, pp. 66–68.

ficiency, in proportion to its I/L. Physically this is allowing the frame to move laterally into a state of equilibrium, all joints being fixed against rotation.

The fifth and sixth steps correspond to the second and third, distributing the unbalanced moments and recording the carry-overs. This results again in a state of imaginary restraint against side sway, the column moments in a story being again less than the shear times the story height. Another correction is made; again the joints are unlocked one at a time; and so the process continues until a satisfactory precision is attained.

The convergence of results by this method is apt to be slow and, although the moments found after perhaps four rounds, should be sufficiently accurate, they may not be so close to final result as the much quicker solution by Bowman's method. Several suggestions have been made for hastening convergence* but the student will do well to master the general procedure before attempting these.



Section Ratio	ab 0.75	ad 0.25	da
FEM	+0	+100	+100
Dist	−75	−25	0
C.O	0	0	−13
Corr'n	−	+19	+19
Dist	−14	−5	0
C.O	0	0	−2
Corr'n	−	+4	+3
Dist	−3	−1	0
Σ	−92	+92	+107

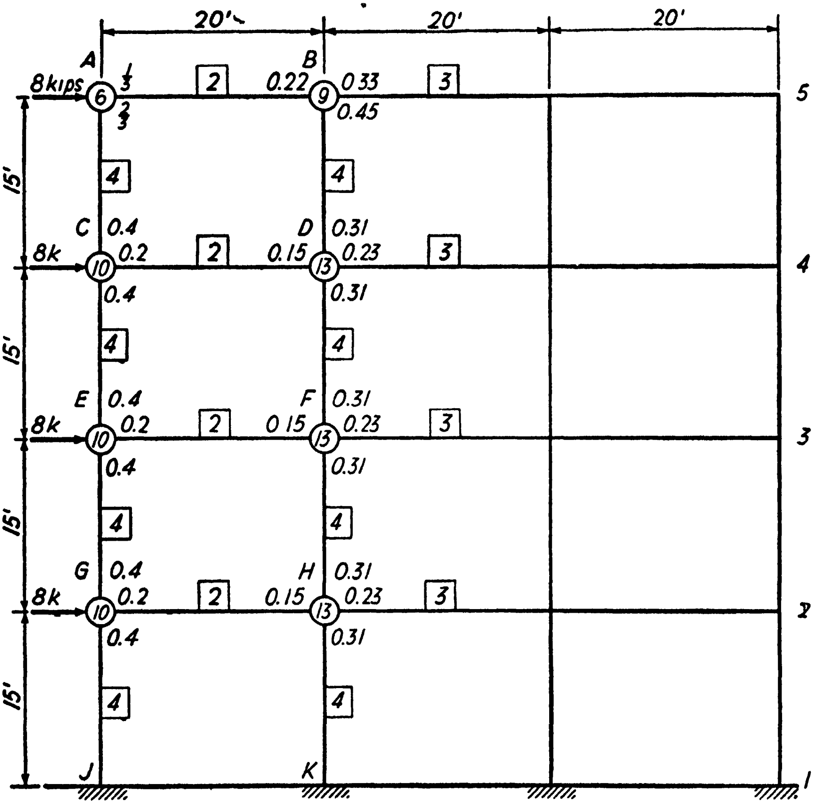
FIG. 19–4

Example 19–4. (Same as Ex. 12–8, page 222.) Compute the joint moments in this bent by Morris' variation of Cross's moment distribution method.

Discussion. As in Ex. 12–8, advantage was here taken of symmetry and the consequent stiffening of the horizontal member, with a resulting 1 : 3 distribution of moment between column and girder at their intersection. Since the two columns are alike, each carries one-half of the shear. If the bent is allowed to deflect under the 20-kip load with the girder assumed as infinitely stiff (consequent translation of joints a and b without rotation) points of inflection are at mid-height of columns and the column end moments equal the shear times the half-length, $10 \times 10 = +100$ k-ft, acting clockwise on the joints. After distribution and carry-over the total of the column end moments is +324 k-ft, less than the shear times story height, 400 k-ft, indicating the presence of a restraining force offsetting in part the given load. This restraint was considered to be removed, permitting free lateral movement, again without rotation of joints a and b. This brought an increase of column end moment, a total of 76 k-ft, one-half to each column. A second distribu-

* For example, by Professor Cross on pp. 228–230, Continuous Frames of Reinforced Concrete.

tion with carry-over brought the total column moment +386 k-ft, requiring a correction of +7 k-ft per column. On unlocking the joints and distributing, the column end moments totaled +398 k-ft, which was considered satisfactory within the degree of precision used. A final distribution terminated the operation.



M-(top story):
$8 \times 15 = 120$ k-ft or 60 k-ft for $\frac{1}{2}$
or 15 k-ft per joint

	Exterior Beams	Interior Beams	Columns
$CK = \frac{CI}{L}$	$4 \times 1 = 4$	$6 \times 1 = 6$	$4 \times 2 = 8$
Relative K=	2	3	4

FIG. 19-5

Example 19-5. Compute the stresses caused by the lateral loads on the four-story frame shown, using moment distribution. (This is the frame which has already been analyzed by the portal method, Ex. 19-1; by the cantilever method, Ex. 19-2; and by Bowman's method, Ex. 19-3.)

Discussion. The application of moment distribution here illustrated follows the suggestions of Professor Clyde T. Morris (see reference above):

1. Calculate the moments in the columns due to the lateral forces, considering the joints fixed against rotation, but free to deflect laterally. The sum of the moments at the top and bottom of all the columns of a story is equal to the shear in the story multiplied by the story height, and, as the

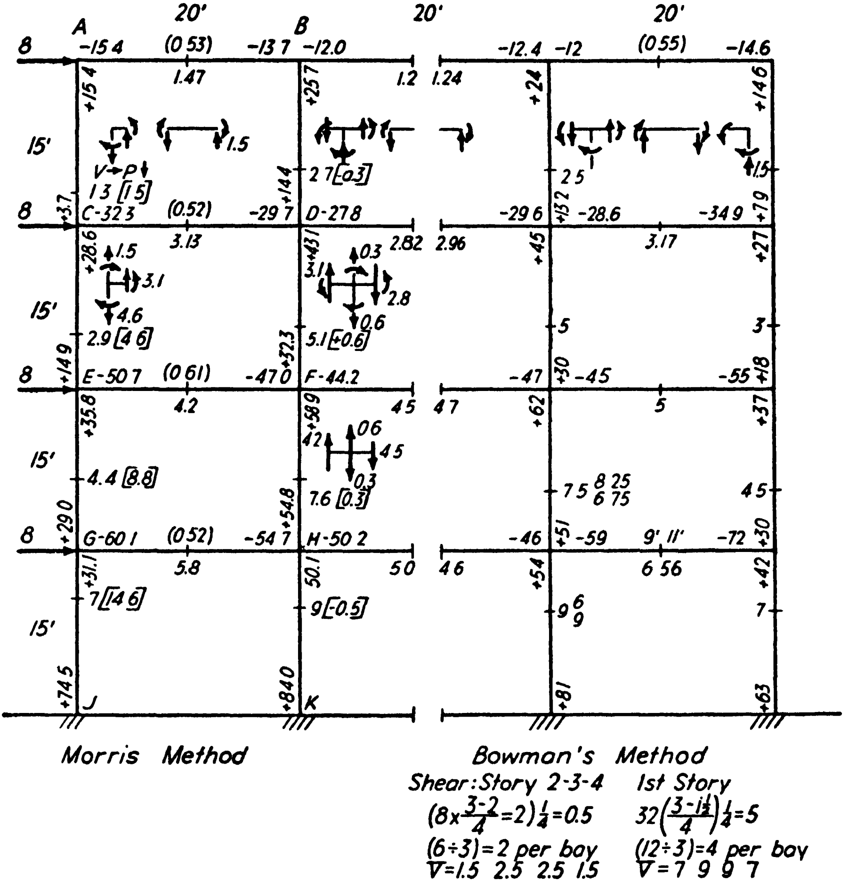


FIG. 19-6

deflections of the columns in a story due to the lateral forces are equal, the column moments and shears are proportional to the I/L^2 values of the columns. (See computations on pages 434–435.)

2. Distribute the moments at the joints, considering them free to rotate but not changing their location.

3. Carry over the distribution moments using a carry-over factor of ½.

4. Balance the column moments in each story by making their sum equal to the shear in the story times the story height.

Repeat steps 2-3-4 for as many cycles as may be necessary.

MOMENT DISTRIBUTION FOR EX 19-5

Line No.	Operation	AB		AC		BA		BB		BD		CA		CD		CE	
		1/3		2/3		0.22		0.33		0.45		2/5		1/5		2/5	
1	Orig. M	0		+15.0		0		0		+15.0		+15.0		0		+30.0	
2	Dist.	-5.0		-10.0		-3.3		-5.0		-6.7		-18.0		-9.0		-18.0	
3	Carry Over	-1.7	-6.7	-9.0	-4.0	-2.5	-5.8	0	-5.0	-7.0	+1.3	-5.0	-8.0	-3.4	-12.4	-15.0	-3.0
4	Correct			+18.3						+18.3		+18.3				+32.0	
5	Dist	-2.5		-5.1		-1.9		-2.9		-4.0		-10.8		-5.4		-10.7	
6	C O	-0.9	-10.1	-5.4	+3.8	-1.3	-9.0	0	-7.9	-4.7	+10.9	-2.5	-3.0	-2.3	-20.1	-8.8	+9.5
7	C			+11.0						+11.0		+11.0				+19.9	
8	D.	-1.6		-3.1		-1.1		-1.6		-2.3		-6.9		-3.5		-6.9	
9	C O.	-0.6	-12.3	-3.5	+8.2	-0.8	-10.9	0	-9.5	-2.9	+16.7	-1.6	-0.5	-1.4	-25.0	-5.4	+17.1
10	C.			+6.8						+6.8		+6.8				+12.2	
11	D.	-0.9		-1.8		-0.7		-1.0		-1.4		-4.2		-2.2		-4.2	
12	C O.	-0.3	-13.5	-2.1	+11.1	-0.4	-12.0	0	-10.5	-1.8	+20.3	-0.9	+1.2	-0.9	-28.1	-3.2	+21.9
13	C.			+4.1						+4.2		+4.1				+7.3	
14	D.	-0.6		-1.1		-0.4		-0.7		-0.9		-2.6		-1.3		-2.5	
15	C O.	-0.2	-14.3	-1.3	+12.8	-0.3	-12.7	0	-11.2	-1.1	+22.5	-0.5	+2.2	-0.5	-29.9	-1.8	+24.9
16	C			+2.6						+2.6		+2.6				+4.2	
17	D	-0.3		-0.8		-0.3		-0.4		-0.5		-1.6		-0.8		-1.6	
18	C O	-0.2	-14.8	-0.8	+13.8	-0.2	-13.2	0	-11.6	-0.7	+23.9	-0.4	+2.8	-0.3	-31.0	-1.1	+26.4
19	C.			+1.6						+1.6		+1.5				+2.6	
20	D.	-0.2		-0.4		-0.2		-0.2		-0.3		-0.9		-0.5		-0.9	
21	C O.	-0.1	-15.1	-0.5	+14.5	-0.1	-13.5	0	-11.8	-0.4	+24.8	-0.2	+3.2	-0.2	-31.7	-0.6	+27.5
22	C.			+0.9						+0.9		+0.9				+1.4	
23	D.	-0.1		-0.2		-0.1		-0.1		-0.2		-0.5		-0.3		-0.5	
24	C O	-0.1	-15.3	-0.3	+14.9	-0.1	-13.7	0	-11.9	-0.3	+25.2	-0.1	+3.5	-0.1	-32.1	-0.4	+28.0
25	C.			+0.6						+0.6		+0.5				+0.9	
26	D	-0.1		-0.1		0		-0.1		-0.1		-0.3		-0.2		-0.3	
27	Result	-15.4		+15.4		-13.7		-12.0		+25.7		+3.7		-32.3		+28.6	

Operation	DC		DB		DD		DF		EC		EF		EG		Line No.
	0.15		0.31		0.23		0.31		0.40		0.20		0.40		
Orig. M	0		+15.0		0		+30.0		+30.0		0		+45.0		1
Dist.	-6.7		-14.0		-10.3		-14.0		-30.0		-15.0		-30.0		2
Carry Over	-4.5	-11.2	-3.4	-2.4	0	-10.3	-11.6	+4.4	-9.0	-9.0	-5.7	-20.7	-21.0	-6.0	3
Correct			+18.2				+32.0		+31.9				+47.9		4
Dist	-4.6		-9.5		-7.1		-9.5		-17.7		-8.8		-17.6		5
C O	-2.7	-18.5	-2.0	+4.3	0	-17.4	-7.6	+19.3	-5.3	-0.1	-3.6	-33.1	-10.6	+13.7	6
C			+11.0				+19.9		+19.9				+26.5		7
D.	-2.8		-5.8		-4.2		-5.8		-10.8		-5.3		-10.8		8
C O.	-1.8	-23.1	-1.2	+8.3	0	-21.6	-4.5	+28.9	-3.5	+5.5	-2.2	-40.6	-5.6	+23.8	9
C.			+6.8				+12.2		+12.2				+14.9		10
D.	-1.7		-3.6		-2.6		-3.6		-6.3		-3.2		-6.3		11
C O.	-1.1	-25.9	-0.7	+10.8	0	-24.2	-2.7	+34.8	-2.1	+9.3	-1.3	-45.1	-3.0	+29.4	12
C.			+4.2				+7.3		+7.3				+8.3		13
D.	-1.0		-2.2		-1.6		-2.2		-3.7		-1.8		-3.7		14
C O.	-0.7	-27.6	-0.5	+12.3	0	-25.8	-1.5	+38.4	-1.2	+11.7	-0.7	-47.6	-1.5	+32.5	15
C			+2.6				+4.2		+4.3				+4.5		16
D	-0.6		-1.3		-0.9		-1.3		-2.2		-1.1		-2.1		17
C O	-0.4	-28.6	-0.3	+13.3	0	-26.7	-0.9	+40.4	-0.8	+13.0	-0.4	-49.1	-0.9	+34.0	18
C.			+1.5				+2.6		+2.6				+2.6		19
D.	-0.4		-0.8		-0.6		-0.7		-1.2		-0.7		-1.2		20
C O.	-0.2	-29.2	-0.1	+13.9	0	-27.3	-0.5	+41.8	-0.4	+14.0	-0.2	-50.0	-0.5	+34.9	21
C.			+0.9				+1.4		+1.4				+1.5		22
D.	-0.2		-0.5		-0.3		-0.5		-0.7		-0.4		-0.7		23
C O	-0.2	-29.6	-0.1	+14.2	0	-27.6	-0.3	+42.4	-0.3	+14.4	-0.1	-50.5	-0.3	+35.4	24
C.			+0.5				+0.9		+0.9				+0.9		25
D	-0.1		-0.3		-0.2		-0.2		-0.4		-0.2		-0.5		26
Result	-29.7		+14.4		-27.8		+43.1		+14.9		-50.7		+35.8		27

Continued on following page

First Correction (Line 4)

4th Story	3rd Story	2nd Story	1st Story
AC=-4.0	CE=-3.0	EG=-6.0	GJ=+18.0
BD=+1.3	DF=+4.4	FH=+5.5	HK=+27.5
CA=-8.0	EC=-9.0	GE=-12.0	J=+39.0
DB=-2.4	FD=-0.2	HF=+0.9	K=+43.7
-13.1 } 60	-7.8 } 120	-11.6 } 180	+128.2 } 240
Corr. +73.1	+127.8	+191.6	+111.8
Each +18.3	+31.95	+47.9	+27.95

Second Correction (Line 7)

4th	3rd	2nd	1st
+3.8	+9.5	+13.7	+24.8
+10.9	+19.3	+29.9	+38.8
-3.0	-0.1	+5.9	+56.3
+4.3	+11.8	+24.5	+63.3
+16.0	+40.5	+74.0	+183.2
+44.0	+79.5	+106.0	+56.8
+11.0	+19.9	+26.5	+14.2

Third Correction (Line 10)

4th	3rd	2nd	1st
+8.2	+17.1	+23.8	+27.9
+16.7	+28.9	+43.1	+44.4
-0.5	+5.5	+15.8	+64.9
+8.3	+19.8	+37.9	+73.2
+32.7	+71.3	+120.6	+210.4
+27.3	+48.7	+59.4	+29.6
+6.8	+12.2	+14.9	+7.4

Fourth Correction (Line 13)

4th	3rd	2nd	1st
+11.1	+21.9	+29.4	+29.4
+20.3	+34.8	+50.4	+47.2
+1.2	+9.3	+21.6	+69.3
+10.8	+24.9	+45.5	+78.3
+43.4	+90.9	+146.9	+224.2
+16.6	+29.1	+33.1	+15.8
+4.15	+7.3	+8.3	+3.95

MOMENT DISTRIBUTION FOR EX 19-5

Line No.	Operation	FE		FD		FF		FH		GE		GH		GJ		HG		HF		HH		HK		J		K		Line No.
		0 15		0 31		0 23		0 31		0 40		0 20		0 40		0 15		0 31		0 23		0 31		1		1		
1	Orig M	0		+300		0		+450		+450		0		+600		0		+450		0		+60.0		+600		+600		1
2	Dist	-113		-232		-173		-232		-420		-210		-420		-158		-325		-242		-325		0		0		2
3	Carry Over	-75	-188	-70	-02	0	-173	-163	+55	-150	-120	-79	-289	0	+180	-105	-263	-116	+09	0	-242	0	+275	-210	+390	-163	+437	3
4	Correct			+319				+479		+479				+280				+479				+280		+279		+279		4
5	Dist	-73		-152		-113		-152		-212		-106		-212		-81		-167		-123		-167		0		0		5
6	C O.	-44	-305	-47	+118	0	-286	-83	+299	-88	+59	-40	-435	0	+248	-53	-397	-76	+245	0	-365	0	+388	-106	+563	-83	+633	6
7	C.			+199				+265		+265				+142				+265				+142		+142		+142		7
8	D.	-44		-90		-66		-90		-112		-56		-111		-42		-86		-64		-86		0		0		8
9	C O	-27	-376	-29	+198	0	-352	-43	+431	-54	+158	-21	-512	0	+279	-28	-467	-45	+379	0	-429	0	+444	-56	+649	-43	+732	9
10	C			+122				+149		+149				+74				+149				+74		+74		+74		10
11	D	-26		-53		-40		-53		-59		-30		-59		-23		-46		-35		-46		0		0		11
12	C O	-16	-418	-18	+249	0	-392	-23	+504	-32	+216	-12	-554	0	+294	-1[illegible]	-505	-27	+45.5	0	-464	0	+472	-30	+693	-23	+783	12
13	C			+73				+83		+83				+40				+83				+40		+39		+39		13
14	D.	-15		-31		-23		-30		-31		-16		-32		-12		-25		-19		-25		0		0		14
15	C O	-09	-442	-11	+280	0	-415	-12	+545	-18	+250	-06	-576	0	+302	-08	-525	-15	+498	0	-483	0	+48.7	-16	+716	-12	+810	15
16	C			+43				+45		+46				+21				+46				+21		+21		+21		16
17	D.	-08		-17		-13		-18		-17		-09		-17		-06		-14		-10		-14		0		0		17
18	C O	-06	-456	-07	+299	0	-428	-07	+565	-11	+268	-03	-588	0	+306	-05	-536	-09	+521	0	493	0	+494	-09	+728	-07	+824	18
19	C.			+26				+26		+27				+12				+27				+12		+12		+12		19
20	D.	-05		-10		-07		-10		-10		-05		-10		-04		-08		-05		-08		0		0		20
21	C O	-03	-464	-04	+311	0	-435	-04	+577	-06	+279	-02	-595	0	+308	-02	-542	-05	+535	0	-498	0	+498	-05	+735	-04	+83.2	21
22	C			+14				+15		+15				+07				+15				+07		+07		+07		22
23	D.	-02		-06		-04		-06		-06		-05		-05		-02		-05		-03		-05		0		0		23
24	C O	-02	-468	-02	+317	0	-439	-03	+583	-04	+284	-01	-599	0	+310	-02	-546	-03	+542	0	-501	0	+500	-03	+739	-03	+836	24
25	C			+09				+09		+09				+04				+09				+04		+04		+04		25
26	D.	-02		-03		-03		-03		-03		-02		-03		-01		-03		-01		-03		0		0		26
27	Result	-470		+323		-442		+589		+290		-601		+311		-547		+548		-502		+501		+743		+840		27

Continued from previous page

Fifth Correction (Line 16)

4th	3rd	2nd	1st
+128	+249	+325	+302
+22.5	+384	+54.5	+487
+2.2	+11.7	+250	+716
+123	+280	+498	+810
+498	+103.0	+1618	+2315
+102	+170	+182	+85
+2.6	+425	+455	+21

Sixth Correction (Line 19)

4th	3rd	2nd	1st
+138	+264	+340	+306
+239	+404	+56.5	+494
+28	+130	+268	+728
+133	+299	+521	+824
+538	+1097	+1694	+235.2
+6.2	+103	+106	+48
+155	+26	+265	+12

Seventh Correction (Line 22)

4th	3rd	2nd	1st
+145	+275	+349	+308
+248	+418	+577	+498
+3.2	+140	+279	+735
+139	+311	+535	+832
+564	+1144	+1740	+2373
+36	+56	+60	+27
+0.9	+14	+1.5	+0.7

Eighth Correction (Line 25)

4th	3rd	2nd	1st
+149	+28.0	+354	+310
+252	+424	+583	+50.0
+35	+144	+28.4	+739
+142	+317	+542	+836
+578	+1165	+176.3	+238.5
+22	+35	+37	+1.5
+0.55	+09	+0.9	+04

For illustration nine rounds of distribution were carried through, all possible for the degree of precision employed. The results are, accordingly, more nearly the true values than those given by the avowedly approximate methods, and are consistent with the actual elastic distortion of the structure. For the sake of easy comparison these final values are placed on the left half of the frame in Fig. 19–6 and the Bowman values on the right. Evidently the latter are sufficiently in agreement for practical design.

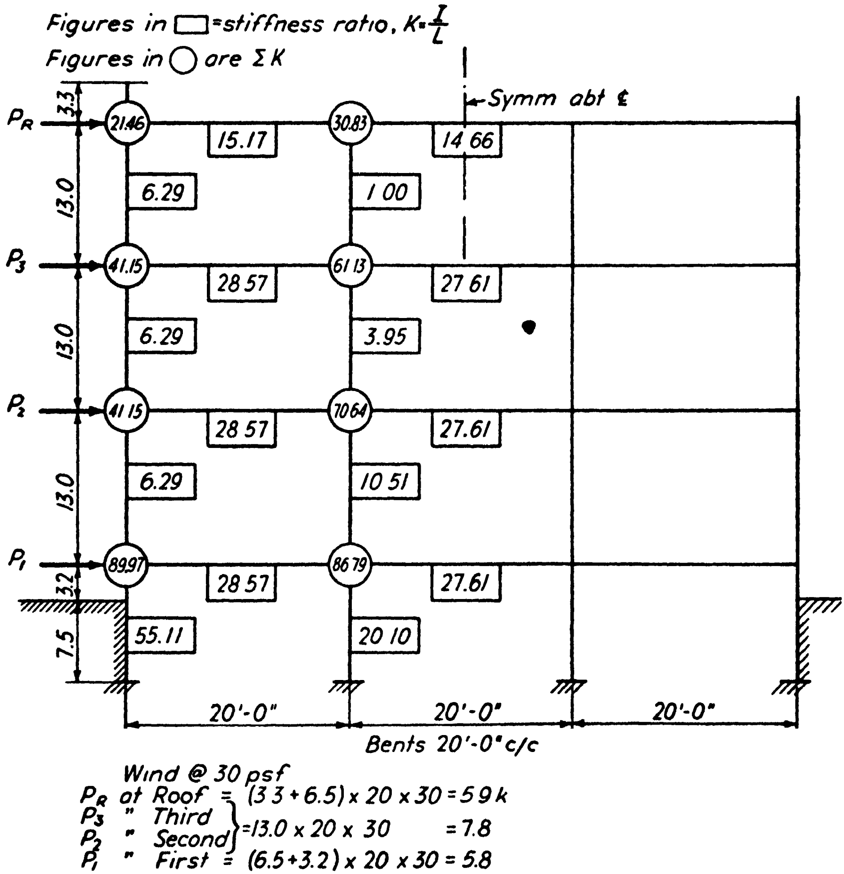
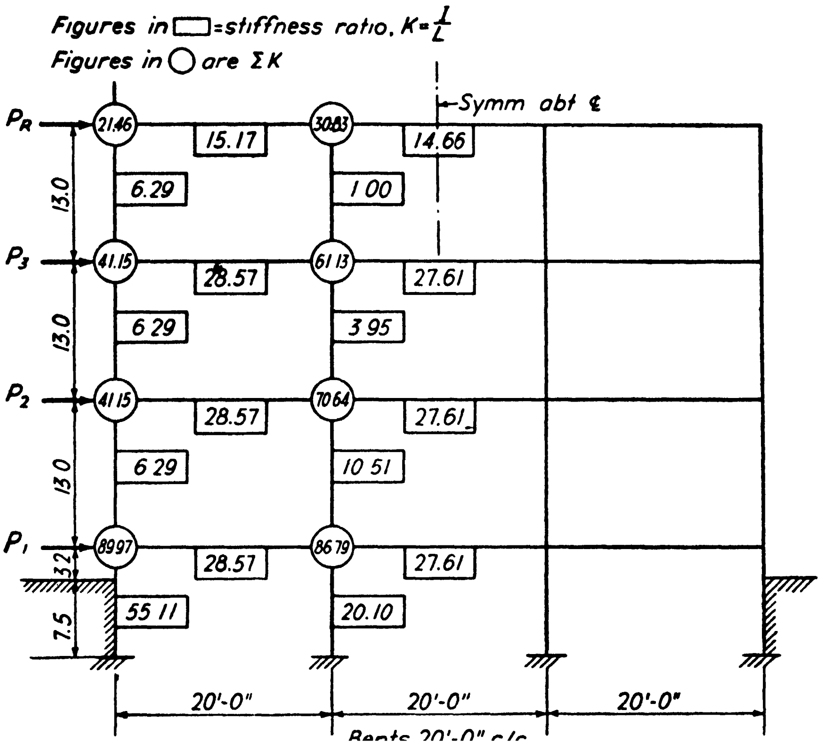


FIG. 19–7

Sometimes the slope deflection and moment distribution methods, as well as others not here described, are called *exact* methods. This is an improper use of the term, particularly when these methods are employed with reinforced concrete. These methods are basically approximate in that they do not take account of the changes of length of members, a negligible matter in buildings of moderate size but one of importance in tall frames. The effect of walls, interior partitions, and deviations of

the actual frame from the assumptions made for analysis are sufficient to change the actual stresses greatly from those given by any analysis.



Computation of Horizontal Forces				
Floor	Ext. Cols	Int. Cols	Entire Floor	Hor. = 10% of Vert
Roof	2 @ 33.6	2 @ 46.1	159.4	P_R = 15.9
Third	2 @ 50.6	2 @ 81.8	264.8	P_3 = 26.5
Second	2 @ 48.4	2 @ 78.0	252.8	P_2 = 25.3
First	2 @ 46.1	2 @ 74.2	240.6	P_1 = 24.1

PROB. 19-2

For convenience the frames discussed in this chapter have been somewhat idealized and simplified. In Chapter XVIII we have already discussed such matters as the obtaining of frame center lines, moments of inertia, and similar factors. The following problems apply the same principles as the examples of this chapter, but introduce the obtaining

of necessary facts from the actual structure. For that reason the problems are partially solved to aid in assembling the data.

Problem 19–1. Compute the shears, direct stresses, and moments in the bent along column line 3 of the building shown on Fig. 19–7 for a horizontal wind pressure of 30 psf by (*a*) the portal method, (*b*) the cantilever method, (*c*) Bowman's method, and (*d*) Morris' variation of moment distribution. See the accompanying figure for necessary data.

Problem 19–2. For the bent of Prob. 19–1 compute the shears, direct stresses, and moments for a horizontal force of $0.1g$, representative of earthquake effect, by all four methods of Prob. 19–1. See the accompanying figure.

19–5. Side-Sway under Gravity Loading. It is important that the designer understand that lateral joint movement will result from unsymmetrical loading conditions normal to the line of motion. The neglect of this side-sway may lead to very large errors. Moments computed for gravity loading with side-sway present but neglected in the computations will be found to indicate the presence of unbalanced column shears. This means that these moments are consistent with the given loading and certain lateral joint forces which prevent sway, forces directly determined from the moments. The moments due to the opposites of these restraining lateral forces may be found by whatever method best suits the situation and combined with the moments first found. The resulting moments are those consistent with the gravity loading alone on a freely lurching frame. This procedure is illustrated in Prob. 12–8, page 223.

CHAPTER XX

ARCHES

20–1. The first arches were curved structures with converging reactions, made of wedge-shaped stone blocks so proportioned that the line of action of the resultant normal stress at any section was wholly compression. The modern arch of plain concrete likewise must be designed so that the line of resistance lies within the middle third. Somewhat smaller sections may be used if reinforced with steel since such an arch can carry a large bending moment with tension in one face.

The most common type of reinforcement is a series of longitudinal bars in top and bottom of the section, following the curve of the arch ring. Large concrete arches often employ a structural steel arch as reinforcement. The great advantage of this is that it is possible to

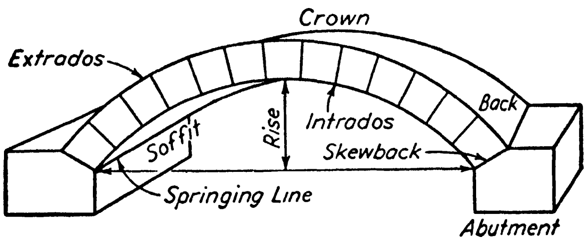


Fig. 20–1

design the steel arch to carry the weight of the forms and the wet concrete of the rib, thus avoiding the use of false work and making more economical use of the steel. Arches thus reinforced are often made with hinges at the supports and sometimes with one also at the crown. The more common type of concrete arch is built without hinges and the discussion of this chapter is limited to that variety.

Hinges serve to fix the point of application of the force acting through them and to eliminate bending. The three-hinged arch is statically determinate and is not subject to stress due to temperature changes and to settlement of the abutment. The hingeless type, like all indeterminate structures, is acted upon by both these influences.

The technical names of the principal parts of an arch are given in Fig. 20–1. An arch may consist of a single ring called the barrel or of

two or more parallel ribs. The haunch is that portion of the rib or barrel mid-way between crown and springing. The spandrel is the space between the upper surface of the arch, the back, and the roadway. Barrel arches are of two sorts: the filled spandrel with the roadway built on earth filled in above the arch which is built with side or spandrel walls; the open spandrel with columns or cross walls built on the back of the arch to carry the beams and slab of the floor system.

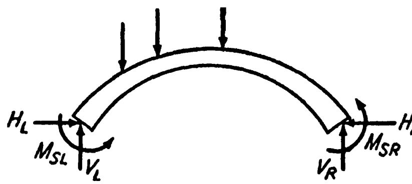


Fig. 20-2

20-2. Arch Analysis. A hingeless arch is indeterminate to the third degree, there being six unknown elements of reaction, as shown by Fig. 20-2. In order to analyze the stresses it is necessary to find three equations in addition to the three of equilibrium for a non-concurrent coplanar force system. This may be done in various ways by consideration of the relations between the elastic deformations of the arch and the internal and external stresses. The so-called different methods of arch analysis are largely different ways of arranging the same fundamental equations and different ways of handling simplifying devices for saving labor. These fundamental equations may be derived either by least work or by the equations expressing the deflections of a curved beam; this should not surprise the student for presumably he has been taught that basically the two methods are one.* The derivation and arrangement presented later are those given by Professor C. M. Spofford in his Theory of Structures, which uses the method of least work.

In order to apply this or any other method of analysis based on the elastic properties of the arch it is first necessary to assume a section of known dimensions. The labor of an exact analysis is so great that it is desirable to have available some simple method for arriving at a trial

* Castigliano's first law states that the deflection of any loaded point of a structure equals the first partial derivative of the internal work with respect to the load at the point. In the case of a redundant reaction the deflection is zero. Accordingly the first derivative with respect to that unknown, the redundant reaction, equals zero. But this is the condition which establishes the minimum for the function, justifying the least work argument. The student can easily extend this to cover the case of a redundant bar in a truss. The least work theorem is known as Castigliano's second law.

section that will require little if any change upon closer study. Mr. Victor S. Cochrane* has developed a simple and speedy method of applying the elastic theory to symmetrical hingeless arches which is somewhat approximate but sufficiently accurate for the final design of structures of moderate span where great refinement is not attempted. Mr. Charles S. Whitney† has prepared a similar adaptation of the elastic theory, also simple and rapid of use, which even exceeds the longer methods in accuracy when applied to arches proportioned in accordance with his fundamental equations. Mr. Whitney's method is presented in part only and very briefly in Art. 20–6.

Stone arches are usually analyzed by the line of thrust or static method, of which there are several variations.‡ The most favored of these assumes that for any loading the crown thrust is the minimum consistent with equilibrium. For symmetrical loading on arches with a central angle§ of not more than 90 to 120° this criterion would indicate a line of resistance passing through the upper middle-third point at the crown and the lower middle-third point at the springing. If this line of resistance lies within the middle third at all sections the arch is considered to be satisfactory for this loading. The criterion for unsymmetrical loading seems to amount to this: If a line of resistance can be drawn within the middle third at all points the arch is satisfactory.

The static method is used by many engineers for the preliminary and even for the final design of monolithic arches of reinforced concrete. Those who are impressed with the uncertainties of the elastic theory due to doubtful preliminary assumptions consider this practice satisfactory for structures of moderate span. The two methods give results in reasonably close agreement. The static method takes no account of temperature stresses and abutment movements.

20–3. Proportions. The axis (the center line of the arch ring) should conform very closely to the dead load line of resistance, thus eliminating as far as possible bending under the permanent and major part of the load and reducing stresses to the least possible. The shape of this

* Victor S. Cochrane, "Design of Symmetrical Hingeless Arches," in Proceedings of the Engineers' Society of Western Pennsylvania, Nov., 1916. Convenient curves for the solution of Mr. Cochrane's equations are given in Hool and Johnson's Concrete Engineers' Handbook, McGraw-Hill Book Co.

† Charles S. Whitney, "Design of Symmetrical Concrete Arches," Trans. A.S.C.E., 1925, Vol. LXXXVIII, p. 931. Complete tables and diagrams for this method are given in Hool and Whitney's Concrete Designers' Manual, 2nd ed., McGraw-Hill Book Co.

‡ The principal methods are discussed simply and clearly in Baker's Treatise on Masonry Construction.

§ For arches with a central angle equal to or greater than these limits a plane of maximum rupture develops at about this location in the ring which should be treated as the actual springing line section.

curve would be parabolic if the dead load were uniformly distributed. Since the intensity of this load increases toward the ends of the span, the theoretical curve lies above a parabola and, in preliminary computations of weight for arches of moderate ratio of rise to span, may be assumed as the segment of a circle through the crown and the two springings. With the dead weight of the structure known with reasonable accuracy, a satisfactory curve for the axis may be obtained by passing an equilibrium polygon through the crown and springing points of the axis. This assumes no bending moment at these two sections; this is not far from the truth if the arch is constructed in such a manner that the shrinkage of the concrete on setting has no effect. Shrinkage is an uncertain factor which is not computed except in the case of long-span structures where construction is carried on in such a manner as to eliminate this element as far as possible.

There is no thoroughly satisfactory way of estimating in advance of computation the proper thickness of the arch ring at the crown. Comparison with existing arches of admittedly good design is an excellent course.* Several formulas have been proposed but none is known to the writers which includes all the factors. The following empirical expression, devised by Mr. F. F. Weld,† gives very conservative results which may be useful in preliminary weight computations:

$$d_c = \sqrt{L} + \frac{L}{10} + \frac{w_L}{200} + \frac{w_c}{400}$$

where d_c = crown thickness in inches
L = clear span in feet
w_L = live load in pounds per square foot
w_c = dead load at crown in pounds per square foot.

The thickness of the arch at the springing is usually about twice or more that at the crown, with a range from about 1.5 to 3.

20–4. Loads. The live loads used in the design of an arch rib or barrel are the same as those for any other highway or railway bridge. Many arches are designed for a uniform live load on the deck, of sufficient intensity per square foot to be equivalent to the maximum concentrations expected. For a railway arch this is the distributed weight of locomotives and train; for highway arches the distributed weight of

* The greatest compendium in this field is Grandes Voûtes, in six large volumes by Paul Séjourné, Professeur à l'École Nationale des Ponts et Chaussées, Paris, et Ingénieur en Chef des Ponts et Chaussées. Professor Séjourné gives pictures and the details of design and construction of most of the masonry and concrete arches of 40-meter span and more, built in all countries up to 1916.

† *Engineering Record*, Nov. 4, 1905.

trolley cars and from 50 to 150 psf on the roadway, 40 to 100 psf on the sidewalk — figures that vary with the type of traffic and the length of span.

Impact allowances vary from 15 to 30 per cent for open spandrel arches; for filled spandrel structures a smaller allowance is made or the item is omitted entirely.

The dead loads on an arch may be computed from these data: concrete at 150 pcf, earth at 100 or 120 pcf, the latter figure being used when there is possibility that the fill may be saturated. It is customary to ignore the horizontal component of earth thrust on the back of the arch ring.

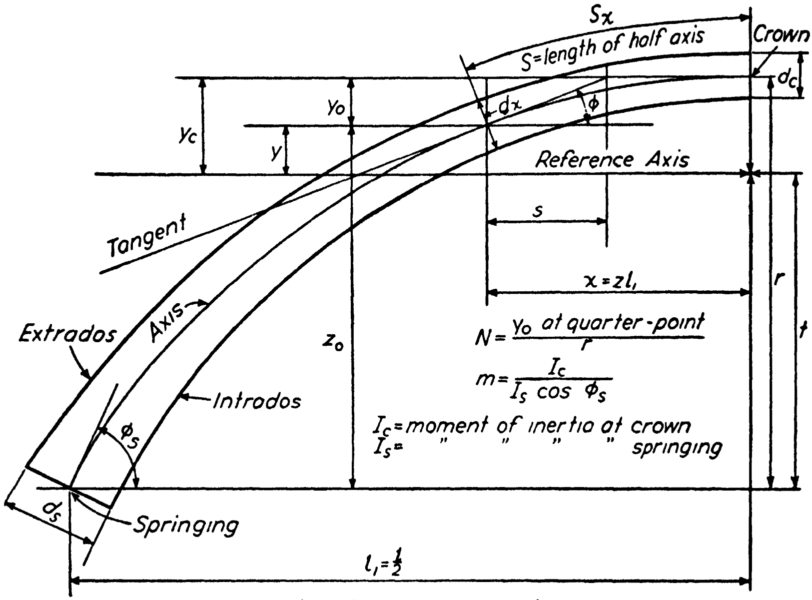
20–5. Deformation Effects. A reinforced concrete arch rib under load is in compression from end to end and consequently is shorter than when unloaded. This rib shortening is resisted by the fixed abutments and tensile and bending stresses are set up which modify the principal load stress effects. These rib-shortening stresses are commonly computed separately but they may be included in the primary stress computation if preferred, as is seen later in this chapter.

Shrinkage and plastic flow of the concrete also shorten the arch rib with resulting stress effects which are very difficult of evaluation since there are many variables entering into the phenomena. The general effects are easily comprehended, those incidental to change of rib length plus those local effects on stress distribution already considered for a compression member (Art. 8–3). In column design the transfer of stress from concrete to steel incident to flow was noted and the design formulas abandoned the elastic theory as base. Similarly in the case of arches the Committee on Plain and Reinforced Concrete Arches of the A.C.I. recommends* that the use of unit stress calculations be abandoned and that a new method of evaluating the strength of a section be used, based on ultimate strength. Plastic flow thus ceases to be a matter of direct consideration. Shrinkage may be properly taken into account, according to this Committee recommendation, by considering it equivalent to a 15°F temperature drop.

In the past it has been common to proportion arches in the northern part of the United States for the stresses set up by a range of about 80°F, the temperature drop being usually taken as larger than the rise. The recommendation of the A.C.I. Committee is for a range of 60 per cent of the maximum and minimum recorded temperatures at the locality, rise and fall being taken as equal. This assumes that plastic flow eliminates the effect of the heat rise while the concrete is hardening and permits the arch to adjust itself to mean temperature.

* See Art. 20–11.

20-6. Whitney's Method.* In order to obtain formulas by actual integration, mathematical expressions for the arch axis and the variation of rib thickness are needed. Using those of Strassner† in the general elastic equations for symmetrical hingeless arches, Charles S. Whitney derived definite formulas for reactions. From these he constructed



Notation for Arch Analysis

FIG. 20-3

influence lines, tables, and diagrams which eliminate all involved calculation. With these helps it is a quick and simple matter to study the effect of variations in crown and springing line thickness and determine the most economical and satisfactory section.

The best axis for any arch within their range may be determined by the values given in Tables 20-1 and 20-2, which give the coordinates of points on the curve and the intercepts necessary to establish the slope of the axis at each tabulated point. In addition to the information regarding notation given in Fig. 20-3 the following are used: w_s = dead load per linear foot at springing; w_c = dead load per linear foot at crown. The formula which is the basis of these tables was derived

* Trans., A.S.C.E., 1925. Mr. Whitney is chairman of the A.C.I. Arch Committee referred to in the previous article.

† Neuere Methoden zur Statik der Rahmentragwerke und der elastischen Bogenträger.

for an arch with distributed loads. To use these data for one with an open spandrel w_c and w_s should be computed as though the actual loads supported by the arch, including the spandrel columns, were carried to the arch by a spandrel wall or filling.

The variation of rib thickness is given by the following equation (see Fig. 20-3):

$$d_x = d_c c \sqrt[8]{1 + \tan^2 \phi} \qquad [20\text{-}1]$$

where

$$c = \frac{1}{\sqrt[3]{1 - (1 - m)z}} \qquad [20\text{-}2]$$

Tables 20-3 and 20-4 are given to facilitate the solution of these expressions.

Two factors are sufficient to determine definitely any arch whose proportions conform to these data: N, the ratio y_0/r at the quarter-point, and m, the ratio $I_c/I_s \cos \phi_s$, factors which designate respectively the form of the arch axis and the form of the rib.

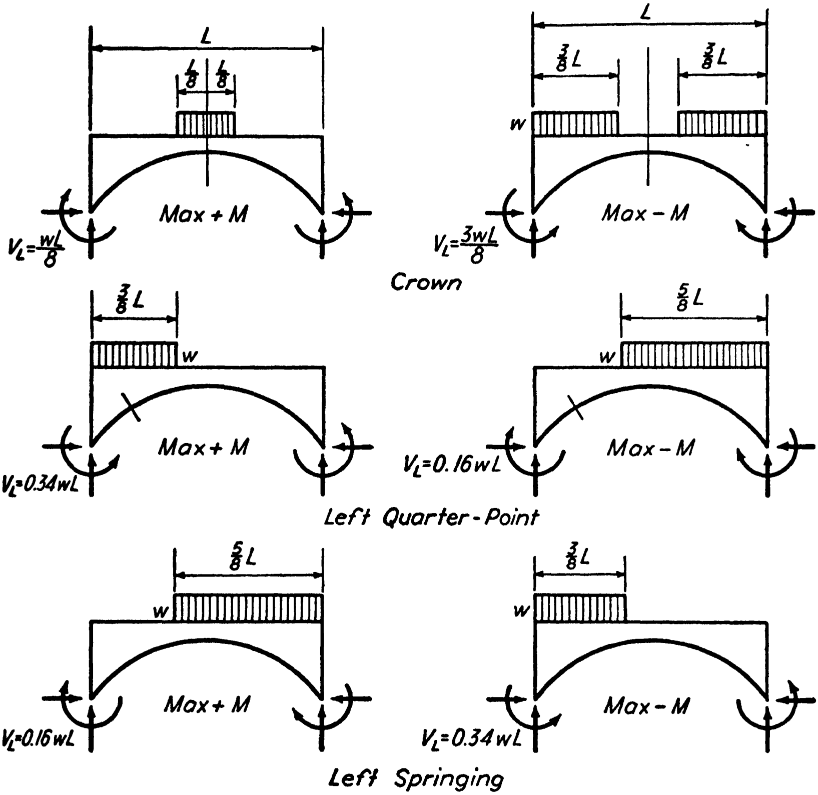
Space forbids the reproduction of the influence lines which are presented in Mr. Whitney's paper, which show clearly the position of the loading for maximum stress at the critical sections of an arch, crown, springing, and quarter-point. With the aid of these influence lines Figs. 20-6 to 20-11 were prepared; they give coefficients for obtaining the maximum values of live load moment at the crown, quarter-point, and springing, with the corresponding horizontal component of crown thrust. In order to obtain the arch stresses it is necessary in addition to know the vertical component of reactions for each loading. In the absence of the influence lines approximate values of these may be taken from Fig. 20-4.

Temperature stress may be computed by aid of Fig. 20-12, which gives the value of the horizontal reaction induced by temperature variation. A fall in temperature causes the arch to contract and tend to draw away from the abutments, setting up the horizontal pull and negative springing line moment shown in Fig. 20-5a. The forces acting on the half-arch are shown in Fig. 20-5b, with the pull and the positive moment at the crown replaced by a single force acting a distance y_c below the crown. Fig. 20-12 gives simply the value of H_T, and Table 20-5, which gives the values of y_c, is printed here.

Any load, live or dead, causes compression in an arch rib and consequently a general shortening of the fibers. This shortening sets up a stress of exactly the same sort as a fall of temperature as shown in Fig. 20-5 and must be combined with the live and dead load stresses to

give the true stress. This effect is called rib shortening. The amount of horizontal pull on the abutments thus induced may be expressed as

$$H_{RS} = \text{approximately } - Hu' \tag{20-3}$$



Position for Maximum Stress
Live Load = w lb per ft on rib

FIG. 20–4

where H is the thrust produced by the loading, live or dead or the two together, and

$$u' = \frac{I_c}{A_c C'_m C r^2} \tag{20-4}$$

Here A_c is the cross-sectional area at the crown and C and C'_m are coefficients given by Table 20–6 and Fig. 20–13, respectively.

Mr. Whitney makes the following statement which should be carefully noted by all making use of this material in design.

The data in the paper can be used by one who is not an expert in arch analysis, provided the designing is done under the direction of an engineer who is thoroughly familiar with both the practical and theoretical aspects of arch design. The design of arches should not be entrusted to a novice. The writer's paper is limited to the mathematical considerations of design and does not attempt to treat the equally important practical considerations. No tables or diagrams should be used without a thorough knowledge of their basis and limitations.

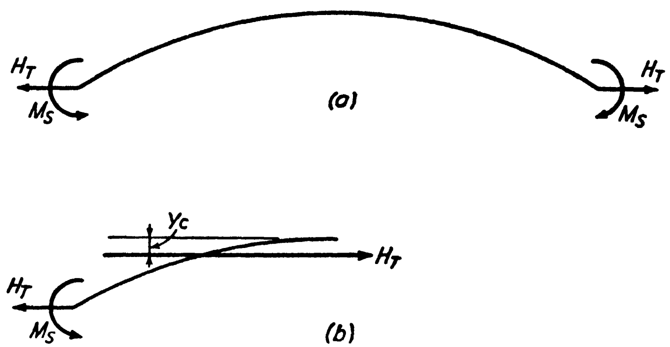


Fig. 20-5

It is hoped that the reproduction of this portion of Mr. Whitney's paper will lead many to make a thorough study of this very important contribution to the literature of arches.

20-7. Method of Least Work.* All analyses of the hingeless arch based on its elastic action assume that the abutments are rigid and immovable so that the span remains unchanged and there is no rotation of the tangent to the axis at the springing. This is probably never exactly true and in consequence another possible source of error is introduced at the start into any such analysis in addition to those common to all reinforced concrete design: lack of uniformity of material; uncertain value of the modulus of elasticity; uncertain action of concrete in tension; and the effect of shrinkage and plastic flow. Consequently an exact theory for the arch, although interesting mathematically, is of doubtful worth actually. By making certain reasonable assumptions the application of the theorem of least work to arch analysis is greatly simplified. The simplifying assumptions are these:

(*a*) The distribution of stress over the cross section of a curved bar is the same as though the bar were straight. This assumes that the

* This treatment is that given by Professor C. M. Spofford in Theory of Structures, McGraw-Hill Book Co. It is based on the method given by Müller-Breslau in Zeitschrift des Architekten und Ingenieur Vereins, Hannover, 1884.

TABLE 20-1

ARCH AXIS COORDINATES

$\frac{w_s}{w_c}$	N	Values of $\frac{y_0}{r}$										
		Point 0 (Springing line)	Point 1	Point 2	Point 3	Point 4	Point 5 (Quarter-point)	Point 6	Point 7	Point 8	Point 9	Point 10 (Crown)
		$z = 1.0$	$z = 0.9$	$z = 0.8$	$z = 0.7$	$z = 0.6$	$z = 0.5$	$z = 0.4$	$z = 0.3$	$z = 0.2$	$z = 0.1$	$z = 0.0$
1.000	0.250	1.000	0.8100	0.6400	0.4900	0.3600	0.2500	0.1600	0.0900	0.0400	0.0100	0.0000
1.167	0.245	1.000	0.8060	0.6339	0.4834	0.3539	0.2450	0.1563	0.0878	0.0390	0.0098	0.0000
1.347	0.240	1.000	0.8019	0.6277	0.4769	0.3478	0.2400	0.1527	0.0856	0.0380	0.0095	0.0000
1.543	0.235	1.000	0.7977	0.6214	0.4701	0.3416	0.2350	0.1493	0.0835	0.0370	0.0092	0.0000
1.756	0.230	1.000	0.7934	0.6151	0.4632	0.3353	0.2300	0.1458	0.0814	0.0360	0.0090	0.0000
1.987	0.225	1.000	0.7890	0.6087	0.4563	0.3293	0.2250	0.1424	0.0792	0.0350	0.0087	0.0000
2.240	0.220	1.000	0.7847	0.6022	0.4494	0.3229	0.2200	0.1386	0.0771	0.0340	0.0085	0.0000
2.514	0.215	1.000	0.7801	0.5957	0.4425	0.3167	0.2150	0.1351	0.0749	0.0330	0.0082	0.0000
2.814	0.210	1.000	0.7755	0.5891	0.4355	0.3104	0.2100	0.1315	0.0728	0.0320	0.0080	0.0000
3.141	0.205	1.000	0.7709	0.5824	0.4285	0.3041	0.2050	0.1281	0.0707	0.0310	0.0077	0.0000
3.500	0.200	1.000	0.7662	0.5757	0.4215	0.2978	0.2000	0.1245	0.0686	0.0300	0.0074	0.0000
3.893	0.195	1.000	0.7615	0.5689	0.4145	0.2914	0.1950	0.1209	0.0665	0.0290	0.0072	0.0000
4.324	0.190	1.000	0.7567	0.5621	0.4073	0.2851	0.1900	0.1176	0.0644	0.0281	0.0070	0.0000
4.801	0.185	1.000	0.7518	0.5551	0.4000	0.2787	0.1850	0.1140	0.0623	0.0271	0.0067	0.0000
5.321	0.180	1.000	0.7469	0.5481	0.3927	0.2723	0.1800	0.1106	0.0602	0.0262	0.0065	0.0000
5.898	0.175	1.000	0.7420	0.5410	0.3854	0.2659	0.1750	0.1072	0.0582	0.0252	0.0062	0.0000
6.536	0.170	1.000	0.7367	0.5337	0.3781	0.2595	0.1700	0.1037	0.0562	0.0243	0.0059	0.0000
7.244	0.165	1.000	0.7313	0.5264	0.3707	0.2531	0.1650	0.1002	0.0541	0.0233	0.0057	0.0000
8.031	0.160	1.000	0.7259	0.5190	0.3632	0.2466	0.1600	0.0968	0.0521	0.0224	0.0055	0.0000
8.906	0.155	1.000	0.7205	0.5116	0.3557	0.2399	0.1550	0.0934	0.0501	0.0215	0.0053	0.0000
9.889	0.150	1.000	0.7151	0.5040	0.3480	0.2332	0.1500	0.0901	0.0483	0.0206	0.0050	0.0000

TABLE 20-2

Intercepts to Determine Position of Tangents to Arch Axis

$\frac{w_s}{w_c}$	N	Values of $\frac{s}{l_1}$										
		Point 0 (Springing line)	Point 1	Point 2	Point 3	Point 4	Point 5 (Quarter-point)	Point 6	Point 7	Point 8	Point 9	Point 10 (Crown)
		$z = 1.0$	$z = 0.9$	$z = 0.8$	$z = 0.7$	$z = 0.6$	$z = 0.5$	$z = 0.4$	$z = 0.3$	$z = 0.2$	$z = 0.1$	$z = 0.0$
1.000	0.25	0.5000	0 4500	0 4000	0 3500	0 3000	0 2500	0 2000	0 1500	0 1000	0 0500	∞
1.347	0.24	0 4743	0 4311	0 3865	0 3410	0 2943	0 2466	0 1982	0.1492	0.0998	0 0500	∞
1.756	0.23	0 4503	0 4129	0 3734	0 3319	0 2884	0 2432	0.1965	0 1485	0 0996	0 0500	∞
2.240	0.22	0 4279	0 3957	0.3607	0 3229	0 2825	0.2397	0.1946	0.1478	0 0993	0 0499	∞
2.814	0.21	0 4070	0 3792	0 3482	0 3140	0 2765	0 2360	0 1926	0 1469	0.0991	0 0499	∞
3.500	0.20	0 3872	0 3634	0 3360	0 3051	0 2704	0 2323	0 1905	0 1459	0.0988	0 0498	∞
4.324	0.19	0.3686	0 3482	0 3241	0 2962	0 2643	0 2285	0.1884	0 1449	0 0985	0 0498	∞
5.321	0.18	0.3510	0 3335	0 3125	0 2875	0 2583	0 2246	0.1862	0 1439	0 0983	0.0497	∞
6.536	0.17	0 3342	0 3194	0 3011	0 2787	0 2520	0 2206	0 1840	0 1429	0 0980	0 0497	∞
8.031	0 16	0 3182	0 3058	0 2899	0 2700	0 2457	0 2164	0 1817	0 1419	0 0976	0 0496	∞
9.889	0.15	0 3030	0 2926	0.2788	0 2613	0 2393	0 2121	0 1792	0 1407	0.0972	0 0496	∞

TABLE 20-3

Determination of Rib Thickness — Values of $\frac{l^2}{r^2}\tan^2\phi$

$\frac{w_s}{w_c}$	N	Values of $\frac{l^2}{r^2}\tan^2\phi$										
		Point 0 (Springing line)	Point 1	Point 2	Point 3	Point 4	Point 5 (Quarter-point)	Point 6	Point 7	Point 8	Point 9	Point 10 (Crown)
		$z = 1.0$	$z = 0.9$	$z = 0.8$	$z = 0.7$	$z = 0.6$	$z = 0.5$	$z = 0.4$	$z = 0.3$	$z = 0.2$	$z = 0.1$	$z = 0.0$
1.000	0.25	16.000	12.960	10.240	7.840	5.760	4 000	2 560	1 440	0 640	0 160	0
1.347	0.24	17.780	13 846	10 552	7.821	5 583	3 788	2.377	1 316	0 580	0.145	0
1.756	0.23	19.722	14 769	10.852	7.791	5 403	3 578	2 200	1 199	0 523	0 130	0
2.240	0.22	21.841	15.730	11.149	7.748	5 220	3 371	2 029	1 088	0 468	0 116	0
2.814	0.21	24.150	16.731	11 443	7 694	5 034	3.166	1 864	0 983	0 417	0.102	0
3.500	0.20	26.676	17 780	11.734	7 629	4 844	2.964	1.706	0 883	0 369	0 089	0
4.324	0.19	29.441	18.885	12 020	7.552	4 650	2 765	1.554	0 789	0 325	0 076	0
5.321	0.18	32.476	20 047	12.297	7.463	4 450	2.569	1.408	0.701	0.284	0 065	0
6.536	0.17	35.812	21.269	12 564	7.360	4 243	2.376	1.268	0.619	0.246	0.056	0
8.031	0.16	39.494	22 553	12 820	7.238	4 029	2.187	1.135	0 541	0.211	0.048	0
9.889	0.15	43.572	23 899	13 063	7 095	3 809	2.001	1 008	0 467	0.180	0 041	0

TABLE 20-4

Determination of Rib Thickness — Values of c

Value of $c = \dfrac{1}{\sqrt[3]{1 - (1 - m)z}}$

m	Point 0 (Springing line)	Point 1	Point 2	Point 3	Point 4	Point 5 (Quarter-point)	Point 6	Point 7	Point 8	Point 9	Point 10 (Crown)
	$z = 1.0$	$z = 0.9$	$z = 0.8$	$z = 0.7$	$z = 0.6$	$z = 0\ 5$	$z = 0.4$	$z = 0.3$	$z = 0.2$	$z = 0.1$	$z = 0.0$
1 0	1 000	1 000	1 000	1 000	1 000	1 000	1 000	1 000	1 000	1 000	1.000
0 8	1 077	1 068	1 060	1 052	1 044	1.036	1 028	1 021	1 014	1 007	1 000
0 6	1 186	1 160	1 137	1 116	1 096	1 077	1 060	1 044	1 028	1 014	1 000
0.5	1 260	1 221	1 186	1 154	1 126	1 101	1 077	1 056	1 036	1 017	1 000
0.4	1.357	1 295	1 244	1 199	1 160	1 126	1 096	1 068	1 044	1 021	1 000
0 3	1 494	1 393	1 315	1 252	1 199	1.154	1 116	1 082	1 052	1 024	1 000
0 25	1 587	1 454	1 357	1 282	1 221	1 170	1 126	1 089	1 056	1 026	1 000
0 20	1 710	1 529	1 405	1 315	1 244	1 186	1 137	1 096	1 060	1 028	1 000
0 15	1 882	1 621	1 462	1 352	1 268	1.203	1 149	1 103	1 064	1 030	1 000

TABLE 20-5

VALUE OF y_c

N	$\frac{w_s}{w_c}$	Value of $\frac{y_c}{r}$					
		$m = 0.15$	$m = 0.20$	$m = 0.25$	$m = 0.30$	$m = 0.40$	$m = 0.50$
0.25	1.000	0.2101	0.2222	0.2333	0.2436	0.2619	0.2778
0.24	1.347	0.2044	0.2163	0.2273	0.2374	0.2556	0.2713
0.23	1.756	0.1985	0.2103	0.2212	0.2312	0.2491	0.2647
0.22	2.240	0.1926	0.2043	0.2150	0.2250	0.2427	0.2580
0.21	2.814	0.1867	0.1983	0.2089	0.2187	0.2362	0.2513
0.20	3.500	0.1808	0.1922	0.2027	0.2124	0.2296	0.2446
0.19	4.324	0.1748	0.1861	0.1964	0.2060	0.2230	0.2378
0.18	5.321	0.1688	0.1799	0.1901	0.1996	0.2164	0.2309
0.17	6.536	0.1628	0.1738	0.1838	0.1931	0.2097	0.2240
0.16	8.031	0.1567	0.1675	0.1774	0.1865	0.2029	0.2170
0.15	9.889	0.1506	0.1612	0.1709	0.1799	0.1960	0.2099

TABLE 20-6

RIB SHORTENING

N	$\frac{w_s}{w_c}$	Value of C ($E = 2{,}000{,}000$)					
		$m = 0.15$	$m = 0.20$	$m = 0.25$	$m = 0.30$	$m = 0.40$	$m = 0.50$
0.25	1.000	0.0329	0.0370	0.0410	0.0448	0.0520	0.0588
0.24	1.347	0.0320	0.0361	0.0400	0.0437	0.0509	0.0577
0.23	1.756	0.0312	0.0352	0.0390	0.0428	0.0498	0.0566
0.22	2.240	0.0302	0.0342	0.0380	0.0417	0.0486	0.0553
0.21	2.814	0.0294	0.0332	0.0370	0.0407	0.0475	0.0541
0.20	3.500	0.0285	0.0323	0.0361	0.0396	0.0466	0.0530
0.19	4.324	0.0276	0.0314	0.0351	0.0386	0.0455	0.0519
0.18	5.321	0.0267	0.0305	0.0341	0.0376	0.0443	0.0507
0.17	6.536	0.0258	0.0295	0.0331	0.0366	0.0432	0.0496
0.16	8.031	0.0250	0.0286	0.0322	0.0356	0.0422	0.0484
0.15	9.889	0.0241	0.0277	0.0312	0.0346	0.0411	0.0472

C varies as *E*

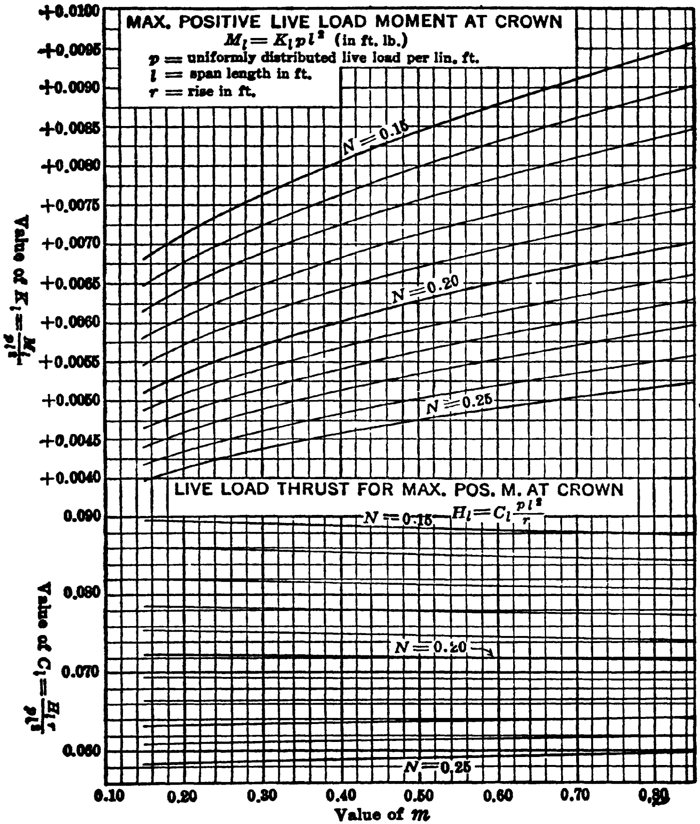


Fig. 20-6

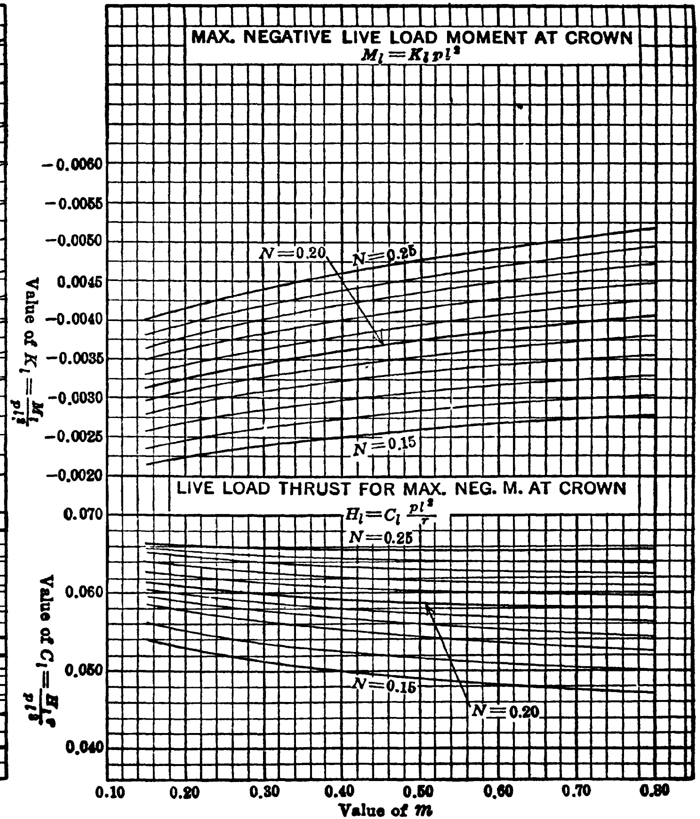


Fig. 20-7

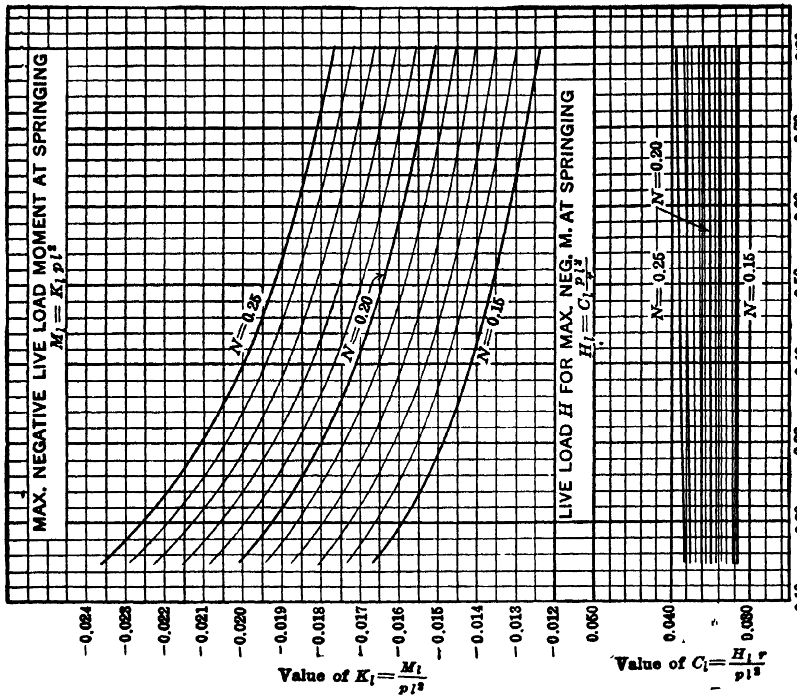


FIG. 20-9

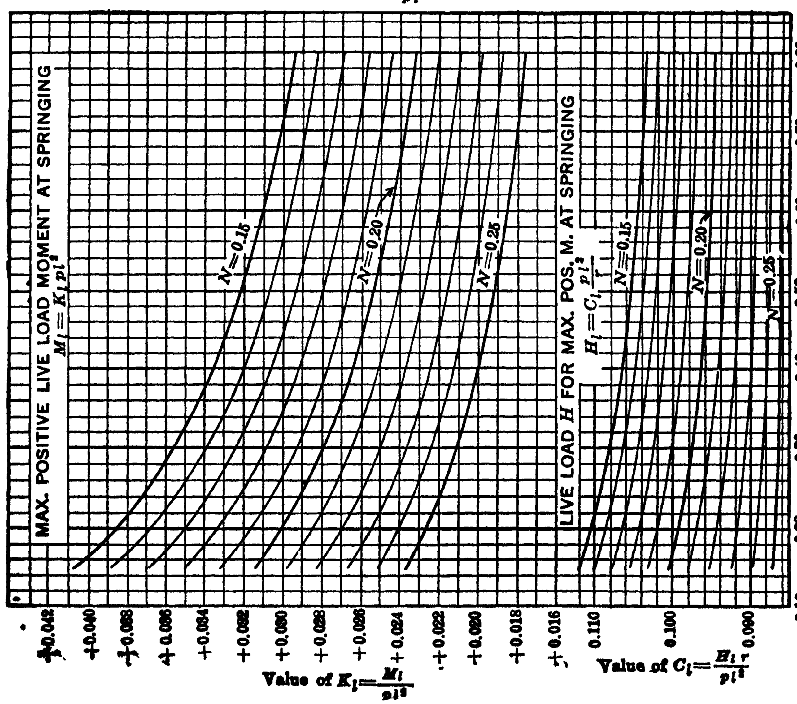


FIG. 20-8

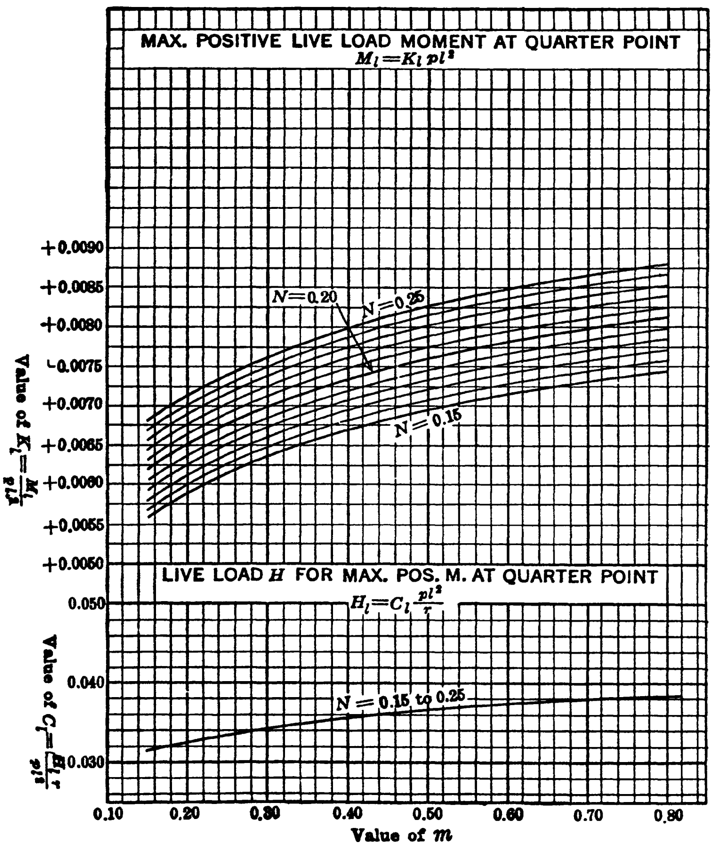


FIG. 20-10

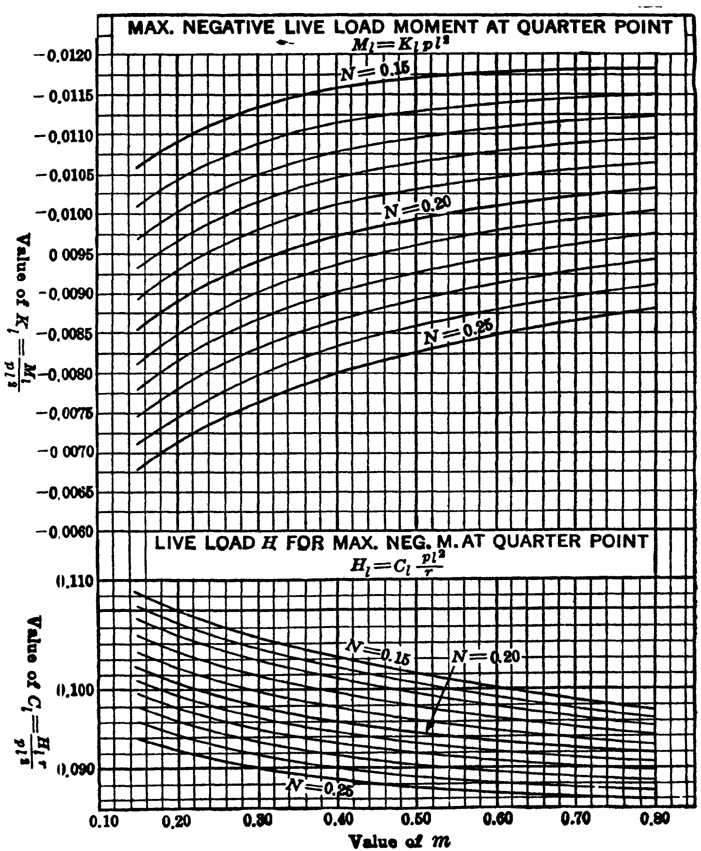
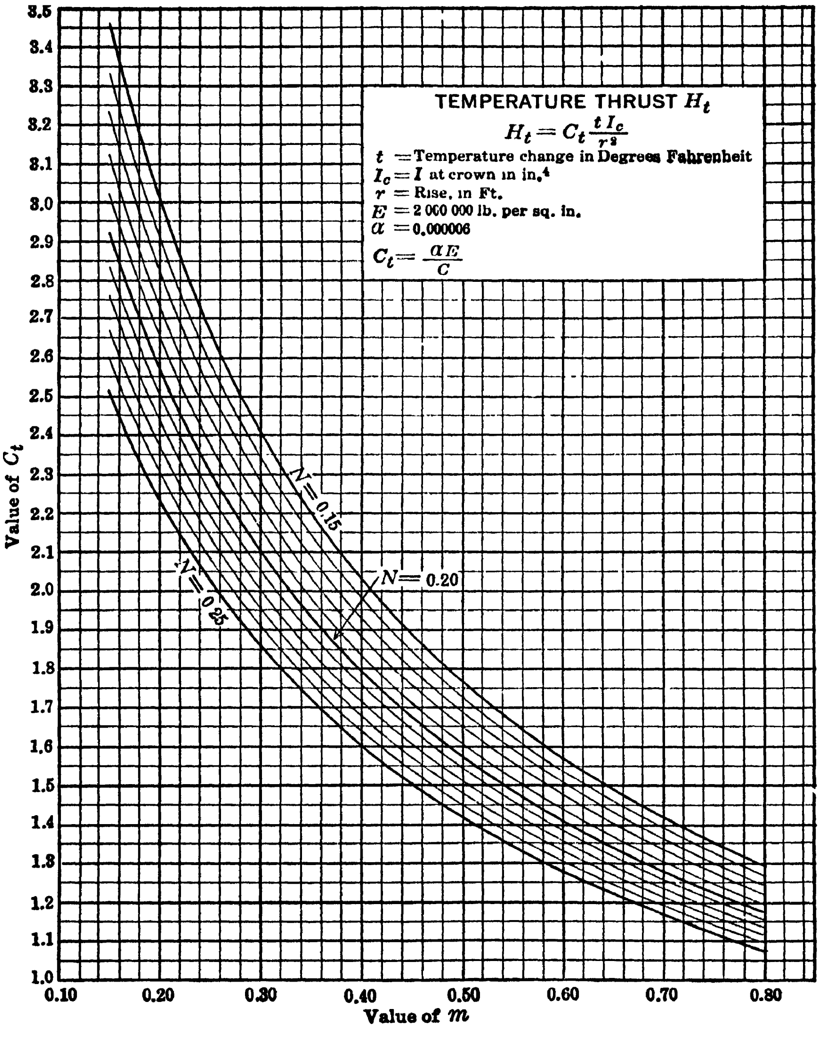


FIG. 20-11



TEMPERATURE THRUST H_t
$H_t = C_t \frac{t I_c}{r^2}$
t = Temperature change in Degrees Fahrenheit
$I_c = I$ at crown in in.^{4}
r = Rise, in Ft.
E = 2 000 000 lb. per sq. in.
α = 0.000006
$C_t = \frac{\alpha E}{C}$
$N = 0.15$
$N = 0.20$
$N = 0.25$
Value of C_t
3.5
3.4
3.3
3.2
3.1
3.0
2.9
2.8
2.7
2.6
2.5
2.4
2.3
2.2
2.1
2.0
1.9
1.8
1.7
1.6
1.5
1.4
1.3
1.2
1.1
1.0
0.10
0.20
0.30
0.40
0.50
0.60
0.70
0.80
Value of m

FIG. 20-12

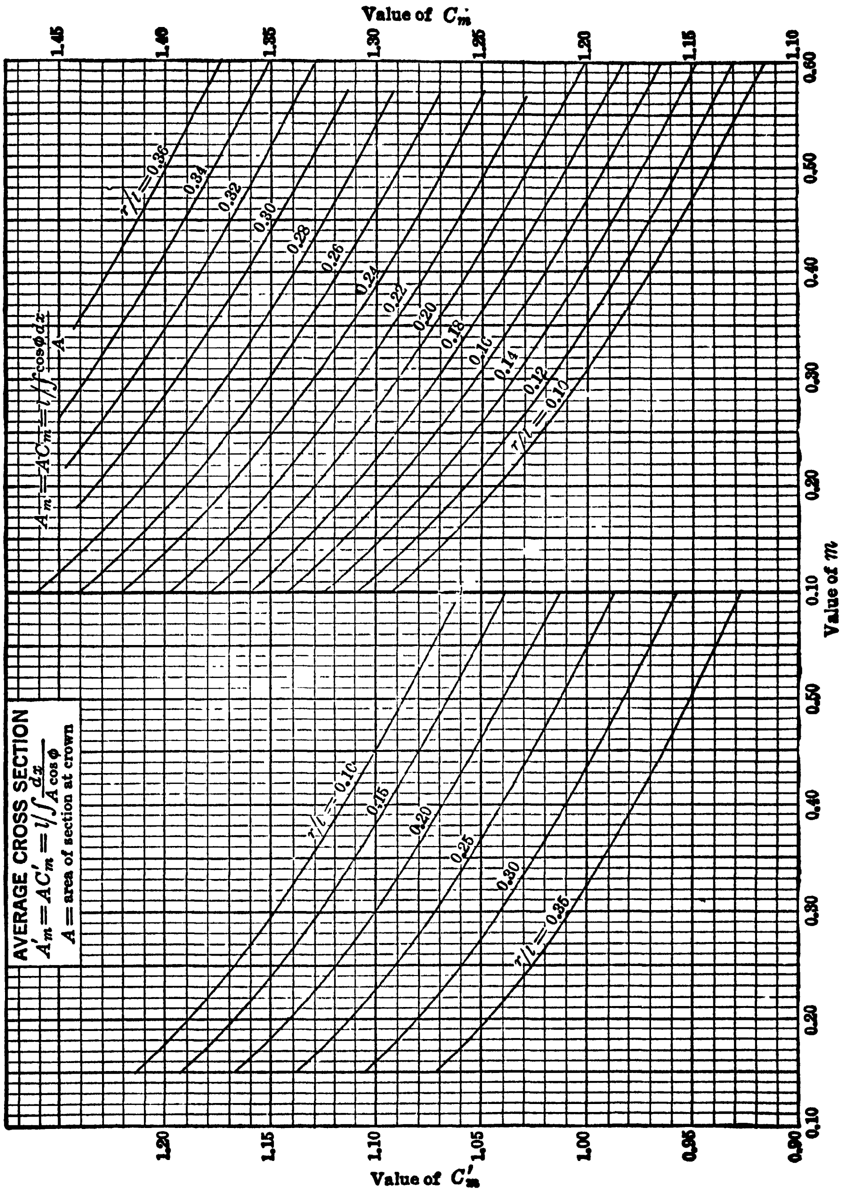

Value of C_m
1.45
1.40
1.35
1.30
1.25
1.20
1.15
1.10
0.60
0.50
0.40
0.30
0.20
0.10
Value of m
$A_m = AC_m = l/\int \frac{\cos\phi\, dx}{A}$
$r/l = 0.36$
0.34
0.32
0.30
0.28
0.26
0.24
0.22
0.20
0.18
0.16
0.14
0.12
$r/l = 0.10$
AVERAGE CROSS SECTION
$A'_m = AC'_m = l/\int \frac{dx}{A\cos\phi}$
A = area of section at crown
$r/l = 0.10$
0.15
0.20
0.25
0.30
$r/l = 0.35$
0.50
0.40
0.30
0.20
0.10
1.20
1.15
1.10
1.05
1.00
0.95
0.90
Value of C'_m

FIG. 20-13

familiar expression, $f = (P/A) \pm (My/I)$, applies instead of the exact formula which involves the radius of curvature.

(*b*) The total direct or axial thrust at all sections of the arch is the same and equal to the horizontal component of crown thrust.

(*c*) The work due to shear may be neglected, as is done universally in the study of indeterminate structures.

The justification of these assumptions may be briefly stated. The distribution of stress over the cross section of a curved bar whose depth

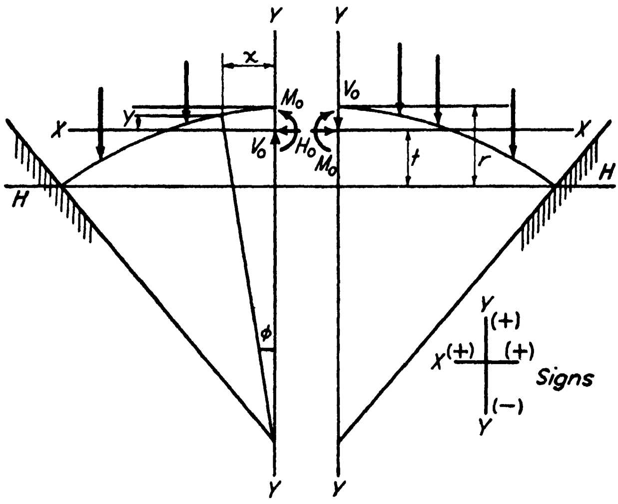


FIG. 20–14

is 0.06, or less, of the radius of curvature is affected to a negligible degree by the curvature. The work due to direct thrust is so small an element in indeterminate structures that it is often neglected. So an approximation in the magnitude of the direct thrust is entirely proper. The magnitude of the error was found to be a fraction of 1 per cent in a 100-ft arch with a rise of 20 ft.

In addition to the notation already made use of (Fig. 20-3) that of Fig. 20-14 and also the following will be used: XX and YY are two reference axes, the first located so that $\int \frac{y\,dS}{EI} = 0$ and the second being an axis of symmetry.

m_L = moment (pound-feet) at any point (x, y) on the left half axis of the loads between that point and the crown, the left half of the arch being considered as a cantilever beam fixed at the abutment with the right half replaced by its equivalent M_o, H_o, and V_o.

m_R = same for right half of arch.
M_c = actual bending moment at crown (foot-pounds) $= M_o - H_o(r - t)$.
M_L = actual bending moment at any point on left half of arch axis.
M_R = actual bending moment at any point on right half of arch axis.
S = length of arch axis.
W = total work in entire arch.
I = moment of inertia (ft^4) at any section normal to axis.
A = area (square feet) at any section normal to axis.
E = modulus of elasticity (pounds per square foot).

M_o, H_o, and V_o are all taken as positive when acting in the direction shown in Fig. 20-14.

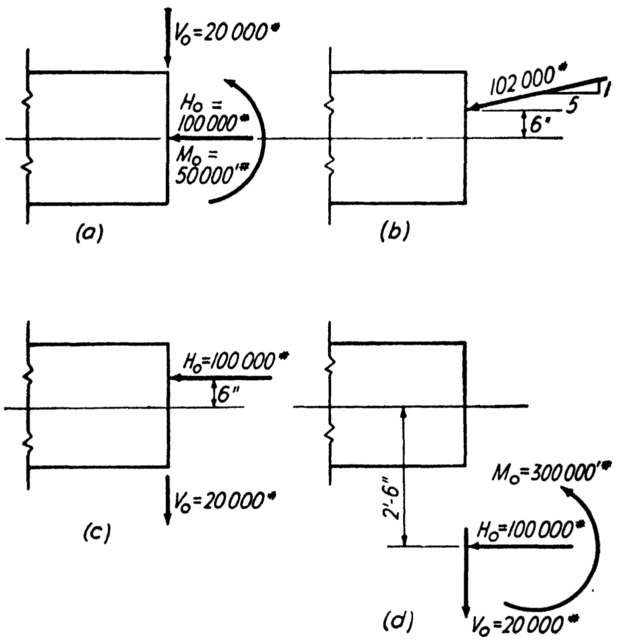


FIG. 20-15

In Fig. 20-14 is shown a loaded arch cut into halves by a section at the crown, with the forces exerted by each portion of the arch upon the other represented by M_o, H_o, and V_o, the moment, horizontal component of thrust, and the shear acting at the level of the reference axis XX.* These three quantities being known for any loading, the principles of statics are sufficient for the determination of the moment, thrust, and

* In Fig. 20-15 are shown four ways of representing the action of the right portion of a given arch upon that to the left.

shear at any other section of the arch. By the theorem of least work it may be stated that M_o, H_o, and V_o always have the least value consistent with equilbirum. They may be determined, therefore, by writing an expression for the total work done in the arch in terms of the loads and these three unknowns, differentiating the expression with regard to each unknown in turn, placing each partial derivative equal to zero and solving for M_o, H_o, and V_o.

The total work done is (see Art. 12-4):

$$W = \int_0^{S/2} \frac{M_L^2\,dS}{2EI} + \int_0^{S/2} \frac{M_R^2\,dS}{2EI} + 2H_0^2 \int_0^{S/2} \frac{dS}{2AE}$$

where

$$M_L = M_o - m_L - H_o y + V_o x$$
$$M_R = M_o - m_R - H_o y - V_o x$$

The partial derivatives are:

$$\frac{\partial W}{\partial M_o} = \int_0^{S/2} (M_o - m_L - H_o y + V_o x)\frac{dS}{EI} + \int_0^{S/2} (M_o - m_R - H_o y - V_o x)\frac{dS}{EI}$$

$$\frac{\partial W}{\partial H_o} = \int_0^{S/2} (M_o - m_L - H_o y + V_o x)(-y)\frac{dS}{EI} + \int_0^{S/2} (M_o - m_R - H_o y - V_o x)(-y)\frac{dS}{EI} + 2H_o \int_0^{S/2} \frac{dS}{AE}$$

$$\frac{\partial W}{\partial V_o} = \int_0^{S/2} (M_o - m_L - H_o y + V_o x)(x)\frac{dS}{EI} + \int_0^{S/2} (M_o - m_R - H_o y - V_o x)(-x)\frac{dS}{EI}$$

The XX axis is located so that $\int \frac{y\,dS}{EI} = 0$ and all terms containing this integral multiplied by a constant equal zero. Combining terms and placing each partial derivative equal to zero gives

$$\frac{\partial W}{\partial M_o} = 2M_o \int_0^{S/2} \frac{dS}{EI} - \int_0^{S/2} \frac{(m_L + m_R)\,dS}{EI} = 0$$

$$\frac{\partial W}{\partial H_o} = \int_0^{S/2} \frac{(m_L + m_R)y\,dS}{EI} + 2H_o \int_0^{S/2} \frac{y^2\,dS}{EI} + 2H_o \int_0^{S/2} \frac{dS}{AE} = 0$$

$$\frac{\partial W}{\partial V_o} = \int_0^{S/2} \frac{(m_R - m_L)x\,dS}{EI} + 2V_o \int_0^{S/2} \frac{x^2\,dS}{EI} = 0$$

Solving

$$M_o = \frac{\displaystyle\int_0^{S/2} \frac{(m_L + m_R)\,dS}{I}}{\displaystyle 2\int_0^{S/2} \frac{dS}{I}} \tag{20-5}$$

$$H_o = \frac{\displaystyle -\int_0^{S/2} \frac{(m_L + m_R)y\,dS}{I}}{\displaystyle 2\int_0^{S/2} \frac{y^2\,dS}{I} + 2\int_0^{S/2} \frac{dS}{A}} \tag{20-6}$$

$$V_o = \frac{\displaystyle\int_0^{S/2} \frac{(m_L - m_R)x\,dS}{I}}{\displaystyle 2\int_0^{S/2} \frac{x^2\,dS}{I}} \tag{20-7}$$

The XX axis is properly located if

$$t\int \frac{dS}{I} = \int \frac{z_0\,dS}{I}$$

$$t = \frac{\displaystyle\int \frac{z_0\,dS}{I}}{\displaystyle\int \frac{dS}{I}} \tag{20-8}$$

These expressions are difficult to integrate and working formulas are obtained by substituting for the infinitesimal dS lengths of finite value, ΔS:

$$M_o = \frac{\displaystyle\Sigma\frac{(m_L + m_R)}{I}}{\displaystyle 2\Sigma\frac{1}{I}} \tag{20-5a}$$

$$H_o = \frac{\displaystyle -\Sigma\frac{(m_L + m_R)y}{I}}{\displaystyle 2\Sigma\frac{y^2}{I} + 2\Sigma\frac{1}{A}} \tag{20-6a}$$

$$V_o = \frac{\displaystyle\Sigma\frac{(m_L - m_R)x}{I}}{\displaystyle 2\Sigma\frac{x^2}{I}} \tag{20-7a}$$

$$t = \frac{\Sigma \frac{z_0}{I}}{\Sigma \frac{1}{I}} \qquad [20\text{-}8a]$$

In all cases the summation is over the half arch.

Rib Shortening. Equations 20-6 and 20-6a contain in the denominator a summation representing the effect of the work of direct thrust. The physical significance of this term should be carefully noted. As

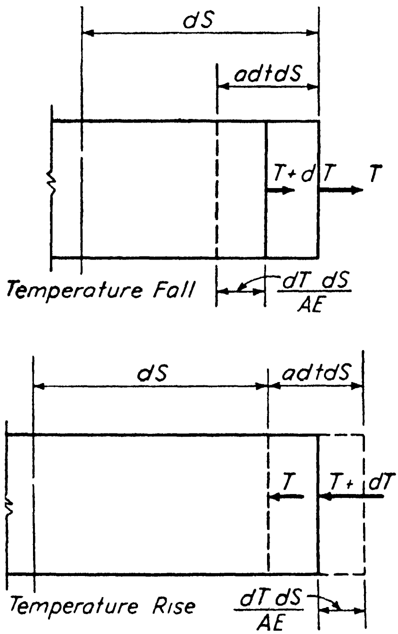


FIG. 20-16

the arch shortens from end to end under compression it tends to pull away from the abutments to which it is firmly attached, thus setting up a general tension with bending, positive in character at the crown, negative at the supports. Some arch designers compute this effect separately; it is of course simpler to include it in the general computation as here.

A drop in temperature has the same effect as rib shortening.

Temperature Stresses. The total work done in an arch by the forces set up by a change of temperature is

$$W = 2\int_0^{S/2} \frac{M^2\,dS}{2EI} + 2\int_0^{S/2} \frac{T^2\,dS}{2AE} - 2\int_0^{S/2} T\alpha t\,dS$$

where M and T are the moment and thrust due to temperature only, α is the coefficient of expansion, and t is the degrees of temperature change. The two terms giving the work done by the thrust may be understood by reference to Fig. 20–16, which shows a short length, dS, of an arch. It is assumed that a thrust, T, due to temperature rise is in action. A further rise of $dt°$ would cause a lengthening of $\alpha dt \times dS$ if there were no restraint, with a negative increment of work equal to $-\int\int T\alpha\, dt\, dS$. The restraint causes the thrust to increase to $T + dT$ and there is a shortening of $(dT \times dS)/AE$. The work done during this movement is

$$\int\int\left(T + \frac{dT}{2}\right)\frac{dT \cdot dS}{AE} = \int\int\left(T \cdot dT + \frac{dT^2}{2}\right)\frac{dS}{AE}$$

the term in dT^2 being extremely small and negligible. The expression for total work is $W = -\int T\alpha\, t\, dS + \int \frac{T^2\, dS}{2AE}$ as given. In the above expression for work $M = M_0 - H_0 y$ and $T = H_0 \cos\phi$. Substituting these values and differentiating the work with regard to the two variables in turn gives

$$\frac{\partial W}{\partial M_0} = 2\int_0^{S/2} \frac{M_0\, dS}{EI} - 2H_0 \int_0^{S/2} \frac{y\, dS}{EI} = 0$$

$$\frac{\partial W}{\partial H_0} = -2\int_0^{S/2} \frac{(M_0 - H_0 y)y\, dS}{EI} + 2\int_0^{S/2} \frac{H_0 \cos^2\phi\, dS}{AE}$$
$$-2\int_0^{S/2} \alpha t \cos\phi\, dS = 0$$

Since $dS = \frac{dx}{\cos\phi}$, $2\int_0^{S/2} \alpha t \cos\phi\, dS = \alpha t l$. Solving the first equation gives $M_0 = 0$ since $\int \frac{y\, dS}{EI} = 0$. The solution of the second is:

$$H_0 = \pm \frac{\alpha t l E}{2\int_0^{S/2} \frac{y^2\, dS}{I} + 2\int_0^{S/2} \frac{\cos^2\phi\, dS}{A}} \qquad [20\text{–}9]$$

For a rise of temperature the sign will be $+$, indicating compression.

A convenient working expression for H_0 may be obtained by substituting unity for $\cos^2\phi$ and by assuming that l, the span, equals the

arch axis in length, equals $2n\Delta S$, where n is the number of divisions of the half axis.

$$H_0 = \frac{\alpha tnE}{\Sigma \frac{y^2}{I} + \Sigma \frac{1}{A}} \qquad [20\text{–}9a]$$

This formula may be easily corrected for the second approximation by aid of the following table given by Cochrane in the paper referred to previously:

Lengths of the half-arch axis (S) in terms of the span length:

	Rise ratio $= r/l$				
	0.10	0.15	0.20	0.25	0.30
Open-spandrel Arch	$0.513l$	$0.529l$	$0.551l$	$0.577l$	$0.607l$
Filled-spandrel Arch	$0.515l$	$0.534l$	$0.559l$		

The above table gives accurate values of course only for arches whose axes conform to the equations used in preparing it.

Since the thrust was considered in writing the expressions for work, rib shortening has been taken account of in these formulas. The effect of thrust appears in the terms in $\Sigma \cdot 1/A$ in the denominators of the expressions for H_0 (equations 20–6a and 20–9a). By omitting these terms the horizontal component of crown thrust is obtained with rib shortening not taken account of. The difference between this larger value and that obtained with the term included is the crown thrust due to rib shortening.

20–8. Example of Arch Design. (Computation Sheets A1 to A9.) On Computation Sheet A1 are given the design data for an open-spandrel hingeless arch. The situation at the crossing fixed the clear span and rise at approximately the values shown, which were those finally used. The preliminary sketches and trials to determine these dimensions are not shown.

Throughout this design it is assumed that the arch ring is the main load-carrying element and that the floor system and columns act only to bring load to this main member. This is essentially correct for dead load stresses. Once the columns and floor have set they form an integral whole with the arch proper (unless a very complete set of expansion joints is placed in the deck, an arrangement which is not advisable), and necessarily participate in the work of the arch rib.* The theoretical analysis of the stresses in this system is very complicated and the use of experimental methods with models is advised.

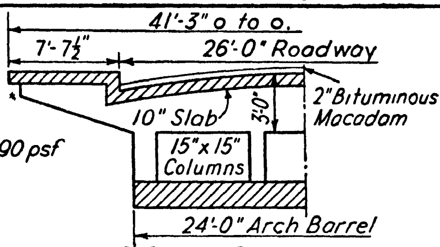
* See " Final Report of the Special Committee on Concrete and Reinforced Concrete Arches," Trans., A.S.C.E., 1935, Vol. 100.

HINGELESS ARCH DESIGN — Sheet A1

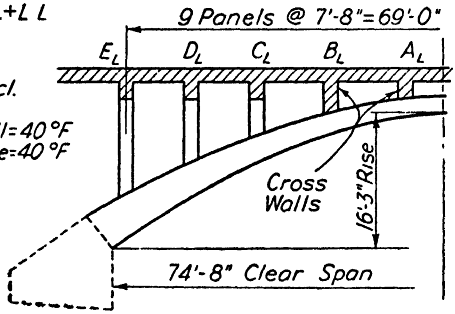
Data

Live Load:
On Floor & Sidewalk = 90 psf
Impact: 25%

Specifications:
$f_c' = 2\,500$ psi
$n = 12$
$\left.\begin{array}{l} f_c = 600 \text{ psi} \\ f_s = 18\,000 \text{ psi} \end{array}\right\}$ D.L.+L.L.
$f_s' = 40\,000$ psi (yield point)
Increase 25% incl. Temperature
Temperature Change: $\left\{\begin{array}{l} \text{Fall} = 40°F \\ \text{Rise} = 40°F \end{array}\right.$
$\alpha = 0.000006$



Half Cross Section



Half Longitudinal Section

Loads on Arch Barrel:
Dead: 10" Slab = 125 psf
Stem $\frac{350 \text{ plf}}{7.67 \text{ ft}} = \underline{46}$
$171 \times 41.25 = 7050$ plf
2" Bituminous Macadam @ 120 pcf
$20 \times 26 = \underline{520}$
$w_d = 7570 \div 24 = 315$ psf

Live + Impact:
Uniform $w_L = \frac{90 \times 41 \times 1.25}{24} = 192$ psf

Prelim Analysis

Whitney's Method - See Art. 20-6
Trial Dimensions: Crown Thickness $15" = d_c$
Springing Line Thickness $28" = d_s$
Span of Arch Axis $76.5' = l$
Rise of Arch Axis $16.1' = r$
Reinforcement 1% at crown
$A_s = 0.01 \times 15 \times 12 = 1.8$ sq in./ft
Use: $\frac{3}{4}\phi$ - 6" c/c each face = 1.76 sq in./ft

Data required for analysis
Moments of Inertia

$$I_c = \frac{12 \times 15^3}{12} + 11 \times 0.88 \times 2 \times \overline{5.5}^2 = 3\,960 \text{ in.}^4$$

$$I_s = \frac{12 \times 28^3}{12} + 11 \times 0.88 \times 2 \times \overline{12}^2 = 24\,700 \text{ in.}^4$$

Crown Section Area: $A_c = 12 \times 15 + 11 \times 0.44 \times 4 = 199$ in.²

Sheet A1

This example follows the method of the elastic theory throughout and assumes to determine the maximum unit stresses set up at the critical sections, assumed to be those at crown and springing: The quarter-point section, or haunch, is often taken as a possible critical point. From the discussion which precedes, the student will realize that the authors do not believe that this hitherto usual analysis yields dependable results. It is believed that the new method proposed by the A.C.I. Committee on Concrete Arches, already referred to, will become that generally accepted. This new method is discussed after the example and data are given to enable the student to redesign on this other basis.

20–8A. Preliminary Analysis. (Computation Sheets A1 to A4.) By use of a formula for crown thickness and by trial computations of the same sort as those carried through on these sheets but made with less precision, it was judged that the trial dimensions shown offered a reasonable solution worthy of more careful study. The area of the reinforcement at the crown was taken at the value usually chosen, about 1 per cent of the cross-sectional area at the crown. The same rods were used throughout the length of the barrel and so the steel ratio at the springing is considerably less than at the crown. The necessary data were computed for the application of Whitney's method. From the first draft of Sheet A4 an estimate was made of ϕ_s and y_0/r for computing values of m and N. Inspection of Figs. 20–7 and 20–8 and consideration of the negative rib shortening moment at the springing made it evident that it was not necessary to compute the maximum negative moment at the crown nor the maximum positive moment at the springing. With the aid of the references on the computation sheets all the details of this analysis should be plain.

The stresses found were considered satisfactory, the slight excess of compressive stress at the springing with temperature effect included not being sufficient to make a change necessary. The crown stresses are very low and a thinner section is possible. However 15 in. was judged as thin as it was desirable to use for good architectural proportions.

It should be remembered that this analysis is approximate only when the arch section differs from the theoretical basis of the method.

20–8B. Arch Axis. In the first draft of Sheet A4 no attempt was made to do more than approximate the shape of the axis and ring. This was sufficient for calculating the dead loads fairly closely. A careful determination of the curve of the axis followed the preliminary analysis. Instead of laying out the dead load equilibrium polygon on paper the elevation of this polygon at each panel point was computed, thus saving time, especially since the moments then computed were required for later steps in the analysis.

HINGELESS ARCH DESIGN	Sheet A2

Prelim Analysis Continued

Constants ·

$$m=\frac{I_c}{I_s \cos\phi_s}=\frac{3960}{24\,700\times 0\,70}=0\,23$$

$$N=\frac{Y_{\text{Quarter·point}}}{r}=\frac{3.55}{16\,1}=0.22$$

See Fig. 20-3·

For Live Loads: $pl = 192\times 76.5 = 14\,700^{\#}$

$pl^2 = 192\times \overline{76\,5}^2 = 1{,}120{,}000\ fp$

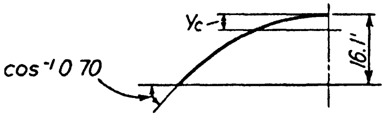
$pl^2/r \Rightarrow \quad 69\,800^{\#}$

Temperature Stresses: (Whitney)

For 40°F. Fall, $n=12$; $E=2{,}500{,}000\,psi$

$$H_T=\left(-2\,3\ \frac{40\times 3\,960}{\overline{16\,1}^2}\right)\frac{15}{12}=1760^{\#}$$

From Fig. 20-12, based on $n=15$



$Y_c = 0.21\times 16.1 = 3.38'$ (From Table 20-5)

$M_c = 1\,760\times 3.38 = +5\,950\ fp$
$M_s = 1\,760\ (16.1-3\,4) = -22\,400\ fp$
$T_s = 1\,760\times 0.70 = -1240^{\#}$

Maximum Positive Moment at Crown. (Whitney):
$pl^2 = 1{,}120{,}000\ fp \quad pl^2/r = 69\,800^{\#} \quad m=0\,23 \quad N=0.22$

	Thrust $=H_0$	Moment $=M_c$	Rib Shortening
Dead Live R S	*25 000 $0\,0665\frac{pl^2}{r}=4\,600$ -300	 $0049pl^2$ **+5 500 $296\times 3\,38 = 1\,000$	$u'=\dfrac{3960\times\frac{2}{25}}{199\times 0\,037\times 1.1\times\overline{16\,1}^2\times 144}$
Σ	29 300	+6500	(Equation 20-4)*** $=0.010$
D + L Temp. R S.	29 600 -1 800 -300	+5500 6000 1 000	$H_{RS}=0\,010\times 29\,600=-296^{\#}$ ***$C=0.037\times\frac{2.5}{2}$
Σ	27 500	12500	

*From trial computations like that following for dead load equilibrium polygon
**From Fig. 20-6

Approximate stress analysis - using Fig. A-17, assuming $\frac{d'}{h}=\frac{1}{10}$

Dead + Live + Rib Shortening: $p = 0\,005$ each face.

Eccentricity · $e=\dfrac{6\,500\times 12}{29\,300}=2.7''$ $\quad \dfrac{e}{h}=\dfrac{2.7}{15}=0\,18$

$$\text{Max.}f_c=\frac{6\,500\times 12}{12\times(15)^2\times 0.105}=280\,psi<600$$

Dead + Live + Temperature + Rib Shortening:

$$e=\frac{12\,500\times 12}{27\,500}=5\,5'' \quad \frac{e}{h}=0.37$$

$$f_c=\frac{12\,500\times 12}{12\times(15)^2\times 0.122}=460\,psi<750$$

SHEET A2

Dividing the moment about the springing of all the dead loads on one half of the arch by the rise of the axis results in the crown thrust. Since there is no moment at any point on the equilibrium polygon, the moment about any panel point of the dead loads between it and the crown, divided by the crown thrust, gives the distance to the polygon from the crown.

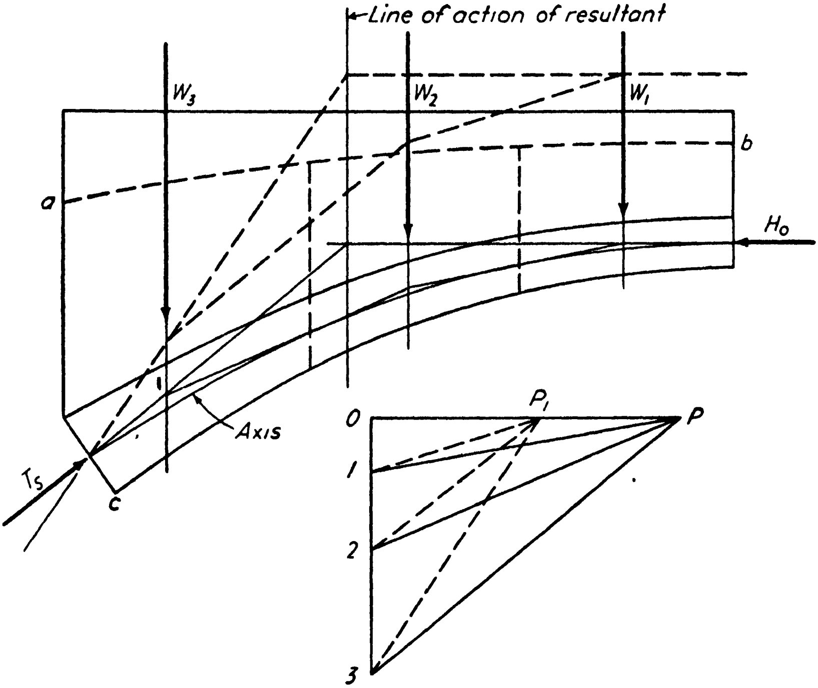


FIG. 20–17

The preliminary proportioning of an arch is illustrated in a general way in Fig. 20–17, which shows the half elevation of a filled-spandrel structure. The material above the arch ring weighs less per cubic foot than the masonry and so for convenience of graphic solution the dash line *ab* is drawn to show the height at which this material would stand if it were compressed to the same density as the masonry. The total dead load is then divided into any desired number of parts, as shown by the vertical dotted lines, the number depending on the span, 8 to 10 being common. For simplicity 3 only were used in the figure. It is equally correct to take the left limit of load through *c* instead of as shown. The line of action of each load division is found graphically and the magnitude computed. An equilibrium polygon is then passed

HINGELESS ARCH DESIGN — Sheet A3

Prelim Analysis Continued

Maximum Negative Moment at Springing (Whitney)

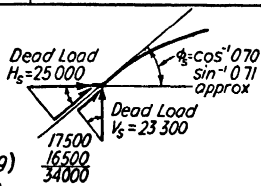
	T_S	M_C	Rib Shortening
Dead	See below +34 000		$H_{RS} = -Hu' = 0.010(25000+2500) = -275$*
Live	" " 5 200	$0.0202\,pl^2 = -22\,700$*	
R.S.	0.104 x 1850 −200	0.104 x 33 500 = −3 500	$\frac{275}{2640} = 0.104$
Σ	+39 000	−26 200	
Temp.	−1 200	−22 400	
Σ	+37 800	−48 600	

*From Fig. 20-9

Data for Max (−M) at Springing:

Dead Load Thrust at Springing:

$T = H \cos \phi_s + V \sin \phi_s$
$= 34\,000$



Live Load Thrust:

$H_L = 0.036 \frac{pl^2}{r} = 2\,500$* (From Fig 20-9)
$V_L = 0.34 \times 192 \times 76.5 = 5\,000$* (From Fig 20-4)
$T_L = 5\,200$*

Stress at Springing:

Dead + Live + Rib Shortening:

$$e = \frac{26\,200 \times 12}{39\,000} = 8.1'' \qquad \frac{e}{h} = \frac{8.1}{28} = 0.29$$

$$p = \frac{2 \times 0.44}{12 \times 28} = 0.0026 \text{ in each face}$$

$$f_c = \frac{26\,200 \times 12}{12 \times (28)^2 \times 0.107} = 310 \text{ psi} < 600$$

Dead + Live + Temperature + Rib Shortening:

$$e = \frac{48\,600 \times 12}{37\,800} = 15.4'' \qquad \frac{e}{h} = \frac{15.4}{28} = 0.55$$

$$f_c = \frac{48\,600 \times 12}{12 \times (28)^2 \times 0.098} = 635 \text{ psi} < 750$$

DEAD LOAD EQUILIBRIUM POLYGON

Load	Arm	Moment	Section	y_0
		0	A	0
A 3 870	7.67	29 700	B	1.19
B 3 985				
7 855	7.67	60 200		
		89 900	C	3.60
C 4 390				
12 245	7.67	94 000		
		183 900	D	7.35
D 4 990				
17 235	7.67	132 000		
		315 900	E	12.64
E 6 035				
23 270	3.74	87 000		16.11
	16.11)	402 900	Springing	
	H =	25 000		

Sheet A3

through crown and springing by the familiar method shown. The crown thrust is horizontal since the arch and loading are symmetrical.

It should be noted that the vertical divisions shown are independent of any divisions made of the arch ring for purposes of analysis.

By use of the data in Art. 20-6 the axis may be plotted directly without the drawing of an equilibrium polygon.

20-8C. Arch Ring. The following table (20-7) is given by Cochrane for laying out the arch ring. It is reproduced here for comparison with Whitney's section and also because of its ease of application. The two thicknesses computed by Whitney's method are somewhat thicker than those recommended by Cochrane and approximate those finally chosen. These two series of trial depths were plotted on Sheet A4 and smooth curves drawn for axis, intrados, and extrados. In a long-span arch it would be desirable to be much more precise in the layout, making the axis fit the dead load equilibrium polygon more exactly and making the thickness agree with the section recommended by Mr. Whitney.

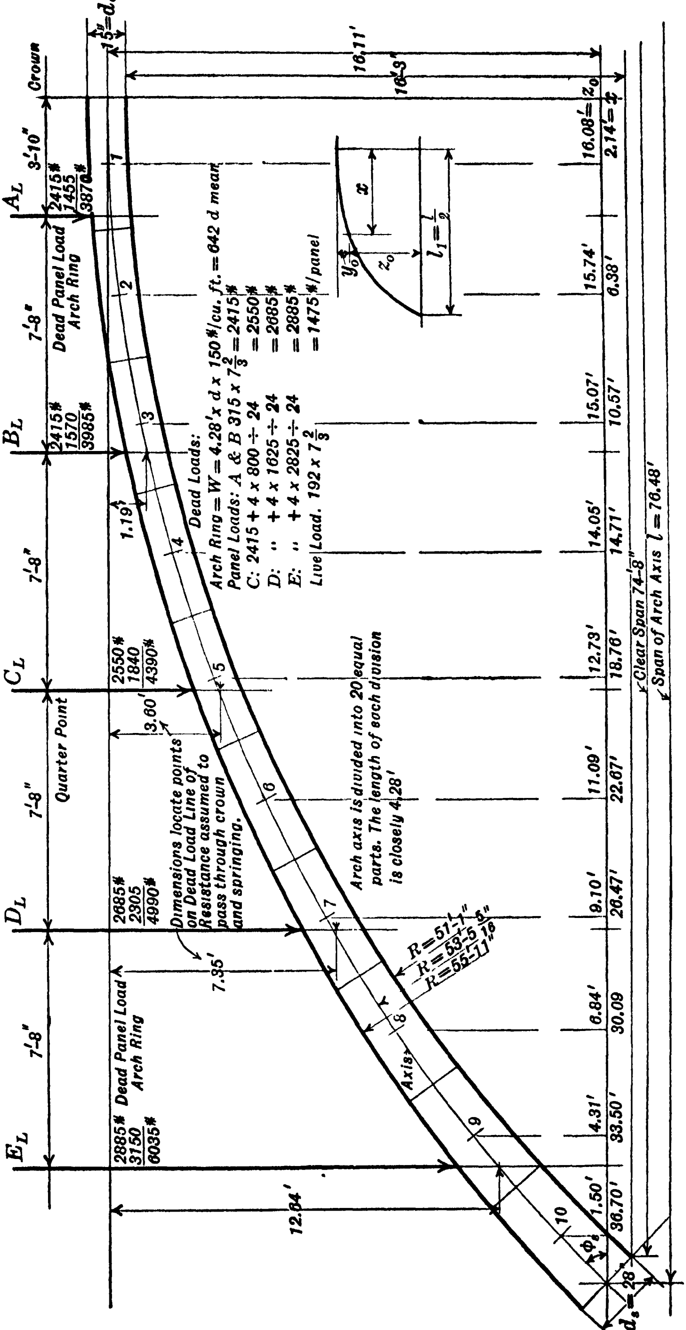
TABLE 20-7

THICKNESS OF TYPICAL ARCHES[1]

Value of $\frac{S_x}{S}$	Values of $\frac{d_x}{d_c}$							
	$\frac{d_s}{d_c}$							
	1.5	1.75	2	2.25	2.5	2.75	3	3.25
0	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
0.05	1.007	1.006	1.005	1.004	1.003	1.002	1.001	1.000
0.15	1.021	1.018	1.015	1.012	1.009	1.006	1.003	1.000
0.25	1.035	1.030	1.025	1.020	1.015	1.010	1.005	1.000
0.35	1.049	1.042	1.035	1.028	1.023	1.021	1.023	1.030
0.45	1.063	1.054	1.048	1.048	1.057	1.070	1.083	1.101
0.55	1.077	1.072	1.085	1.105	1.133	1.165	1.193	1.231
0.65	1.095	1.125	1.168	1.215	1.269	1.328	1.385	1.455
0.75	1.145	1.223	1.311	1.403	1.508	1.625	1.737	1.865
0.85	1.245	1.393	1 547	1.700	1 862	2.025	2.185	2.355
0.95	1.406	1.621	1.837	2.055	2.277	2.495	2.709	2.932
1.00	1.500	1.750	2.000	2.250	2.500	2.750	3.000	3.250

[1] Reprinted from a paper by Victor H. Cochrane, "The Design of Symmetrical Hingeless Concrete Arches," Proceedings of The Engineers' Society of Western Pennsylvania, Nov., 1916.

Of the three curves drawn, the extrados and intrados must be chosen and designated so that construction may follow the desired lines. Usually these curves are made up of one or more circular segments.


Crown
3'-10"
7'-8"
7'-8"
7'-8"
7'-8"
Quarter Point
Dead Panel Load
Arch Ring
Dead Loads:
Arch Ring = W = 4.28' x d x 150#/cu. ft. = 642 d mean
Panel Loads: A & B 315 x 7 2/3 = 2415#
C: 2415 + 4 x 800 ÷ 24 = 2550#
D: " + 4 x 1625 ÷ 24 = 2685#
E: " + 4 x 2825 ÷ 24 = 2885#
Live Load. 192 x 7 2/3 = 1475#/panel
Dimensions locate points on Dead Load Line of Resistance assumed to pass through crown and springing.
Arch axis is divided into 20 equal parts. The length of each division is closely 4.28'
Clear Span 74'-8"
Span of Arch Axis l = 76.48'
Axis

Sheet A4

In this case a single segment fitted the requirements and the result is a segmental or single-centered arch. If an arc of different radius had been used toward the springing it would have been called a three-centered arch. A smooth curve suffices for the axis and it is not necessary to determine its curvature, although it is often done as here.

20–8D. Analysis by Least Work. Arch Constants. (Computation Sheet A5.) From Sheet A4 the values of Z_0, d_x, and x were scaled and recorded as shown in the tables on Computation Sheet A5. In doing this work it is best to plan to do the final work on the layout and the scaling all on the same day. An overnight change of humidity may easily cause sufficient expansion or contraction of the paper to be very troublesome. A desirable scale is about 2 to 3 ft to the inch.

20–8E. Dead Stresses. (Computation Sheet A6.) No explanation of these figures should be necessary except to note that the computations of m_L are omitted.

20–8F. Influence Table. (Computation Sheets A7, A8.) It was assumed that the critical sections were those at crown and springing which Cochrane found to be true for arches conforming at all closely to his assumed proportions.

The only comment necessary for the explanation of this table is to call attention to the figure and formulas following it which explain the headings in the left-hand column. The moment and thrust at the right springing with loads on the left equal those at the left springing with loads at the corresponding points on the right. Therefore it was necessary to show only one-half the load points in the table. All the figures required by the computation are shown.

20–8G. Table of Maximum Stress. (Computation Sheet A8.) The influence table showed the panel points to load for maximum positive and negative moments at crown and springing and the values for a unit load. These coefficients multiplied by the live panel load gave the maximum stresses.

20–8H. Temperature Stress. (Computation Sheet A9.) Considerable doubt exists as to the accuracy of computations for temperature stress on account of the uncertain values of the coefficient of expansion and the modulus of elasticity. The values used here are representative. However, many designers believe that the temperature stresses should be computed for an equal rise and fall from the mean annual temperature.

20–8I. Summary of Unit Stresses. (Computation Sheet A9.) The values here found are well within the limits set and more precise computation is unnecessary. The steel stresses were approximated by aid of the dotted curves for f_s/f_c on Fig. A–17 (Appendix).

A comparison of the results with those of the preliminary analysis

HINGELESS ARCH DESIGN — Sheet A5

Prelim Analysis Continued

Arch Thickness (after Cochrane) $d_s/d_c = 1.87$ See Table 20-7

S_x/S	d_x/d_c	d_x (ft)	S_x/S	d_x/d_c	d_x (ft)
0	1.0000	1.250	0.55	1 079	1.348
0.05	1.0055	1.257	0.65	1.147	1 432
0.15	1.0165	1.270	0.75	1.267	1.580
0.25	1.0275	1.285	0.85	1.470	1.838
0.35	1.0385	1.298	0.95	1.729	2 160
0.45	1.0510	1.314	1.00	1 870	2.333

The section above is somewhat thinner than that used
Length of half-axis = 0.557 x 76.48 = 42.65 ft
(after Whitney)

Point (Sheet A4)	S_x/S	$z = x/38\,24$	$\sqrt[6]{1+\tan^2\phi}$	c	d_c	d_x
4	0 35	0 385	1 014	1.124	15	17.1
8	0.75	0 786	1 066	1.365	15	21.9

Comparison.

	Whitney	Cochrane	Used
4	17.1"	15.6"	16 7"
8	21.9"	19.0"	22.3"

Design by Least Work

Reference Axis:

Sect	z_o (ft)	d_x (ft)	$d_x^3/12$ (ft⁴)	$0\,0336(d_x-0\,33)^2$ (ft⁴)	I (ft⁴)	$1/I$ (ft⁻⁴)	z_o/I (ft⁻³)
1	16.08	1.25	0.163	0.0284	0 1914	5.22	84 1
2	15.74	1.27	0 171	0.0297	0.2007	4 98	78 4
3	15.07	1 31	0.187	0.0323	0 2193	4 56	68.7
4	14 05	1.39	0.224	0.0378	0 2618	3 82	53 7
5	12 73	1.48	0 270	0 0444	0.3144	3 18	40.5
6	11.09	1 58	0 329	0 0525	0 3815	2 62	29.1
7	9.10	1.70	0.410	0.0631	0 4731	2.11	19 2
8	6.84	1.86	0 536	0 0787	0.6147	1.63	11.1
9	4.31	2.03	0 697	0 0971	0 7941	1.26	5.4
10	1.50	2 20	0.888	0.1175	1.0055	0.99	1.5
						30 37	391.7

$$A = d_x + \frac{11 \times 2 \times 0.88}{144} = d_x + 0.1345 \text{ (sf)}$$

$$I = \frac{d_x^3}{12} + 0.1345\left(\frac{d_x}{2} - \frac{1}{6}\right)^2 = \frac{d_x^3}{12} + 0.0336(d_x - 0.33)^2 \text{ (ft}^4\text{)}$$

$$t = \frac{\Sigma \frac{z_o}{I}}{\Sigma \frac{1}{I}} = \frac{391.7}{30.37} = 12.90 \qquad y_c = r - t = 16.11 - 12.90 = 3.21 \text{ (ft)}$$

shows that the differences are small except as regards dead load moment. The true dead load equilibrium polygon does not pass through the crown and springing. It should be realized that on account of the shortening of the arch fibers under load there will always be moment under dead load at the crown and springing of a hingeless arch as usually constructed.

20–8J. Shrinkage and Plastic Flow. In the table below are given the maximum steel and concrete stresses at the crown when those due to dead load, live load, and temperature drop are combined with those due to shrinkage and plastic flow. The total for dead, live, and temperature drop is from Sheet A9; the direct shrinkage effect was computed as in Ex. 8–1*a*, page 113; the direct flow effect was figured by computing fiber stresses due to dead load crown thrust and moment with $n = 12$ and $n = 42$, the difference being the desired result; the rib shortening effect of shrinkage and flow used a unit shortening term in equation 20–5*a* of 0.00023, the sum of $s_a = 0.0002 + 0.000141$ for shrinkage and an assumed mean plastic shortening equal to 2700/30,000,000. This shrinkage and plastic rib shortening resulted in a crown tension of about 750 lb and a positive crown moment of 2400 lb-ft.

	Unit Stresses (psi)	
Stresses at the crown (extrados)	f_c	f_s
Dead + live + temperature drop	+500	+ 6,000
Shrinkage, direct	− 42	+ 4,200
Flow, direct	− 40	+ 3,400
Rib shortening, shrinkage, and flow	+ 40	+ 1,100
Total	+460	+14,700

Computation shows similar effects at the springing with a maximum steel stress of over 25,000 psi, well over any limit set by the usual specification. There is a question, however, whether concern need be felt here over the effect of shrinkage and flow more than in columns, provided the steel is well anchored to prevent buckling.

20–9. Arch Adjustment. Study of the maximum stresses at a series of sections the length of an arch will usually show a considerable range of variation: high stresses at certain critical sections, commonly those at crown, springing and haunch (quarter-point), and relatively low stresses elsewhere. A more economical design would result if means were found to secure greater uniformity, a leveling up and down from the extremes. Two methods have been devised for this purpose coupled with that of lessening the arch shortening stresses. The effectiveness of this latter function has been greatly questioned.*

* C. S. Whitney, "Plain and Reinforced Concrete Arches," *Journal, A.C.I.*, 1932, Vol. 28, p. 479. In rebuttal see the argument of Mr. A. L. Gemeny, *Journal, A.C.I.*, 1933, Vol. 29, p. 87. See also the references relating to the Freyssinet method in a later footnote.

HINGELESS ARCH DESIGN Sheet A6

Design by Least Work Continued

Arch Constants:

Sect	x (ft)	$y=z_0-t$ (ft)	x^2/I (ft^{-2})	y^2/I (ft^{-2})	$1/A$ (ft^{-2})
1	2.14	3.18	23.9	52.8	0.722
2	6.38	2.84	203.0	40.2	0.712
3	10.57	2.17	510	21.5	0.692
4	14.71	1.15	827	5.1	0.657
5	18.76	− 0.17	1120	0.1	0.620
6	22.67	− 1.81	1347	8.6	0.583
7	26.47	− 3.80	1480	30.5	0.545
8	30.09	− 6.06	1474	59.7	0.502
9	33.50	− 8.59	1414	92.9	0.462
10	36.70	− 11.40	1340	129.5	0.428
	38.24		9739	440.9	5.923

$$M_0 = \frac{\Sigma \frac{m_L+m_R}{I}}{2\Sigma\frac{1}{I} = 60.74\ ft^{-4}} \qquad H_0 = \frac{-\Sigma\frac{(m_L+m_R)y}{I}}{2\Sigma\frac{y^2}{I} + 2\Sigma\frac{1}{A} = 893.6\ ft^{-2}}$$

$$V_0 = \frac{\Sigma\frac{(m_L-m_R)x}{I}}{2\Sigma\frac{x^2}{I} = 19{,}478\ ft^{-2}}$$

Temperature Stress:

$$H_0 = \frac{\alpha t n E}{\Sigma\frac{y^2}{I} + \Sigma\frac{1}{A} = 446.8\ ft^{-2}}$$

Dead Stresses:

Sect	$m_L = m_R$	$\frac{m_L+m_R}{I}$	$\frac{(m_L+m_R)y}{I}$
1	0		
2	9.9	99	+295
3	26.1	238	+516
4	54.9	419	(1294) +483
5	86.7	552	−94
6	132.8	696	−1260
7	179.3	758	−2875
8	239.9	782	−4738
9	298.7	753	−6470
10	367.2	731	−8330
		5028	−22473

$M_0 = \frac{5028}{60.74} = 82.8$ k-f

$H_0 = \frac{22473}{893.6} = 25.1$ k (25.4 k without rib shortening)

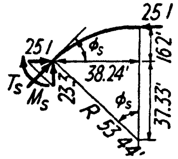
$M_c = 82.8 - 25.1 \times 3.21 = 2200$ fp (1300 fp without rib shortening)

$e = \frac{2200 \times 12}{25100} = 1.1''$ above axis $= 0.09$ ft

$M_s = -402.9 + 25.1 \times 16.20 = 3700$ fp (See Sheet A3)

$T_s = 25.1 \times 0.698 + 23.3 \times 0.715 = 34\ 200\#$

$e = \frac{3700 \times 12}{34200} = 1.3''$ above axis $= 0.11$ ft.



$\sin\phi_s = 0.715$
$\cos\phi_s = 0.698$

The first method devised for this purpose was the use of temporary hinges at crown and springings, which are closed with concrete after all or most of the dead load is in position. The arch acts as a three-hinged arch for dead load and is statically determinate, not affected by arch shortening. Since the major part of the settlement of abutments usually takes place during construction this element is thus largely eliminated. The hinges are so placed as to neutralize most effectively the arch-shortening moments, below the axis at the crown and above it at the springing.

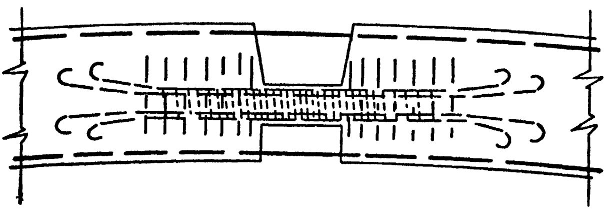


FIG. 20–18

As devised by Considère* such temporary hinges consist of short lengths of reduced cross section, heavily reinforced with longitudinal steel and spirals, designed for such heavy stress that the concrete is ductile and offers little or no resistance to bending. As shown in Fig. 20–18, a sufficient length of the main reinforcement is exposed so that it yields easily to the small movement that accompanies the adjustment.

The more recent method, best known through the work of Freyssinet† in France, consists in inserting hydraulic jacks between the two sections of the arch at the crown and exerting a thrust definitely known in magnitude and point of application. Precast concrete slabs are then inserted to carry the weight of the structure. They are made of such thickness that the previously determined crown thrust is brought into action. This process results in lifting the arch away from the centering which can accordingly be constructed in a simpler manner than usual, without devices for striking. By making the closure at the desired

* W. L. Scott, Reinforced Concrete Bridges, Crosby, Lockwood & Son, London; W. L. Scott, "Two Reinforced Concrete Bridges in France," *Engineering News-Record*, Dec. 6, 1923. It is stated by Mr. J. F. Brett (letter in *Engineering News-Record*, Nov. 5, 1925) that the saving from the use of temporary hinges amounts to 20 to 25 per cent of the cost of the structure.

† Charles S. Whitney, "Long Span Concrete Arch Design in France," *Engineering News-Record*, Sept. 18, 1924; E. Freyssinet, "Le Pont de Villeneuve-sur-Lot," *Le Genie Civil*, July 30, Aug. 6–13, 1921; A. L. Gemeny and C. B. McCullough, "Rogue River Bridge," *Journal, A.C.I.*, 1933, Vol. 29, p. 57; C. B. McCullough and E. S. Thayer, Elastic Arch Bridges, John Wiley & Sons, Inc., 1931, Chapter IX.

HINGELESS ARCH DESIGN — *Sheet A7*

Design by Least Work Continued

Stresses-Crown $A_c = 199 \text{ in.}^2$ $I_c = 3\,960 \text{ in.}^4$

$$\text{Max } f_c = \frac{25\,100}{199} \pm \frac{2200 \times 12 \times 7.5}{3960} = 126 \pm 50 = \begin{cases} 176 \text{ psi Top Fiber} \\ 76 \text{ psi Bottom Fiber} \end{cases}$$

Stresses-Springing $A_s = 355 \text{ in.}^2$ $I_s = 24\,700 \text{ in.}^4$

$$\text{Max } f_c = \frac{34\,200}{355} \pm \frac{3\,700 \times 12 \times 14}{24\,700} = 96 \pm 25 = \begin{cases} 121 \text{ psi Top Fiber} \\ 71 \text{ psi Bottom Fiber} \end{cases}$$

Influence Table-Data:

Sect	$m_L = (m_L + m_R = m_L - m_R)$	$(m_L + m_R)\frac{1}{I}$	$(m_L - m_R)\frac{x}{I}$	$(m_L + m_R)\frac{y}{I}$
		Load at A_L		
2	2.55	12.7	81	+ 36.1
3	6.74	30.7	325	+ 66.7
4	10.88	41.6	612	150.6 + 47.8
5	14.93	47.5	891	− 8.1
6	18.84	49.4	1119	− 89.4
7	22.64	47.8	1264	− 181.7
8	26.26	42.7	1285	− 259.0
9	29.67	37.4	1253	− 321.0
10	32.87	32.7	1200	− 373.0
		342.5	8030	− 1081.6
		Load at B_L		
4	3.21	12.3	181	+ 14.1
5	7.26	23.1	433	− 3.9
6	11.17	29.3	664	− 53.0
7	14.97	31.6	836	− 120.1
8	18.59	30.2	908	− 183.0
9	22.00	27.7	928	− 238.0
10	25.20	25.1	921	− 286.0
		179.3	4871	− 869.9
		Load at C_L		
6	3.50	9.2	209	− 16.6
7	7.30	15.4	407	− 58.6
8	10.92	17.8	536	− 107.9
9	14.33	18.0	603	− 154.6
10	17.53	17.4	639	− 198.4
		77.8	2394	− 536.1
		Load at D_L		
8	3.26	5.3	160	− 32.1
9	6.67	8.4	281	− 72.2
10	9.87	9.8	360	− 111.7
		23.5	801	− 216.0
		Load at E_L		
10	2.20	2.19	80.4	− 24.96

Load at A_L:

$$M_0 = \frac{342.5}{60.74} = 5.64 \qquad V_0 = \frac{8030}{19\,478} = 0.412 \qquad H_0 = \frac{1081.6}{893.6} = 1.210$$

Load at B_L:

$$M_0 = \frac{179.3}{60.74} = 2.95 \qquad V_0 = \frac{4871}{19\,478} = 0.250 \qquad H_0 = \frac{869.9}{893.6} = 0.974$$

Load at C_L:

$$M_0 = \frac{77.8}{60.74} = 1.280 \qquad V_0 = \frac{2394}{19\,478} = 0.123 \qquad H_0 = \frac{536.1}{893.6} = 0.600$$

Load at D_L:

$$M_0 = \frac{23.5}{60.74} = 0.387 \qquad V_0 = \frac{801}{19\,478} = 0.0411 \qquad H_0 = \frac{216.0}{893.6} \quad 0.242$$

Load at E_L:

$$M_0 = \frac{2.19}{60.74} = 0.0360 \qquad V_0 = \frac{80.4}{19\,478} = 0.00413 \qquad H_0 = \frac{24.96}{893.6} = 0.0280$$

SHEET A7

HINGELESS ARCH DESIGN — Sheet A8

Design by Least Work Continued

Influence Table:

	Unit Load at				
	A_L	B_L	C_L	D_L	E_L
M_O	5.64	2.95	1.280	0.387	0.0360
H_O	1.210	0.974	0.600	0.242	0.0280
$V_O = V_R$	0.412	0.250	0.123	0.0411	0.00413
V_L	0.588	0.750	0.877	0.9589	0.99587
$-3.21\,H_O$	−3.89	−3.128	−1.927	−0.777	−0.0899
M_{crown}	+1.75	−0.178	−0.647	−0.390	−0.0539
$12.90\,H_O$	15.61	12.56	7.74	3.120	0.3612
$38.24\,V_O$	15.76	9.56	4.71	1.573	0.1580
	37.01	25.07	13.73	5.080	0.5552
kL	−34.407	−26.740	−19.074	−11.407	−3.741
M_S left	+2.61	−1.67	−5.34	−6.33	−3.186
M_S right	+5.49	+5.95	+4.31	+1.93	+0.239
$H_O \cos\phi_S$	0.845	0.680	0.419	0.169	0.0196
$V_L \sin\phi_S$	0.420	0.536	0.627	0.685	0.712
T_S left	1.265	1.216	1.046	0.854	0.732
$H_O \cos\phi_S$	0.845	0.680	0.419	0.169	0.0196
$V_R \cos\phi_S$	0.295	0.179	0.088	0.030	0.0030
T_S right	1.140	0.859	0.507	0.199	0.0226

$M_C = M_O - 3.21\,H_O$

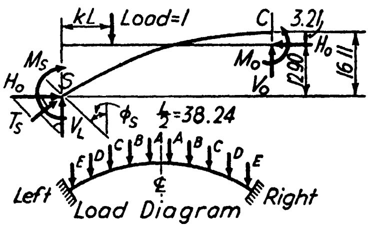
For load on left side

$M_{S_{left}} = M_O + 12.90\,H_O + 38.24\,V_O - kL$

$M_{S_{right}} = M_O + 12.90\,H_O - 38.24\,V_O$

$T_S = V \sin\phi_S + H \cos\phi_S$

$= 0.715\,V + 0.698\,H$ (Sheet A7)



Maximum Live Thrust and Moment at Crown and Springing:
Data from Influence Table

	Loading	Unit Load Values	Live Load Stresses
Moment (+) at Crown (−)	$A_L + A_R$ B, C, D, E right and left	+3.50 −2.54	5 160 pf 3 740
H_O with $+M_C$ " " $-M_C$		2.42 3.69	3 570 # 5 440
Moment at (+) Springing (−)	$A_L + A_R + B_R + C_R + D_R + E_R$ $B_L + C_L + D_L + E_L$	20.5 16.5	30 200 pf 24 400
Thrust with $+M_S$ " " $-M_S$		3.99 3.85	5 880 # 5 680

Live panel load = 192 × 7.67 = 1475 #

HINGELESS ARCH DESIGN

Sheet A9

Design by Least Work Continued

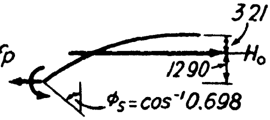
Temperature Stresses:

For 40°F Fall: $H_o = \frac{-0.000006 \times 40 \times 10 \times 2\,500\,000 \times 144}{446.8} = -1930^{\#}$ tension

$M_o = 1930 \times 3.21 = +6\,200$ fp

$T_s = 1930 \times 0.698 = -1350^{\#}$

$M_s = 1930 \times 12.90 = -25\,000$ fp



Shrinkage Reactions (15°F Fall)

$M_c = +2300$ fp

$T_s = -500^{\#}$

$M_s = -9400$ fp

STRESS SUMMARY AND UNIT STRESSES

Crown		Thrust H_o (lb)	Moment M_c (fp)	e [1] (in)	$\frac{e}{h}$ [2]	L [3]	f_c [3] (psi)	f_s [3] (psi)
Max +M	Dead	+ 25 100	+ 2 200					
	Live	+ 3 600	+ 5 200					
	Σ	+ 28 700	+ 7 400	3.1	0.21	0.110	300	3 000 ±
	Temp fall	- 1 900	+ 6 200					
	Σ	+ 26 800	+ 13 600	6.1	0.51	0.122	500	6 000 ±
Max −M	Dead	+ 25 100	+ 2 200					
	Live	+ 5 400	− 3 700					
	Σ	+ 30 500	− 1 500	0.6	0.04			No Tension
	Temp rise	+ 1 900	− 6 200					
	Σ	+ 32 400	− 7 700	2.9	0.19			

[1] $e = \frac{M_c}{H_o}$ [2] $\frac{e}{h} = \frac{e}{15}$ [3] From Fig. A-17, $f_c = \frac{M}{bh^2L} = \frac{M \times 12}{12 \times (15)^2 L} = \frac{M}{225L}$

Data: h = 15" b = 12" p = 0.0050 in each face

d'/h = 0.10 assumed · actual value = 2/15 = 0.133

Springing		Thrust T_s	Moment M_s	e	$\frac{e}{h}$	L	f_c	f_s
Max $+M_s$	Dead	+ 34 200	+ 3 700					
	Live	+ 5 900	+ 30 200					
	Σ	+ 40 100	+ 33 900	10.2	0.36	0.106	410	4000 ±
	Temp rise	+ 1 400	+ 25 000					
	Σ	+ 41 500	+ 58 900	17.0	0.61	0.097	775	13000 ±
Max $-M_s$	Dead	+ 34 200	+ 3 700					
	Live	+ 5 700	− 24 400					
	Σ	+ 39 900	− 20 900	6.2	0.22	0.101	260	2000 ±
	Temp fall	− 1 400	− 25 000					
	Σ	+ 38 500	− 45 900	14.3	0.51	0.099	590	9000 ±

Data: h = 28" b = 12" $p = \frac{15}{28} \times 0.0050 = 0.0027$ in each face

d'/h = 0.10 assumed: actual value $= \frac{2}{28} = 0.070$

temperature definite knowledge is had of the temperature which causes no stress, thus removing, it is asserted, another element of uncertainty in design.

20-10. Plastic Theory of Reinforced Concrete Design.* Since concrete is not truly elastic, the stress-strain curve being a parabola, and since its deformations under load increase with time because of shrinkage and plastic flow, the elastic or straight-line theory has long been recognized as frankly approximate.† Since shrinkage and flow are very indeterminate and impossible of prediction with any accuracy for any given structure, it appears to be a hopeless task to attempt to determine with any exactitude the stresses under any given load. It is possible, however, to derive a simple relationship which will predict with con-

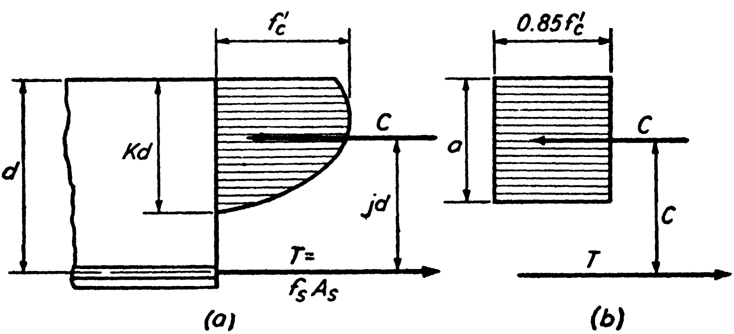


FIG. 20-19

siderable accuracy the ultimate strength of a reinforced concrete beam, a value little affected by shrinkage and flow since there is a large redistribution of stress with the large strains previous to failure. It thus becomes possible to design on the basis of ultimate strength, applying a definite factor of safety suitable to the conditions.

Beams. The stress distribution in a reinforced concrete beam at failure is shown in Fig. 20-19*a*, which employs the familiar standard notation: here f_s = yield point of the steel or the stress producing a total unit strain of 0.004. For the sake of simplicity in deriving a workable expression the actual stress curve for the concrete may be replaced by the rectangle of (*b*) of the figure, with the maximum stress intensity taken as also the average, $0.85f'_c$, a value which we know corresponds to the actual strength of columns (f'_c being the strength of a test cylinder). The moment of resistance may then be expressed as

$$M = 0.85f'_c bac$$

* From the paper of this same name by Charles S. Whitney, Proceedings, A.S.C.E., Dec., 1940, p. 1749.

† See also Art. 7-7.

A study of test results shows that we may take $a/d = 0.537$ and $c/d = 0.732$ which results in

$$M = 0.33f'_c bd^2 \tag{20-10}$$

If the beam is reinforced in compression as well as in tension, the compression steel being placed a distance D from that in tension, the moment of resistance will be increased, becoming

$$M = 0.33f'_c bd^2 + f_s A'_s D \tag{20-11}$$

f_s being the elastic limit stress as before. These equations assume that there is sufficient tensile steel to develop the strength in compression. The required tensile steel area can be found by equating total compression to total tension, the steel developing its yield point strength before failure.

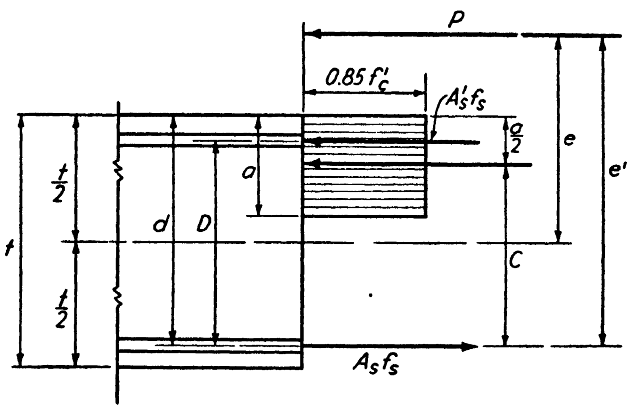


Fig. 20-20

Direct Stress and Bending. Equation 20-11 expresses also in terms of the total compression the strength of a member subjected to direct stress and bending. Such a situation is illustrated in Fig. 20-20, where P represents the resultant eccentric load on the section. We may write

$$M = P\left(e + d - \frac{t}{2}\right) = \frac{1}{3} f'_c bd^2 + f_s A'_s D$$

Solving for P, with $d = (t + D)/2$, the usual situation, gives

$$P = \frac{2A'_s f_s}{\dfrac{2e}{D} + 1} + \frac{btf'_c}{\dfrac{3te}{d^2} + \left(\dfrac{6dt - 3t^2}{2d^2} = 1.178\right)} \tag{20-12}$$

the numerical value of the second term of the second denominator being the value necessary to adjust the expression to the situation for axial

stress, $e = 0$, when the second term should equal $0.85f'_cbt$ in order to agree with test results. This equation assumes sufficient tensile steel to prevent tension failure.

In terms of the tension the strength can be written as follows for the case where the amount of compression reinforcement is large, so that the total compression can be assumed to be carried by the steel without help from the concrete

$$M = P\left(e - \frac{D}{2}\right) = f_sA_sD$$

$$P = f_sA_s\frac{2D}{2e - D} \qquad [20\text{–}13]$$

If the compression reinforcement is not sufficient to justify neglecting the concrete the procedure followed in developing equation 20–12 may be used, omitting the simplifying test values for a and c,

$$M = P\left(e + \frac{D}{2}\right) = 0.85f'_cabc + f_sA'_sD$$

The eccentric force on the section equals

$$P = 0.85f'_cab + f_sA'_s - f_sA_s$$

from which

$$a = \frac{P}{0.85f'_cb} + (p - p')mt$$

where pbt and $p'bt$ have replaced A_s and A'_s, and $m = f_s/0.85f'_c$. By substituting these values of a and m in the equation for M above an expression for P results which takes the following form for the case when $p = p'$: here $p_t = p + p'$:

$$P = 0.85f'_cbt\left[\sqrt{\left(\frac{e}{t} - 0.5\right)^2 + \frac{p_tmD}{t}} - \left(\frac{e}{t} - 0.5\right)\right] \qquad [20\text{–}14]$$

This equation assumes that there is tension on one face and so is not valid for small values of e: Equation 20–12 covers this case.

The application of Equations 20–12 and 20–14 to a given member and comparison with the results of testing these members to destruction are given by Fig. 20–21. It is noteworthy that when the moment is greater than that which would cause failure in the member acting as a beam without longitudinal load the ultimate moment increases with load P; also that in certain ranges a decrease in thrust without corresponding decrease in moment may cause failure. Accordingly, the

assumption of a fixed relationship between thrust and moment is not permissible; the effect of any possible combination of thrust and moment should be considered. A definite factor of safety, for example 2.5,

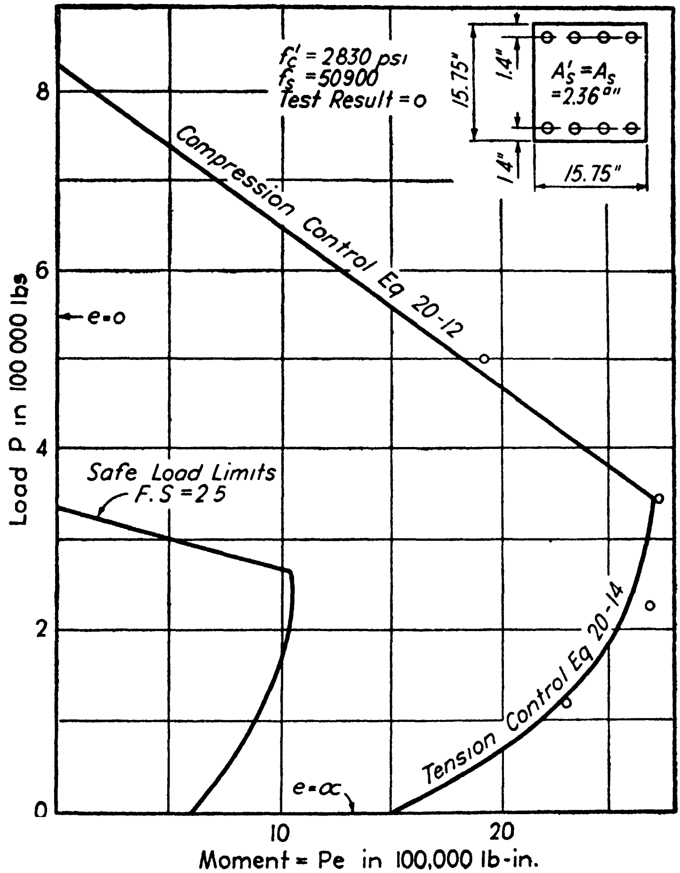


Fig. 20–21

as illustrated in the figure, is realized or exceeded if the load-moment combination falls within the zone shown; here the ordinates of the upper (straight-line) portion of the diagram are 40 per cent of the compression control ordinates and the abscissas of the lower curve are 40 per cent of the tension control values.

20–11. Arch Design by the Plastic Theory. The development of the plastic flow theory of reinforced concrete design leads to a simplification of the complex methods of analysis and control of shrinkage and plastic flow effects in arches. The Committee on Plain and Reinforced Con-

crete Arches of the A.C.I. (Charles S. Whitney, chairman) has recently presented its recommendations* which are here summarized. Briefly, these state that the moments and thrusts due to dead and live loads, to temperature and to shrinkage, shall be computed on the assumption that the arch is an elastic structure; the strength of the section which resists these thrusts and moments shall be determined by the plastic flow theory, a suitable factor of safety relating ultimate strength to actual load effects. The direct effect of plastic flow does not require investigation, as previously noted, since these are eliminated before failure.

The following items of the Committee specifications for arch design are to be noted in addition to what has already been stated. In order to reduce dead load distortion the arch axis should follow the dead load equilibrium polygon closely. A value of 4,000,000 psi is recommended for the concrete modulus except for computation of dead load deflections, where one-third to one-fourth of this value is suggested. The reactions due to temperature shall be calculated for a rise and fall equal to 30 per cent of the local temperature range using a coefficient of expansion of 0.0000055 per degree Fahrenheit. Shrinkage effect is computed by adding 15°F to the temperature drop (without change in the temperature rise).

Strength of Rib. "The ultimate strength of the rib as given by [Equation 20-8] shall not be exceeded at any section by the total bending moment due to the combined effects of dead and live loads, temperature change and shrinkage, in combination with twice the dead load thrust plus three times the thrust due to other causes. The ultimate strength of the rib [as given by equation 20-14] shall not be exceeded by the total thrust due to the combined effects of dead and live loads, temperature change and shrinkage in combination with two and a half times the total bending moment due to all causes."

The student should study the Committee report for discussion of these requirements and for further detailed recommendations.

* *Journal, A C.I.*, Sept., 1940, (Proceedings, Vol. 37), p. 1.

CHAPTER XXI

PLANS AND DETAILS

21–1. Engineering design is expressed by drawings. They must be clear, complete, and free from ambiguity for the sake of the work and must be made as economically as possible. Too often the latter consideration is allowed to outweigh the former.

21–2. Drawings. Concrete drawings have a double purpose, to show the outlines of the concrete and the size, shape, and location of the reinforcement. Information regarding both materials is usually combined on a single drawing. Where there are many complications it may be desirable to have separate " outline " and " reinforcement " drawings. The criterion for separate drawings is whether or not all necessary dimensions and data can be placed on a single drawing without confusion. Frequently schedules can be used to reduce the amount of information that would otherwise appear on the layouts themselves. A floor with many openings, or with machine pedestals, or both, will require a great many dimensions to show properly the size and location of corners in the concrete, and there may not be room left to show the reinforcement clearly. This is also true of large turbine foundations and similar elaborate constructions. Owing to the possibility of error in transferring dimensions, particularly the changes that are commonly made as the design progresses, it is advisable to keep all the information on a single layout unless it becomes absolutely impracticable. Frequently the detailing of the reinforcing steel in bands or panels which are merely marked on the plans and scheduled alongside will help to reduce the amount of lettering on the drawing.

The general method of making concrete drawings is similar to that of making structural steel drawings. But concrete drawings, particularly framing plans showing slab steel, are often more complicated than steel drawings for there are more members and they are more closely spaced than in steel construction. It is, therefore, particularly necessary that they are made carefully and clearly.

An assembly drawing which gives the size, location, and mark of each member (beam, column, wall, etc.) is the first drawing made for concrete as for steel. In steel construction the individual members are then detailed at large scale. In concrete construction it is frequently possible

to arrange the pertinent data of the individual members in a schedule, thus saving time and space. Members that differ considerably from routine construction and those involving peculiar conditions may be detailed at larger scale. Structural steel beams are ordinarily indicated on the framing plans by a single heavy line, and many engineers use a pair of closely spaced parallel lines to represent the sides of a concrete beam, thus differentiating the two materials when they appear on the same drawing.

Since the framing plan is an attempt to represent a three-dimensional structure on a plane surface, it should be supplemented by a rather considerable number of detail sections to a larger scale, showing conditions of clearance, head room, etc., at spandrel walls, elevator shafts, stairs, and wherever there is a possibility of interference or overlap.

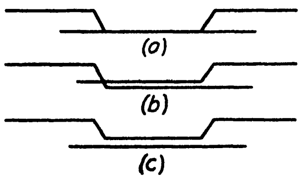
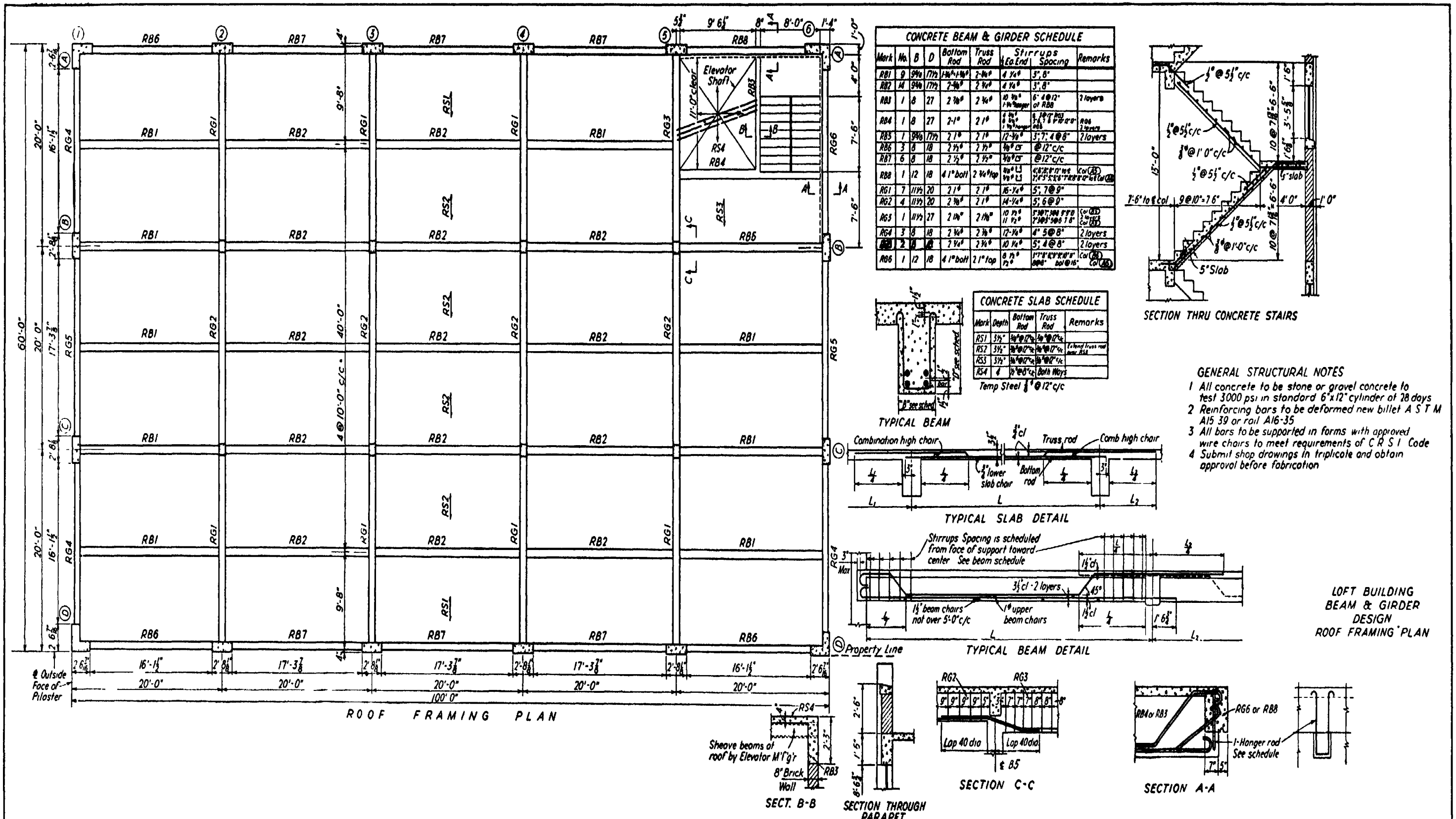


FIG. 21-1

Iuch time is saved in the preparation of aming plans if the required architectural onditions are shown thoroughly dimen-oned and the concrete members worked to conform to these conditions. This is necessary, first, because concrete members are fairly bulky and care must be taken to see that they offer no obstructions to the planning, and second, because concrete is more flexible than other building materials, because it is manufactured on the job to any size and shape required.

The ordinary principles of drafting suffice to show outlines. Third angle projection is standard in this country, though first angle is common in parts of Europe. In showing reinforcement on drawings there are certain conventions which are helpful and reinforcement drawings are, in general, diagrammatic in character. For illustration, the rods shown in Fig. 21-1*a* might in fact be diagrammed as in Fig. 21-1*b* or 21-1*c*, and properly should be shown as in one of these later views. Absolute clarity is of paramount importance. The amount of coverage over the bars can be taken care of by one general note on the drawing. Dotted lines are sometimes used to differentiate the reinforcement from outlines, but because of the possibility of ambiguity full heavy lines are better, and also quicker to draw.

21-3. Purpose of Drawings. In building construction several kinds of drawings are customary, especially on more important work. The architect's plans ordinarily show wall dimensions, columns, partitions, doors, windows, and similar features of the planning. The architect's engineers supplement these with sets of mechanical and structural plans. The former show provisions for plumbing, ventilating, electrical



CONCRETE BEAM & GIRDER SCHEDULE

Mark	No.	B	D	Bottom Rod	Truss Rod	Stirrups ½ Ea. End	Stirrups Spacing	Remarks
RB1	9	9⅝	17½	1-⅝ϕ+1-⅝ϕ	2-¾ϕ	4 ¼ϕ	3", 8"	
RB2	14	9⅝	17½	2-⅞ϕ	2 ¾ϕ	4 ¼ϕ	3", 8"	
RB3	1	8	27	2 ⅞ϕ	2 ¾ϕ	10 ⅜ϕ 1-¾ϕ hanger	6" 4@12" of RB8	2 layers
RB4	1	8	27	2-1ϕ	2 1ϕ	[illegible]	[illegible]	RB6 2 layers
RB5	1	9⅝	17½	2 1ϕ	2 1ϕ	12-⅜ϕ	3", 7", 4@8"	2 layers
RB6	3	8	18	2 ½ϕ	2 ½ϕ	⅜ϕ ⊔	@12" c/c	
RB7	6	8	18	2 ½ϕ	2 ½ϕ	⅜ϕ ⊔	@12" c/c	
RB8	1	12	18	4 1" bott	2 ¾ϕ top	⅜ϕ ⊔ ⅜ϕ ⊔	[illegible]	Col A5 Col A6
RG1	7	11½	20	2 1ϕ	2 1ϕ	16-¼ϕ	5", 7@9"	
RG2	4	11½	20	2 ⅞ϕ	2 1ϕ	14-¼ϕ	5", 6@9"	
RG3	1	11½	27	2 1⅛"	2 1⅛"	10 ½ϕ 11 ½ϕ	[illegible]	Col B5 2 layers Col A5
RG4	3	8	18	2 ¾ϕ	2 ⅞ϕ	12-¼ϕ	4" 5@8"	2 layers
[illegible]	2	[illegible]	[illegible]	2 ¾ϕ	2 ¾ϕ	10 ¼ϕ	5", 4@8"	2 layers
RG6	1	12	18	4 1" bott	2 1" top	8 ½ϕ ½ϕ	[illegible]	Col B6 Col A6

CONCRETE SLAB SCHEDULE

Mark	Depth	Bottom Rod	Truss Rod	Remarks
RS1	3½"	⅜ϕ@12"c/c	⅜ϕ@12"c/c	
RS2	3½"	⅜ϕ@12"c/c	⅜ϕ@12"c/c	Extend truss rod over RS4
RS3	3½"	⅜ϕ@12"c/c	⅜ϕ@12"c/c	
RS4	4	½ϕ@6"c/c	Both Ways	

Temp Steel ¼ϕ @ 12" c/c

GENERAL STRUCTURAL NOTES

1 All concrete to be stone or gravel concrete to test 3000 psi in standard 6"x12" cylinder at 28 days
2 Reinforcing bars to be deformed new billet A S T M A15 39 or rail A16-35
3 All bars to be supported in forms with approved wire chairs to meet requirements of C R S I Code
4 Submit shop drawings in triplicate and obtain approval before fabrication

FIG 21-2

work, heating, air-conditioning, elevators, escalators, and other mechanical equipment. The latter, together with the structural part of the specifications, give all necessary information on the structural skeleton. Since this type of drawing will be used by other engineers in preparing detailed shop drawings for the actual execution of the work, many simplifications and conventions are possible. As these drawings are a part of the contract documents on the basis of which payment is made and charges adjusted, they must be complete and thorough. Every possible question must be answered. However, by the use of notes, typical details, schedules, etc., it is possible to simplify the work. A few notes and typical details such as those on Fig. 21-2 will automatically determine the bending of every bar without putting the architect's office to the trouble of working it all out. The omission of such information would leave it to the detailer's judgment what to supply.

From these framing plans the contractor or, more often, some " bar company " prepares shop drawings showing the number, size, length, mark, location, spacing, and bending details of every piece of reinforcing steel in the project, and showing no more of the rest of the structure than is necessary for the intelligent detailing and placing of these rods in the forms. Here again schedules, notes, and bills of material are used to minimize the drafting work. Since these drawings are used by the placing crews they must be prepared in a manner that will be clear to them. These drawings are then checked by the architect for compliance with the contract requirements and the purposes of the design.

The contractor or his form builder usually prepares details of the formwork showing nothing but the outlines of the concrete and the methods of forming. Frequently drawings or schedules of each member are prepared as explained in Chapter VI and the bulk of the formwork is built on a bench and raised in place ready to fasten together. Meanwhile other trades are preparing shop drawings of all the various details that go to make up a modern building. The architect must correlate all these trades. Often adjustments in the location of members must be made, holes may be required in the concrete, or beams furnished for the support of hangers. Construction joints between the different pours affect the splicing of the reinforcement. All these details are worked out on the shop drawings, illustrations of which are shown.

For state highway bridge work, foundations for machinery, and some types of industrial buildings it is common to use one set of plans showing the concrete outlines, details of interconnected features, and complete information for the reinforcing steel, including shop bills of material

with marks and bending diagrams. This saves some duplication of effort, but can only be applied where the designer is fully conversant with all the details of all materials. It cannot be applied to large building work because considerable latitude is allowed each trade to work out its own best details, and until contracts are let and shop drawings are prepared, many details of construction are not completely determined.

Before starting on a drawing, the detailer must know for whom it is being prepared, what details and information to show, how completely they must be developed, and, especially, what to emphasize. Form drawings show the concrete outline heavy and completely dimensioned with other details that affect the concrete indicated lightly and just sufficiently developed to show their effect. Reinforcing details show the concrete outlines lightly and the rods in heavy lines. General design drawings must show all features, but a careful selection of various types of line, lettering, and cross-hatching can be made to differentiate the materials clearly.

Assembly drawings of concrete work are usually made at $\frac{1}{8}$ or $\frac{1}{4}$ in. to the foot and details at $\frac{3}{8}$, $\frac{1}{2}$, or sometimes $\frac{3}{4}$ in. to the foot. On account of their complexity it is unwise to use too small a scale and crowd the drawings.

21-4. Reinforcement. Slab and wall reinforcement naturally divides into groups or bands of identical parallel rods. Only the two outer rods of such bands are ordinarily shown on a plan, for to show all would confuse the plan to no advantage. These bands are then labeled as shown in Fig. 21-3. Rods which show in several views often should have type marks on them to identify them in the different views, but they should have a complete label and be called for in only one view to avoid duplication in taking off and ordering. It is important that rods be called for in the proper view. They should be listed with the part or member in which they will first be set during construction. For example, an angle rod extending from a wall into a floor, as shown in Fig. 21-4, should be listed in the wall detail rather than in the floor detail, for there will ordinarily be a construction joint at the top of the wall and the rods must be placed in pouring the wall. Other things being equal, it is clearer to list rods in a view where they show in elevation as straight lines as shown in Fig. 21-4, though many detailers lay the rods down in the plane of the paper so as to show the general nature of any bends in the rods. Ambiguity in listing as well as in drawing should be avoided: for example, listing as shown in *a*, Fig. 21-5, should be shown as in *b* or *c*, whichever is meant. Fig. 21-6 shows a part of a concrete floor plan illustrating the foregoing. A wall elevation

would be similar. It is often necessary to show a few cross sections of complicated places in walls and slabs in addition to the plan. The typical reinforcement can usually be shown laid over 90° in the plan, as in the illustration.

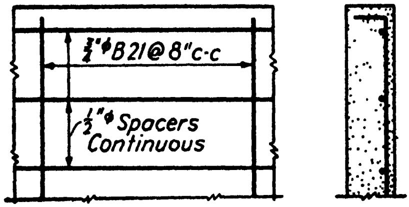


FIG. 21-3

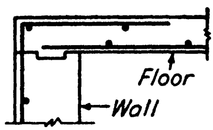


FIG. 21-4

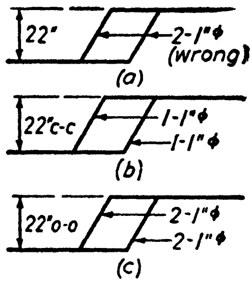


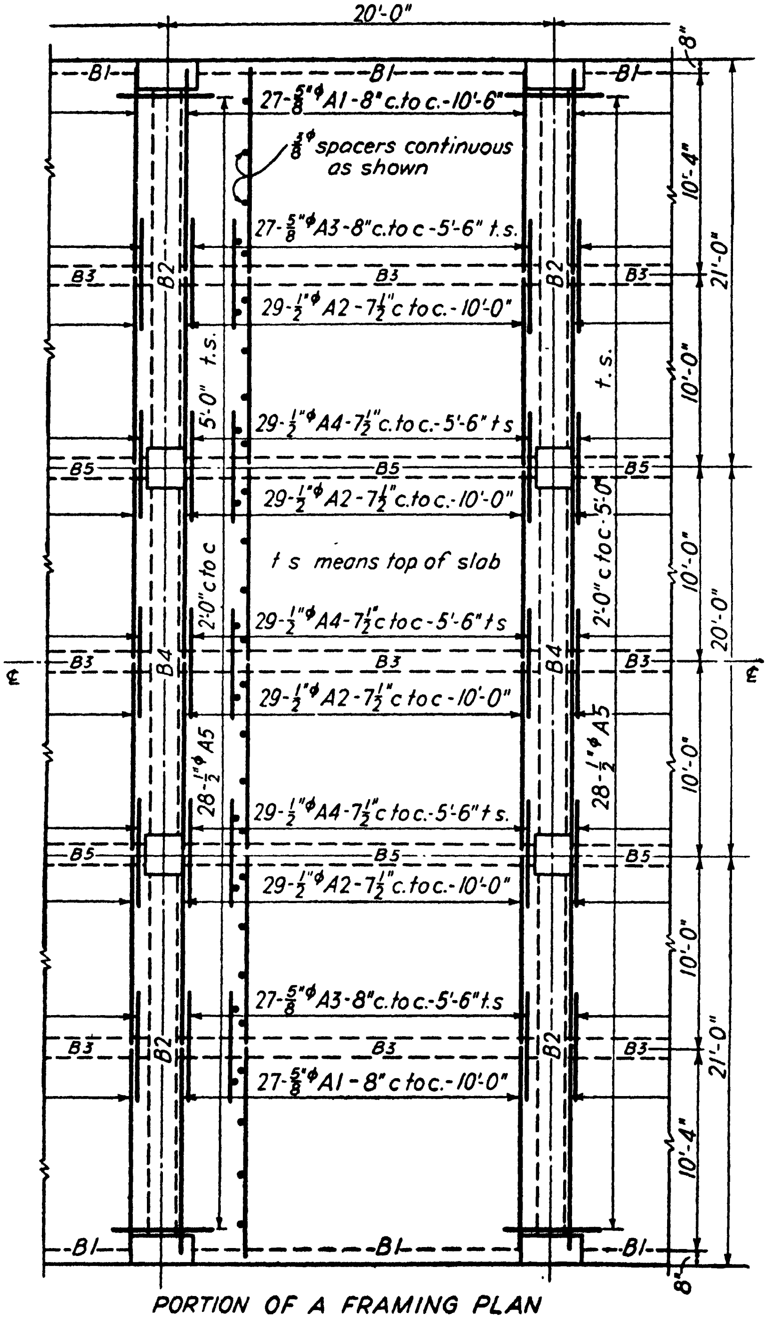
FIG. 21-5

The marking of bars is an aid to identifying material. Straight bars are either not marked at all or marked with a tag on each bundle giving the number, size, and length of bars. Bent bars are marked using prefixes as follows:

F	= footings	1G	= first-floor girders
D	= dowels	1J	= " " joists
W	= walls	1S	= " " slabs
1C	= columns basement to first	1T	= " -story ties
1B	= first-floor beams	1U	= " -floor stirrups

For upper floors the numbering is changed to 2, 3, etc. These prefixes are followed by serial numbers designating the individual bars. Many fabricators use a system designating the bar size, as

¼ in. round	= 200 and up	⅞ in. round	= 700 and up
⅜ in. "	= 300 " "	1 in. "	= 800 " "
½ in. "	= 400 " "	1 in. square	= 850 " "
½ in. square	= 450 " "	1⅛ in. "	= 900 " "
⅝ in. round	= 500 " "	1¼ in. "	= 1000 " "
¾ in. round	= 600 " "		



PORTION OF A FRAMING PLAN

FIG. 21-6

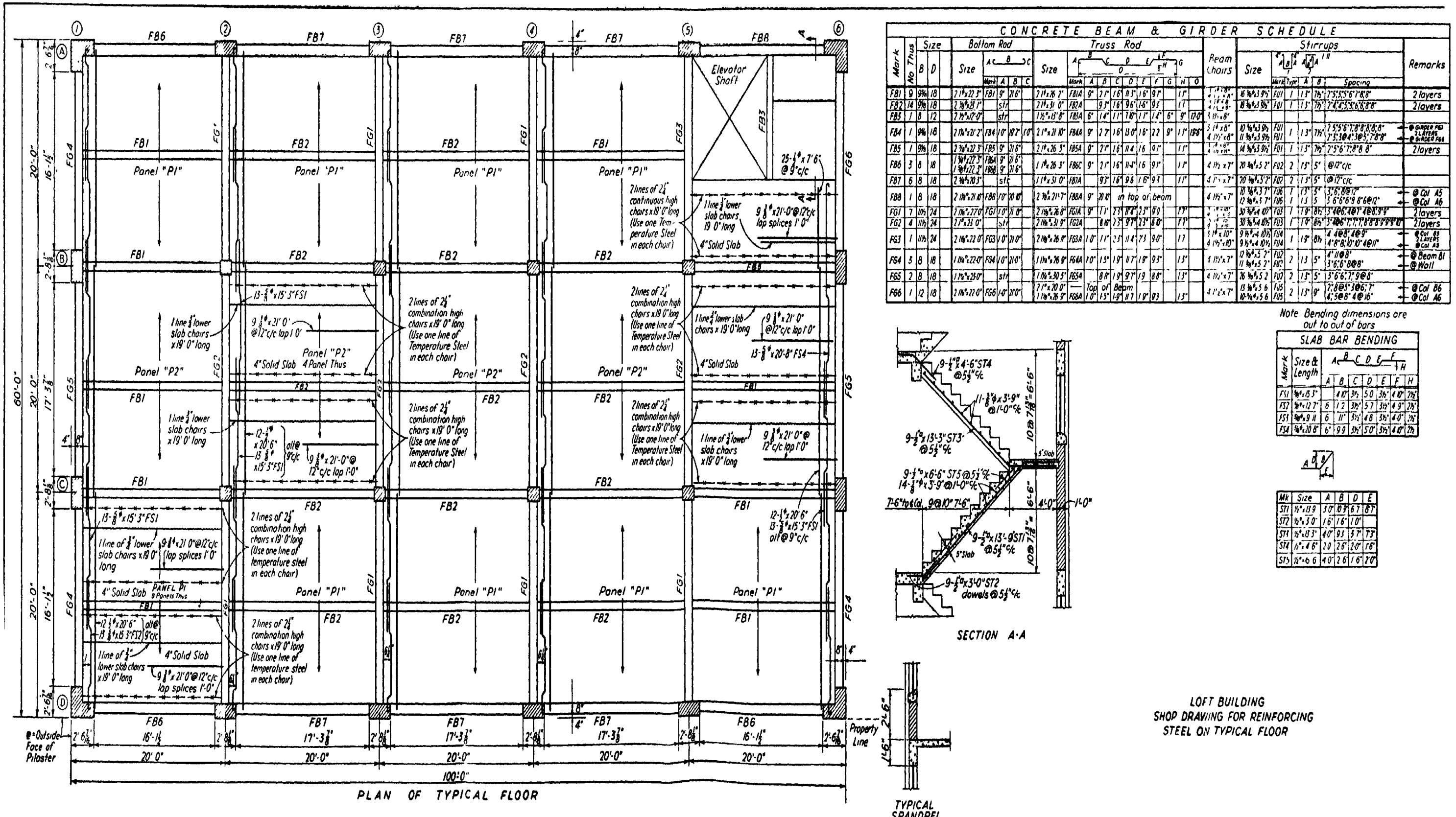

CONCRETE BEAM & GIRDER SCHEDULE
SLAB BAR BENDING
Note Bending dimensions are out to out of bars
SECTION A-A
TYPICAL SPANDREL
PLAN OF TYPICAL FLOOR
LOFT BUILDING
SHOP DRAWING FOR REINFORCING
STEEL ON TYPICAL FLOOR

Fig 21-7

Others use the same number as the mark of the beam, column or joist on the original framing plans. Each bundle of bars should be marked with a large metal tag giving the number, size, length, and mark of the bars in that bundle.

Beams are sometimes completely detailed in elevation and section as shown in Fig. 13-13, page 259. It is important that the detail show the location and angle of bends and the lap of rods beyond the center line. The length of the rods need not ordinarily be figured on the design drawing, nor need the stirrups be dimensioned, as the reinforcing contractor can figure them directly from the concrete dimension. This illustration shows the diagramming of rods over supports with actual location shown in section. Where there are numerous beams which are similar but not identical, as is the case in most building construction, they are preferably detailed by drawing one beam with instructions in place of dimensions, and then tabulating each beam. These methods as applied to simple building work are illustrated on Figs. 21-2 and 21-7.

In ordinary building work it is advisable to detail a typical interior and a typical exterior column in the same general manner that beams are detailed. The balance of the columns are then covered in a schedule similar to that shown in Fig. 21-8.

Irregular structures should be separated and detailed as slab, beam, and column units as far as possible. Where this is not possible, the foregoing principles, aided by good judgment, will give a satisfactory solution. It must be kept in mind that the prime purpose is to make clear the number, size, location, and bending of every bar. A drawing, like a contract, is not something that can be understood, but rather something that cannot be misunderstood.

The one requirement of bending sketches for steel reinforcement is that they be definite. It is not enough to show the height of a bend as in Fig. 21-5*a*; it should be indicated whether the figure is the desired dimension out to out, or center to center, or clear. If the total length occupied by a bar is limited, that limit (with proper allowance for clearance) should be explicitly stated, as otherwise there may be a variation of as much as 2 in. in the wrong direction. The sketches shown in Fig. 21-9 are those used on his order sheet by a large dealer in reinforcing steel; they show the dimensions required for proper bending of the steel. Other companies have similar standards that differ somewhat in detail. It should be remembered that the angle of bend in a bar like Type 1 or 2 is not so sharp as shown, since the bar is bent around a pin, usually with a diameter of about 3 or 4 in. Precision beyond the nearest whole inch in figuring the length C is not needed. This dimension is required only for computing the length of the bar. There is good reason for requiring

COLUMN SCHEDULE

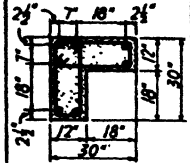
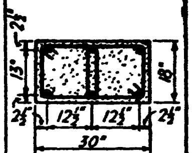
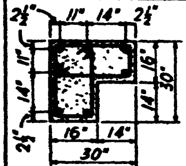
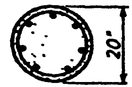
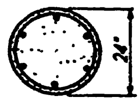
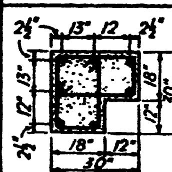
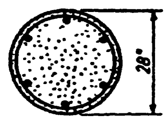
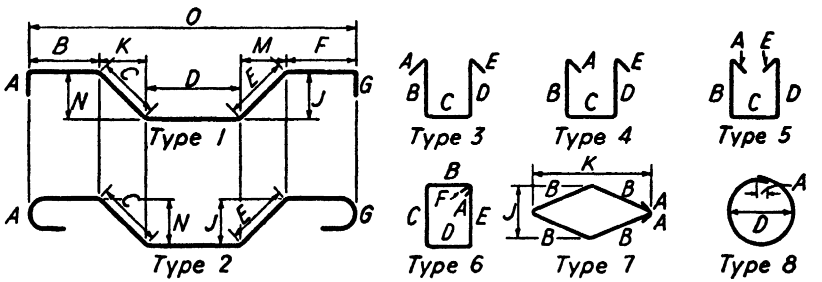
Story / Fin. Gr.	Typical Corner: Steel	Typical Corner: Section	Typical Interior: Steel	Typical Interior: Section	Typical Exterior: Steel	Typical Exterior: Section
Third 12'-0" Fin. Gr.	7-$\frac{5}{8}$"φ-11'-6" 30-$\frac{1}{4}$"φ ⊡-6'-6" 8" c-c in pairs		5-$\frac{3}{8}$"φ-11'-6" 15-$\frac{1}{4}$"φ ⊙-3'-9" 12" diam. 8" c-c		6-$\frac{3}{4}$"φ-11'-6" 30-$\frac{1}{4}$"φ ⊡-5'-0" 8" c-c in pairs	
Second 12'-0" Fin. Gr.	7-$\frac{3}{4}$"φ-14'-6" 30-$\frac{1}{4}$"φ ⊡-7'-6" 8" c-c in pairs		2-$\frac{5}{8}$"φ-10'-0" 5-$\frac{5}{8}$"φ-14'-0" $\frac{3}{8}$" Spiral-16"d 10'-0"-2"p		6-$\frac{3}{4}$"φ-14'-6" 30-$\frac{1}{4}$"φ ⊡-5'-0" 8" c-c in pairs	do
First 12'-0" Fin. Gr.	do	do	6-$\frac{7}{8}$"φ-15'-0" $\frac{3}{8}$" Spiral-20"d 10'-0"-2"p		do	do
Basement 10'-0" Fin. Gr.	7-$\frac{7}{8}$"φ-13'-0" 24-$\frac{3}{8}$"φ ⊡-8'-0" 8" c-c in pairs 7-$\frac{7}{8}$"φ Stubs-6'-0"		6-1"φ-13'-6" $\frac{1}{2}$" Spiral-24"d 8'-0"-3"p 6-1"φ Stubs-6'-8"		6-1"φ-13'-0" 24-$\frac{1}{4}$"φ ⊡-5'-0" 8" c-c in pairs 6-1"φ Stubs-6'-8"	do

FIG. 21-8

that this bend be gradual. Such a bar is heavily stressed and there are heavy bearing stresses brought onto the concrete at these points, tending to split the beam. Hooks, similarly, bring heavy splitting stresses to the concrete and require a large mass for embedment and often cross-reinforcement or an enclosing spiral, if they are to be effective.



All dimensions are out to out. Proper allowances for stretch of bars due to bending are made in shop. Standard wheels, about which bars are bent, are 3" diameter for $\frac{5}{8}$" and smaller; 4" diameter for $\frac{3}{4}$"; 6" diameter for $\frac{7}{8}$" and 1"; 8" diameter for $1\frac{1}{8}$" and $1\frac{1}{4}$".

Item	No	Size	Length	Mark	Weight	Type	A	B	C	D	E	F	G	J	K	M	N	O
1																		
2																		
3																		
4																		

FIG. 21–9

21–5. Details.* The best of designs may be futile if the details are neglected, and furthermore, even if requirements of strength are met, poor details may greatly increase the expense of the work. So careful attention to details cannot be too strongly recommended. Understanding of construction methods, observation, and experience are necessary to make a good detailer. An endeavor is made in this section to point out the general requirements and some of the common problems to be met.

Parallel with the use of arbitrary moment coefficients for determining the stresses in continuous structures has been the employment of rule-of-thumb standards for detailing. Probably every engineer of experience knows of instances where this procedure has resulted in structural damage due to lack of reinforcement in needed places. With the advent of the speedy modern approximate methods of analysis there is no longer possible excuse for such inadequacy. Higher competency is demanded of the detailer nowadays who must be able to determine from the en-

* An excellent summary of detailing is given in "Modern Developments in Reinforced Concrete," No. 2, p. 11, issued by the Portland Cement Association.

gineer's design sheets the required lengths and location of all bars. On important work and for the unusual structure the designer will find it necessary to give careful study to details and prepare sketches for the guidance of his subordinate in the drafting room.

21–5A. Forms. Concrete outlines must be such that they can be formed and the forms removed with reasonable facility. An example of consideration for formwork might be taken as FB4 on Fig. 21–2 which is pocketed for the later construction of stair slabs, and FB1 which is slotted for the steel sash. In both cases the stirrups must be detailed to clear the forms for these recesses. Forms are considered in Chapter VI, which should be read again in this connection.

21–5B. Rod Spacing. A theoretical minimum spacing for beam and slab rods can be deduced from the bond stress and the allowable shear on the concrete between the rods. Except for the largest rods this theoretical spacing is too close for good construction. The spacing must be such that the coarse aggregate will not arch between rods in pouring and cause voids. The clear distance between rods should be at least one and one-quarter times the size of the maximum aggregate,

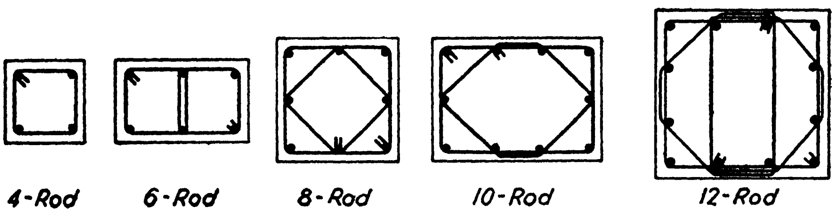


Fig. 21–10

according to the Joint Committee, which also establishes the minimum spacing of bars as $2\frac{1}{2}$ diameters on centers for round and 3 diameters for square bars (J.C. 504). The maximum spacing for slab rods is three times the slab thickness, according to the Joint Committee, as determined by the possibility of a concentrated load punching through between bars, but this is larger than in most common practice; it seldom exceeds one and one-half times or twice the thickness. Maximum spacing is never a consideration in beam rods, but as the steel is often in two or more layers it should be separated by steel spacers, usually 1 in. in diameter. Spacing of vertical column steel should be two to three times the aggregate size in the clear. It is desirable that every rod near the exterior surface of a column be securely tied in such a way that it cannot buckle and spall off the fireproofing. Fig. 21–10 shows satisfactory arrangements of ties for ordinary columns. As column steel is usually assembled into a unit and erected in one operation, such ties

should produce a rigid cage. In large columns it is sometimes impossible to get enough steel in a single layer and maintain proper spacing around the periphery of the spiral. In such cases it is possible to use a smaller spiral inside of the regular one and a second interior band of steel. Other detailers place such extra steel along diameters.

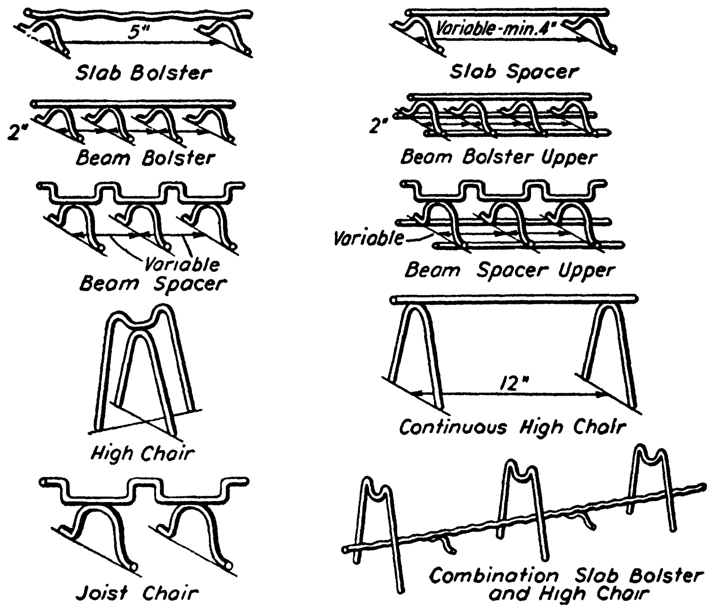


FIG. 21-11

21-5C. Spacers. Secondary reinforcement in slabs, walls, and miscellaneous structures at right angles with the main reinforcing is used for the dual purpose of distributing live load and holding the main reinforcement in place during the pouring of concrete. This latter purpose is of great importance. In ordinary slabs $\frac{3}{8}$ in. round rods 1 ft 0 in. to 2 ft 0 in. c to c are commonly used; in walls spacers are often made one size smaller than the main reinforcement and placed 1 to 2 ft apart, according to conditions. Stirrups in beams and hoops in columns tie the steel together. It is good practice to provide spacer bars under the hooks of stirrups.

Many forms of welded-wire chairs and supports are available to hold reinforcing steel accurately in place, some of which are illustrated in Fig. 21-11. The cost per ton of steel placed is a few dollars at most, and quite definite assurance is had that the steel will be held in the positions desired. For exposed ceiling slabs where no plaster is to be used the legs of chairs that come in contact with the forms are hot-galvanized

to prevent rusting and staining of ceilings. Specifications should call for such chairs to meet the requirements of the Concrete Reinforcing Steel Institute on all important work.

21–5D. Splices. In general it is good practice to minimize as much as possible splicing of reinforcing bars under stress. It is necessary to splice rods in columns and other compression members under stress. This is usually done by lapping them a sufficient distance to develop the stress in bond, ($L = f_s D/4u$ from page 103, which for compression steel stressed, say, 10,000 psi gives $L = 16\frac{2}{3}D$. J.C. 854*c* requires 24 diameters lap on structural and intermediate grades of billet steel and 30 diameters for rail or hard billet steel with a minimum of 1 ft 6 in.). Rods from the lower section to the number of those in the section above extend above the floor. For column verticals $\frac{7}{8}$ in. or larger it is customary to bend the rods to clear those above at the shop, smaller bars are pulled together in a sort of cone in the field. In bending column verticals ample allowance must be made in the offset to allow for bar diameters, and at the corners of columns to include the offset in two directions (Fig. 21–12). Such offsets should be made gradually at a slope of 2 in. per ft. (J.C. 503).

Tension rods carry greater stress than compression rods and when it is necessary to splice them under stress greater lap must be provided. (For 20,000 psi, $L = 33\frac{1}{3}D$, but it is customary to lap tension rods 40 or even 45 diameters.) In pure tension members, such as hangers, it is desirable to provide the above amount of lap and then hook the ends of the rods and encircle the surrounding concrete with a spiral, or closely spaced hoops, to prevent bursting of the concrete (Fig. 21–13). In standpipes and tanks rods were often fastened together with U-bolt cable clamps or by welding, but present practice often merely laps such bars generously (45 diameters) and staggers the splices in adjacent tiers of steel, a practice which has given satisfactory results.

In heavy cantilevers or long-span girders it is often desirable to drop off part of the steel as the bending moment falls off. It is obviously contrary to theoretical considerations to cut off rods under stress because of bond, and it has been customary to loop the ends. A better way is to bend the cut rods across the neutral axis at a flat angle and anchor with a standard hook in the compression side of the member as shown in Fig. 14–3, on page 279. This satisfies both practical and theoretical considerations.

As a general rule splices should be made at construction joints to facilitate field work. That is, rods will project the proper bond distance beyond the joint and then the next tier of rods can butt against the joint. Short dowel rods, called stubs, which extend bond distance each

side of the joint, are used to take steel stress into footings and are often useful in miscellaneous structures, being employed as extra splice rods by many engineers at construction joints in reinforced concrete walls.

Another reason for splicing rods is to keep reasonable lengths for shipping. For trucking or less-than-carload delivery 22 ft is the maximum; for special trucks or full carload shipment lengths may be made from 30 to 40 ft. Lengths up to 60 ft are readily obtainable and can be

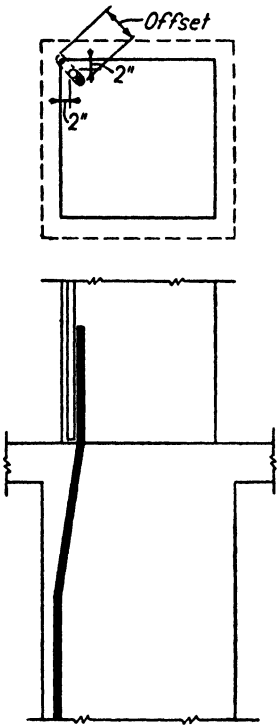


FIG. 21–12

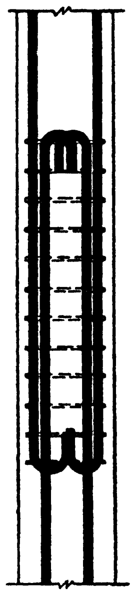


FIG. 21–13

shipped as a double carload or in special cars or by some haulers. Rods up to 90 or more feet have been rolled by special arrangement, but time is required for such special rollings and the handling of such bars in the field is often troublesome.

21–5E. Connections. In detailing reinforcing, interference at joints often needs attention. Small rods, ½ in. and less, can easily be bent in the field or allowed to sag into place. Larger rods are not pliable and interfering layers of steel at beam and girder intersections and beam steel dimensioned so as to intersect column rods should be avoided. On important work, and especially in rigid frame designs, a 1½- or 3-in.

scale layout is necessary to secure clearance at such intersections (Fig. 21-14).

The location of cambers or bends in beam and slab bars is a design consideration, although often left to the detailer. Many design drawings merely give a general instruction to bend up at the quarter-

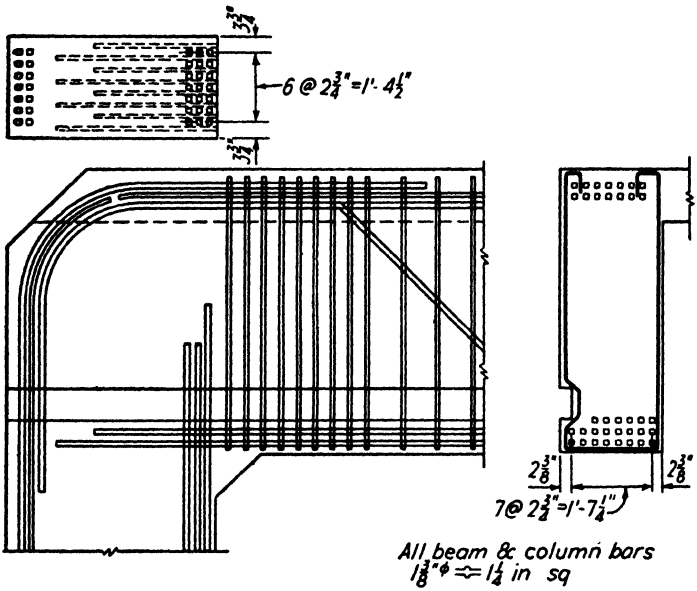


FIG. 21-14

and seventh-points of continuous and non-continuous spans respectively. Other designers schedule the distance from the center of the support to each point of bending down of truss rods.

21-5F. Construction Joints. In ordinary concrete work it is not possible to pour the entire structure in a continuous operation, and so there must be construction joints. The general characteristics of a construction joint are absence of tension value, weakness in shear, and reasonable strength in compression. Joints in floors, therefore, are made, as often as possible, at mid-span. Joints are often toothed or keyed by the insertion of blocks of wood in the first section poured; these are knocked out after the concrete has set. Horizontal joints are subject to the formation of laitance, a scum of the finest cement, which is soft. It is not possible to avoid this entirely, but it can be lessened by avoiding an excess of water. Laitance should always be cleaned off with a wire brush before proceeding to pour the next layer of concrete.

In miscellaneous structures joints should be located in so far as possible

with regard to convenience in splicing steel and setting up the next tier of forms; for example, at a point where there is a horizontal break in the structure. Joints in columns are usually fixed by floors or spandrel beams. Joints in floors are a field problem subject to general specification or the approval of the field engineer; but in many structures they should be considered by the designer and shown on the drawings. For example, in a tunnel of rectangular section it is necessary that there be a construction joint at the top of the floor and at the underside of the roof. It is also necessary that these joints be able to sustain in shear the reaction of the wall due to earth pressure. They must, therefore, be designed to resist pressure from one or both sides.

21-5G. Architectural Concrete. When concrete is used as the exposed building material for exterior walls some special precautions must be followed in detailing. In order to prevent air and moisture from reaching the steel and causing rusting, with consequent increase in size and the spalling off of the covering concrete, it is absolutely essential that a minimum of 1½ to 2 in. of protection on the exterior be specified and *obtained.* To prevent unsightly cracks at the corners of door and window openings, diagonal rods such as in Fig. 21-15 should be used. It should be realized that the presence of openings reduces the amount of wall section available to resist expansion and contraction and longitudinal steel should be added to compensate for the lost resistance. Since concrete shrinks while hardening, horizontal contraction and construction joints should be carefully planned. It is best to pour up to the underside of the window openings and stop at a horizontal joint. The next pour should carry to the head of the openings. The next pour can go either to the bottom of the second tier of openings or, if a floor slab intervenes, to the slab. It is better not to have two horizontal joints in the wall close together, as at the bottom and top of such a floor slab. Many detailers carry the wall unbroken up to the next window sill with a horizontal shelf and dowels for the reception of the floor, which is undoubtedly the best way. Others make one horizontal joint at the top of the slab, but introduce a belt course or an incised joint or similar detail so that the joint is not exposed on the flat surface of a spandrel wall. Such a joint is best made as in Fig. 21-16 with a horizontal strip on the exposed side to minimize laitance and to form a true horizontal surface. Such exposed walls require a considerable amount of reinforcing steel both horizontally and

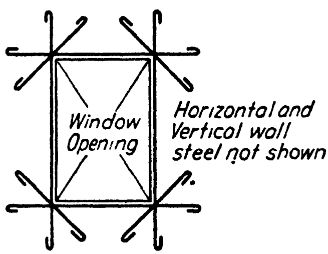
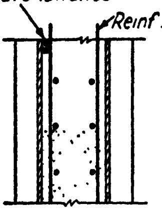


FIG. 21-15

vertically on each face of the wall, say about ¼ to ⅛ of 1 per cent of the total section in each of the four layers. Rods should be hooked generously around all corners, and hooked at all openings. The location of joints must be worked out in advance and the reinforcing steel cut to lengths that are appropriate.



Horizontal Construction Joint in Architectural Concrete Wall

FIG. 21-16

21-6. Examples of Design and Detail Drawings. Illustrations of methods of detailing various types of structures have been shown in the preceding chapters, notably: Fig. 6-10, forms; Computation Sheet W-6, page 169, retaining walls; Fig. 13-7, solid slab; Figs. 13-13 and 13-15, beams; Figs. 15-9 and 15-10, combined footings; Fig. 15-11, cantilevered footings; Fig. 16-1, ribbed slab; Fig. 16-10, two-way slab; Fig. 16-14, two-way flat slab.

Each of these shows only a portion of a structure but does indicate all the necessary information for building forms, for cutting, bending, and placing reinforcing steel, and for finishing the concrete surfaces that do not require forms. The data to be given on a complete set of plans would parallel these exactly except that, when duplication of members warrants, schedules under a type sketch may save time when compared with individual details, and except that general notes on the plans may replace some of the information shown on the individual details.

As examples of the methods of delineating reinforced concrete buildings two drawings are given: Fig. 21-2, which shows a roof framing plan for the building designed in Chapter XVII as it might be prepared by an architect or a consulting engineer, covering thoroughly all the requirements of the design, but mainly in the form of instructions or notes to the detailer who will later make "shop drawings," and Fig. 21-7, which shows a framing plan for this same building as it would be prepared by a contractor or a bar company giving the exact details of each and every piece of material. Ordinarily both sets of plans are prepared for any structure of good size.

The quality of concrete as determined by its 28-day strength and its slump should always be recorded, as well as that of the reinforcing steel. On the general design type of drawing notes are freely used to give the amount of concrete protection over the bars, the style and size of hooks, the amount of lap at splices, and any other restrictions to be observed in detailing. Sketches of standard beam, slab, and column sections are shown which give clearly the methods to be observed in bending rods. Notes are used to indicate special anchorage, size and spacing of column

ties, projection of dowels, offsetting of bars, and any special features of the design.

The complete shop drawing method of delineation requires considerably more time in its preparation. Standardized notes and details can be used to less advantage, although the competent detailer will minimize his work to the utmost consistent with absolute clarity. For example, there is no possibility here of merely marking special anchorage. The detailer makes the required provisions in the length and bending of the bars and from that point on it is simply a question of placing bars as shown. In this connection it should be noted that if an extra length of embedment is required for some reason on one end only of a bar, the skillful detailer will either add to *both* ends of the bar, so that when symmetrically placed in the forms there will certainly be embedment on the required end, or will very plainly dimension the end of the bar from a center line or face of a concrete member to insure that it is properly placed and that the embedment is not on the wrong end. It is quite difficult to inspect reinforcing steel in place, because of both the large number of small members closely packed together, and the fact that as soon as the reinforcement is securely wired in place the contractor is ready to pour concrete. For that reason it is well to make all bar arrangements as nearly symmetrical as possible and as simple as practicable, even at the expense of a small amount of unneeded material. To be a good detailer field experience in the using of the plans prepared and knowledge of the understanding and capacity of the men who will do the actual work are required.

These two drawings embody some very good standard notes and details that will repay close study by the reader. Another type of note is illustrated on Figs. 16-10 and 16-14 (pages 342 and 349), planned to aid in the work of installing the steel by preventing the necessity of threading any bars through steel already in place. If the notes are followed each bar is laid in place on top of bars already there. These notes also help the detailer because, if followed, they tell him in which layers different bars are located so that he can truss them to the proper depths.

In scheduling bars note that lengths are, in general, cut to the nearest 3 in. whereas truss depths and similar dimensions are given to the nearest inch or sometimes half-inch. No greater accuracy is ordinarily obtainable with the standard equipment for shearing and bending. The lengths of bottom beam and slab bars are obtained as the distance center of supports (or face to face of supports plus 1 ft) increased by any allowance required for special anchorage and any further lengths to insure that the bar is available as compression reinforcement for nega-

tive moment as illustrated on Fig. 13–13. The lengths of bent beam and slab rods equal the distance center to center of supports plus 0.8 times the out-to-out depth to allow for trussing plus the extensions into each adjoining span which are frequently taken to the quarter-points, but may be the full length if the adjoining span is short and negative moment can exist the full length.

For the method of design illustrated in Chapter XVII using arbitrary moment coefficients, it was considered satisfactory to bend up beam rods at the quarter-points of the span. If several bars were bent up the first were trussed at the quarter-points and successive bars nearer to the supports. The ends of these bars were extended through the supports and to the quarter-points of the adjacent spans. At non-continuous ends, bars were bent up at the seventh point. In some cases of greater refinement, plots of standardized moment curves using the arbitrary positive and negative coefficients were kept handy and bending was done in accordance with them.

The methods of design in Chapter XVIII review this same building by several more exact methods. A brief comparison indicates that the tendency is to reduce positive and increase negative moments, requiring more top reinforcement at the supports and possibly necessitating compressive reinforcement at this point. With these larger negative moments the inflection points move farther out into the spans; in fact, where shorter spans occur between longer ones, there may be negative moment all the way across. Under these methods considerable bending is transferred to the columns. The detailing of structures designed by more exact methods cannot proceed by any rule-of-thumb method. The detailer should visualize each joint as consisting of four cantilever beams as illustrated in Fig. 13–14*f* (page 260) and see that sufficient steel is provided at all points as shown on Fig. 13–12 (page 257). In obtaining bar lengths it is sufficiently accurate to treat the parabolic moment curve as a straight line between the maximum negative value and the point of inflection. For positive steel, however, it is much better to use the properties of a parabola between inflection points. Because the shear is the derivative of the bending moment, wherever there are rapid changes in moment high shears will result. At such points the detailer should be on the alert to see that adequate stirrups and ties are provided. The detailer of designs made by more exact methods must use extreme caution to see that reinforcing steel is arranged properly to care for all tensile stresses.

CHAPTER XXII

ECONOMY IN DESIGN

22-1. Good engineering design is, of course, economical design. With reinforced concrete, as with any other engineering material, really good design must of necessity be the product not only of a knowledge of the principles involved but also of considerable experience in their application. Sundry formulas have been published in the past which purported to give economical proportions of concrete members when the relative costs per unit volume of concrete and steel were known. In the main they were interesting rather than useful. Their lack of practical value was due to the fact that they did not include all the variables. In the opinion of the authors it is not practical to devise a workable formula of general application for economical proportions of concrete members.

22-2. Factors To Be Considered. The direct cost of a concrete member may be divided into three items: concrete, steel, and forms. These vary independently of each other. In addition to these, the effect of changed concrete dimensions on the cost of other parts of the building or structure must be determined. In considering a building column, for example, the value of the extra floor space taken by a lightly reinforced column must be considered in comparing its cost with that of a smaller hooped column.

The minimum clear story height of a building is usually determined by considerations other than structural. In comparing economy of section of typical floor beams, therefore, the additional length of columns, partitions, and walls must be debited against deeper beam sections. If the beam under consideration is typical for several floors this may be a large item.

One or more dimensions of concrete members are often directly determined by other than structural requirements. If a large proportion of the beams of a floor are typical, the others, except for secondary headers or trimmers at stairs and shafts, will be made the same depth. This will make simpler forming because the shores or supports for the forms will be the same length; the ceiling will look better and it will be more convenient for supporting shafting or pipes. A beam may have its width limited to that of a partition or, if a spandrel, may have both dimensions determined by architectural considerations.

22-3. Methods of Comparison. The method to be used in determining economical proportions is to make several designs, compute the cost of each, and select the cheapest which fulfills all fixed conditions, or determine whether some advantage such as appearance or the convenience of a more expensive method is sufficient to offset its extra cost over the cheapest. The design need not be complete in all details, nor do the unit costs need to be precise; relative costs are all that are needed. This is no royal road but it is the only one which leads to satisfactory results.

In applying this method it is also necessary to consider when and how to apply it. It would be obviously absurd to make such analysis of each member of a building and equally absurd to omit entirely such computations of an important structure. In practice, consideration of a typical bay of a reasonably regular building or a few isolated sections of an irregular building is sufficient.

Certain members should be considered as a group in computations of this character because of their interdependence; for example; a two-or three-span beam across a building or a complete column. On a structure figured for wind or other loads in addition to the floor loads it will usually be best to consider a complete bay or bent as a unit in estimating comparative costs.

This consideration of a complete bay is also necessary in studying the most difficult problem of this type, namely, special or patented types of construction. The great difficulty of this type of problem lies in the personal equation of the contractor, for different contractors will often disagree entirely on costs of unusual or patented systems. Since the economy of patented systems often hinges on savings in certain unit costs, or extra labor which offsets material savings, the best that the engineer can do in such a case is to check over his unit prices with one or more of the contractors who are expected to figure the work, get bids for any patented material or forms, and get estimated costs on any special work the contractor has to do. It is better to get prices from the contractor rather than from a salesman, although the latter may at times be a great help.

A good understanding of making comparative cost studies is obtained from a very careful consideration of the following examples which compare costs for alternative designs of individual members, and of the comments that accompany them. These range over a fair variety of possible alternatives, such as maximum versus minimum reinforcement in tied columns, rich versus lean mixes for tied columns, tied versus spirally hooped columns, thin versus thick slabs, lean versus rich mixes for solid slabs, deep versus shallow beams, and whether there is

ECONOMIC STUDY: COLUMNS
Example 22-1

Sheet ES1

Floor Suppd	Load	Design ($p = 0.01\pm$) $P = 540\,A_g + 12\,800\,A_s$	Design ($p = 0.04\pm$) $P = 540\,A_g + 12\,800\,A_s$
R	46.1	$(9\frac{5}{8})^2$ = 50.0 $4\text{-}\frac{5}{8}^{\phi}$ = 15 9 (1.34%) 65.9	$(9\frac{5}{8})^2$ = 50.0 $4\text{-}\frac{5}{8}^{\phi}$ = 15.9 (1.34%) 65.9
3	127.9	$(13\frac{1}{2})^2$ = 98.4 $4\text{-}\frac{7}{8}^{\phi}$ = 30 7 (1.32%) 129.1	$(11\frac{1}{2})^2$ = 71.4 $8\text{-}\frac{7}{8}^{\phi}$ = 61.4 (3.63%) 132.8
2	205.8	$(17\frac{1}{2})^2$ = 165 4 $4\text{-}1^{\phi}$ = 40 2 (103%) 205.6	$(14\frac{1}{2})^2$ = 113.5 $6\text{-}1\frac{1}{8}^{\square}$ = 96 8 (3.60%) 210.3
1	280.0	$(20\frac{1}{2})^2$ = 226 9 $10\text{-}\frac{3}{4}^{\phi}$ = 56 3 (1.05%) 283.2	$(16\frac{1}{2})^2$ = 147.0 $10\text{-}1^{\square}$ = 128.0 (367% 275.0

Comparative Costs:

$p = 0.01\pm$

Forms
4 x 0.80 x 13.5 = 43.2 sf
4 x 1.13 x 13 5 = 61.0
4 x 1.63 x 13.5 = 88.0
4 x 1.88 x 11.5 = 86 5
278 7 @ 25¢ = \$69.68

Concrete
$(0.80)^2$ x 13.5 = 8.7 cf
$(1.13)^2$ x 13.5 = 17.3
$(1.63)^2$ x 13.5 = 35.9
$(1.88)^2$ x 11.5 = 40.6
27)102.5
3.80 cy @ \$9 50 = \$36.10

Steel
4 x 1.043 x 13.5 = 55.5 lb
4 x 2.044 x 15.7 = 128.4
4 x 2 670 x 16.0 = 170.9
10 x 1.502 x 17.1 = 256.9
611.7 @ 5¢ = \$30 59
\$136 37

Floor Space
$(1.13)^2$ = 1.28
$(1.63)^2$ = 2.66
$(1.88)^2$ = 3.53
7.47 @ \$2.50 = \$18.68
\$155.05

$p = 0\ 04\pm$

Forms
4 x 0.80 x 13.5 = 43.2 sf
4 x 0.96 x 13 5 = 51 8
4 x 1.21 x 13.5 = 65.3
4 x 1 38 x 11 5 = 63 5
223 8 @ 25¢ = \$55.95

Concrete
$(0\,80)^2$ x 13 5 = 8 7 cf
$(0.96)^2$ x 13.5 = 12.5
$(1.21)^2$ x 13.5 = 19.8
$(1.38)^2$ x 11.5 = 22.1
27)63.1
2.34 cy @ \$9.50 = \$22.23

Steel
4 x 1 043 x 13 5 = 55.5 lb
4 x 2 044 x 15.7 = 256 8
6 x 4 303 x 16.3 = 420.8
10 x 3.400 x 19.0 = 646.0
1379.1 @ 5¢ = \$68.96
\$147.14

Floor Space
$(0.96)^2$ = 0.92
$(1.21)^2$ = 1.47
$(1.38)^2$ = 1.90
4.29 @ \$2.50 = \$10.73
\$157.87

any economy in compressive reinforcement for beams. The next article discusses the economics of complete floor systems as opposed to the individual members discussed immediately below.

Example 22-1. Is it more economical to use the minimum or the maximum allowable percentage of reinforcement in designing the column stack of Ex. 13-4, page 269, if unit costs are: forms 25 cents psf, concrete \$9.50 per cu yd, and steel 5 cents per lb?

Solution. According to the 1940 J.C. Code upon which Ex. 13-4 was based, the allowable percentage of longitudinal steel may vary from 1 to 4 per cent. On Computation Sheet ES1 are recorded the loads from Ex. 13-4 and the design for the two conditions. The percentages of steel in each case are computed for comparison. Here column sizes are taken to the nearest half-inch, and steel combinations are chosen closely to reflect accurately the differences in cost. A quantity take-off is then made for each condition. Bar lengths include a 30 diameter lap above each floor line, and a 60 diameter length for footing dowels. Concrete and form quantities are taken from floor line to floor line. The quantities are then extended at the given units and added to obtain the total cost. The difference in cost is not a large item but, if saved on each stack of columns without detriment to the use of the building, it is worth while. Note that the formwork is the largest individual item of cost, frequently about equal to the sum of the concrete and steel. The second design reduces the cost of the relatively inexpensive concrete and of the forms and increases the cost of the expensive reinforcing steel.

Finally an adjustment is made for the value of the floor space displaced by the column. This is taken at \$2.50 psf, which is reasonable as structures of this sort vary in cost from a little over \$2 to considerably over \$3. Whether or not to include this value in the comparison depends upon the use of the building. For warehouse storage every cubic foot of space is valuable, and the column displacements detract from the storage capacity. In a mercantile building it is possible that columns could be included in bundle-wrapping and service spaces without affecting the value of the building but, on the other hand, they might occur in the space behind counters in such a manner that the entire counter arrangement would have to be spread out to give more passage space. In this case the larger columns not only displace their occupied area, but a strip the whole length of the counter, though, in such case, the columns could be made rectangular and of minimum width perpendicular to the counter.

Example 22-2. Is it more economical to use concrete of 2000, 3000, 3750, or 5000 psi for the column stack of Ex. 13-4 (page 269)?

Solution. On Computation Sheet ES2 are recorded the design loads for each floor and comparative designs for each case. In pricing, the forms and steel are taken at the same units in all three cases, but the concrete varies in each instance, mainly because of the difference in cement content. Such variations in unit prices must be accurately and carefully done or the comparative costs are of no value. Note that in spite of the higher cost of concrete and the fact that the roof columns are of minimum size in all cases, the richer mix is the most economical even without considering the saving in floor space. This is practically always true, and columns are ordinarily of as rich a mix as can be obtained with absolute surety.

ECONOMIC STUDY: COLUMNS
Example 22-2 Sheet ES2

Floor Supptd	Load	2000 psi $360A_g + 12\,800A_s$	3000 psi $540A_g + 12\,800A_s$	3750 psi $675A_g + 12\,800A_s$	5000 psi $900A_g + 12\,800A_s$
R	46.1	$(9\frac{5}{8})^2 = 33.4$ 4-$\frac{5}{8}\phi$ = 15.9 49.3	$(9\frac{5}{8})^2 = 50.0$ 4-$\frac{5}{8}\phi$ = 15.9 65.9	$(9\frac{5}{8})^2 = 62.5$ 4-$\frac{5}{8}\phi$ = 15.9 78.4	$(9\frac{5}{8})^2 = 83.4$ 4-$\frac{5}{8}\phi$ = 15.9 99.3
3	127.9	$(16)^2 = 92.1$ 6-$\frac{3}{4}\phi$ = 33.8 125.9	$(13\frac{1}{2})^2$ = 98 4 4-$\frac{7}{8}\phi$ = 30 7 129 1	$(11\frac{1}{2})^2 = 89.3$ 4-1ϕ = 40.2 129.5	$(10\frac{1}{2})^2 = 99.2$ 6-$\frac{3}{4}\phi$ = 33.8 133.0
2	205.8	$(20\frac{1}{2})^2 = 151.3$ 10-$\frac{3}{4}\phi$ = 56.3 207.6	$(17\frac{1}{2})^2 = 165.4$ 4-1ϕ = 40.2 205.6	$(16)^2$ = 172 8 6-$\frac{3}{4}\phi$ = 33 8 206 6	$(14)^2$ = 176 4 6-$\frac{3}{4}\phi$ = 33.8 210.2
1	280 0	$(24)^2$ = 207 4 8-1ϕ = 80 4 287.8	$(20\frac{1}{2})^2 = 226.9$ 10-$\frac{3}{4}\phi$ = 56 3 283 2	$(18\frac{1}{2})^2 = 231.0$ 4-1° = 51.2 282.2	$(16\frac{1}{2})^2 = 245.0$ 4-1ϕ = 40 2 285.2

2000 psi	3000 psi	3750 psi	5000 psi
Forms 4x0.80x13.5=43 2 4x1.33x13 5=71 8 4x1.71x13 5=92 3 4x2.00x11 5=92 0 299 3 sf	 4x0 80x13 5=43.2 4x1 13x13 5=61.0 4x1 46x13 5=78.8 4x1 71x11 5=78 7 261 7 sf	 4x0 80x13 5=43 2 4x0 96x13 5=51.8 4x1 33x13 5=71 8 4x1.54x11.5=70 8 237 6 sf	 4x0 80x13 5=43 2 4x0 88x13.5=47.5 4x1.17x13 5=63 2 4x1 38x11.5=63 5 217 4 sf
Concrete $(0.80)^2$x13 5= 8 7 $(1.33)^2$x13 5=23 9 $(1.71)^2$x13 5=39 5 $(2.0)^2$x11 5=46 0 27)118 1 4 37 cy	 $(0\ 80)^2$x13 5= 8 7 $(1\ 13)^2$x13 5=17 2 $(1\ 46)^2$x13 5=28 8 $(1\ 71)^2$x11 5=33 6 27)88 3 3.27 cy	 $(0\ 80)^2$x13 5= 8 7 $(0\ 96)^2$x13 5=12 4 $(1.33)^2$x13 5=23 9 $(1\ 54)^2$x11.5=27 3 27)72 3 2 68 cy	 $(0\ 80)^2$x13.5= 8 7 $(0\ 88)^2$x13.5=10 5 $(1.17)^2$x13.5=18 5 $(1\ 38)^2$x11.5=21 9 27)59.6 2 21 cy
Steel 4x1043x13.3=55 5* 6x1.502x15 4=138 8 10x1502x15 4=231 3 8x2670x19.0=405 8 831 4	 4x1043x13 3=55 5* 4x2.044x15 7=128 4 4x2670x16 0=170 9 10x1502x17 1=256 8 611 6	 4x1043x13 3=55 5* 4x2670x16 0=170.9 6x1502x15 4=138 8 4x3400x19 0=258 4 623 6	 4x1043x13.3=55 5* 6x1502x15.4=138.8 6x1502x15.4=138.8 4x2670x19.0=202.9 536.0
Floor Space $(1.33)^2$=1 77 $(1.71)^2$=2 92 $(2.00)^2$=4.00 8.69	 $(1\ 13)^2$=1.28 $(1.46)^2$=2.13 $(1.71)^2$=2.92 6 33	 $(0\ 96)^2$=0.92 $(1\ 33)^2$=1.77 $(1\ 54)^2$=2 37 5.06	 $(0.88)^2$=0 77 $(1.17)^2$=1.37 $(1.38)^2$=1.90 4.04
Cost Comparison 299 3 sf @ 25¢ = \$74.83 4.37 cy @ \$8.50 = 37.15 831.4* @ 5¢ = 41 57 \$153.55	 261.7 sf @ 25¢ = \$65.43 3.27 cy @ \$9.50 = 31.07 611.6* @ 5¢ = 30.58 \$127.08	 237.6 sf @ 25¢ = \$59 40 2 68 cy @ \$10 20 = 27.34 623 6* @ 5¢ = 31.18 \$117.92	 217.4 sf @ 25¢ = \$54.35 2 21 cy @ \$11.00 = 24.31 536.0* @ 5¢ = 26.80 \$105.46
Floor Space 8.69 @ \$2.50 21.73 \$175.28	 6.33 @ \$2.50 15.83 \$142.91	 5.06 @ \$2.50 12.65 \$130.57	 4.04 @ \$2.50 10.10 115.56

SHEET ES2

Example 22-3. Refer to Ex. 13-5 (page 270) and determine whether the spirally reinforced concrete column there designed is more or less economical than the rectangular column of Ex. 13-4. (Sheet ES3.)

Solution. The designs have already been made in Chapter XIII; all that remains to be done here is the pricing. Note that spiral reinforcement is taken at a higher unit price than the main reinforcing steel as it always costs more. The cost of the round steel column mold would have to be verified by quotations from subcontractors who lease and erect these forms.

It will be noted that with the relatively light loads involved there is no particular economy in spirally reinforced columns.

Example 22-4. The slab designed in Ex. 13-1 (page 235) was taken as 4 in. thick because any thinner slab would not adequately take care of the negative moment. Would it be more economical to use a thicker slab and less reinforcing steel, having in mind that there is a large slab area so that a small saving per square foot would be worth while?

Solution. On Computation Sheet ES4 designs have been made for slab thicknesses of 4, 4½, 5, and 5½ in. Note that in each case the proper dead load has been used. For this comparison the same size of reinforcing steel is used throughout and the spacing is varied to suit. The same is done with the top rods over supports. The weight of the positive steel has been increased 27 per cent to allow for the extension into the adjoining spans for negative moment and for the length used up in trussing the bars. The negative steel is multiplied by the percentage its length is of the total span length. No account is taken here of the effect of the increased weight on the beams, columns, and footings. Had the results indicated that the heavier slab was the more economical as far as slab alone is concerned then this further study would have been necessary.

Example 22-5. In Ex. 22-4, the concrete in the thicker slabs is not working at capacity, and a lower grade concrete might be substituted. Would a saving result if this were done?

Solution. On Computation Sheet ES4 the previous slab designs are reconsidered, taking the R values, selecting a concrete that is sufficient for the flexural stresses, and proportioning steel to suit. Here the differences in cost hinge largely on the variations in unit cost of concrete as the cement content changes and, for any given locality, a careful checking of these units is necessary before a conclusion is drawn. This comparison relates only to the cost of the slab itself, and does not take into account increases required in the supporting framework to carry the increased dead load. Such studies require more complete treatment as illustrated in the next article.

These examples illustrate the method of making cost comparisons for different designs of the same member. Unit costs vary considerably not only among contractors under identical conditions, but with the same contractor as reflections of changing market and labor conditions. The designer is interested in general overall unit costs for the time and place of his particular structure. A slightly higher or lower level will not affect the conclusions if all units vary proportionately. If certain items are raised while others are lowered the conclusions may be upset. The obtaining of reliable figures requires care because, even in the case

ECONOMIC STUDY : COLUMNS				Sheet ES3
Tied Columns				Ex 22-3
19"x19"	Forms	$4 \times 1.58 \times 11.42 = 72.2$ sf @ 25¢ =	\$18.05	
	Concrete	$(1.58)^2 \times 11.42 \times \frac{1}{27} = 1.06$ cy @ \$9.50 =	10.07	
10-1"φ	– Steel	$10 @ 2.670 \times 18.92 = 505.2$# @ 5¢ =	25.26	
$\frac{3}{8}$φ to 12"c/c	– Ties	$12 @ 0.376 \times 5.83 = 26.3$# @ 5¢ =	1.32	
			\$54.70	
Spiral Column				
18"φ	Forms - Sub-contract		\$10.00	
	Concrete	$\frac{\pi \times 1.5^2}{4} \times 11.42 \times \frac{1}{27} = 0.75$ cy @ \$9.50 =	7.13	
6-1$\frac{1}{8}$"	Steel	$6 @ 4.303 \times 19.86 = 512.7$# @ 5¢ =	25.64	
$\frac{3}{8}$φ-1$\frac{3}{4}$p-15"φ	Spiral	$\left[\frac{11 \times 12}{1\frac{3}{4}} \pi 1.25 \times 0.376 + 2 \times 11 \times 0.83\right] = 129.7$# @ 6$\frac{1}{2}$¢ =	8.43	
			\$51.20	

Sheet ES3

of the most carefully kept cost figures, there is always the decision as to where items are to be charged. Loading, unloading, handling materials, watching forms during concreting, service, supervision, and many other factors are very difficult to prorate accurately. The student can determine by using different sets of unit costs in the examples just what effect such variations would have.

22-4. Comparison of Complete Systems. A larger problem is presented in deciding what type of framing to use for a given structure. In the early part of Chapter XVII, in studying the building designed there it was explained that several possible framing systems might be economical for these loads and spans, such as: (*a*) solid slab, beam, and girder, (*b*) clay-tile and joist, (*c*) ribbed slab, (*d*) two-way slab, and (*e*) flat slab of two-way type, to mention only the main possibilities. It is proposed to take here a typical interior square panel and compare the costs of the floor system only for each of these schemes. It happens that the necessary designs have already been made, and this saves considerable work here. A word of caution is necessary about drawing conclusions regarding the entire structure merely from a typical interior panel as here considered. Many factors enter the cost of the entire structure, such as the heights of partitions, amount of exterior sash, lengths of heating pipes, roof drains, and all vertical services such as elevator and stairs, and the differences in weight of building that would affect the column sizes and footing areas, even the amount of painting, heating, and lighting required. The following examples are intended to illustrate method rather than to afford a conclusion by direct comparison.

Example 22-6. Prepare comparative cost studies for one typical interior square panel of floor system only for the building on Fig. 17-2 if designed by each of the five systems above, neglecting temporarily the effect on walls, partitions, columns, or footings.

Solution. The designs upon which this comparison is based are found: (*a*) solid slab, beam, and girder, Chapter XVII, Computation Sheets BG; (*b*) clay-tile and joist, Chapter XVI, Computation Sheet RS1; (*c*) ribbed slab, Chapter XVI, Computation Sheet RS2; (*d*) two-way slab, Chapter XVI, Computation Sheet TWS-1; (*e*) flat slab of two-way type, Chapter XVI, Computation Sheet FS1.

It then becomes simply a matter of taking off the quantities for each design, pricing them at suitable unit costs, and obtaining totals (this is done on Computation Sheet ES5). Under the heading " Concrete " the total yardage in one 20 ft 0 in. square interior panel is computed, the volume contained in the flaring column capital of the flat slab design being postponed for consideration in comparing the columns. The formwork is taken off in two items: the less expensive slab soffits and the more expensive beam forms. Pricing the quantities under each scheme at suitable units results in an approximate cost per panel of the floor system only, making no allowance for the differences in story heights required to obtain the same clear head room. Thus there

ECONOMIC STUDY: SLABS
Examples 22-4 and 22-5 — Sheet ES4

	4" Slab	4½" Slab	5" Slab	5½" Slab (Ex 22-4)
w	188	194	200	207
$+M$	1350	1393	1436	1486
d	3	3½	4	4½
R	150	114	90	73.4
p	0.0083	0.0063	0.0050	0.0040
A_s	0.0249	0.0221	0.0200	0.0180
Rods	½φ-8" c/c	½φ-9" c/c	½φ-10" c/c	½φ-11" c/c
$-M$	1942	2004	2066	2138
$-R$	216	164	129	106
$-p$	0.0124	0.0093	0.0071	0.0059
$-A_s$	0.0372	0.0326	0.0284	0.0266
Top Rods	½φ-16" c/c	½φ-19" c/c	½φ-24" c/c	½φ-24" c/c
Forms	1 sf $0.150	1 sf $0.150	1 sf $0.150	1 sf $0.150
Concrete	$\frac{4}{12\times27}$ @ $9.50 = 0.117	$\frac{4\frac{1}{2}}{12\times27}$ @ $9.50 = 0.132	$\frac{5}{12\times27}$ @ $9.50 = 0.147	$\frac{5\frac{1}{2}}{12\times27}$ @ $9.50 = 0.161
Rods	1.50# @ 5¢ = 0.075 $0.342	1.32# @ 5¢ = 0.066 $0.348	1.17# @ 5¢ = 0.059 $0.356	1.08# @ 5¢ = 0.054 $0.365
	$1.27\times0.668\times\frac{12}{8}=1.27$ $0.45\times0.668\times\frac{12}{16}=0.23$ 1.50	$1.27\times0.668\times\frac{12}{9}=1.13$ $0.45\times0.668\times\frac{12}{19}=0.19$ 1.32	$1.27\times0.668\times\frac{12}{10}=1.02$ $0.45\times0.668\times\frac{12}{24}=0.15$ 1.17	$1.27\times0.668\times\frac{12}{11}=0.93$ $0.45\times0.668\times\frac{12}{24}=0.15$ 1.08
$-f_c$	1150	930	800	700 (Ex 22-5)
f'_c	2600	2100	1780	1560
Use	3000	2500	2000	2000
Forms	$0.150	$0.150	$0.150	$0.150
Concrete	$\frac{4}{12\times27}$ @ $9.50 = 0.117	$\frac{4\frac{1}{2}}{12\times27}$ @ $9.00 = 0.125	$\frac{5}{12\times27}$ @ $8.50 = 0.132	$\frac{5\frac{1}{2}}{12\times27}$ @ $8.50 = 0.144
Rods	0.075 $0.342	0.066 $0.341	0.059 $0.341	0.054 $0.348

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would be adjustments to cover the differences in heights of the supporting columns, the interior partitions and the exterior walls.

The above economic studies of small portions of a building are presented merely to show the method of attack. In actual practice a decision can only be arrived at by considering the structure as a whole. The selection of the most economical type of framing requires a great deal of experience and study. The spacing of column centers, the question of which span is to be the principal one when the column centers are not the same in both directions, the variations in the costs of different materials and operations in different parts of the country, the effect upon the exterior walls, interior partitions, and mechanical trades must be evaluated. Just as illustrations of what may be accomplished, a few instances are cited below.

In the case of a retail mercantile building approximately 200 by 250 ft with basement and 6 stories, studies involving the total overall cost of all equipment and services indicated variations as great as $75,000 between the different methods of framing. This is against a total cost of approximately $2,000,000.

For another commercial building approximately 40 by 120 ft, 8 stories and basement, similar studies showed a saving of $35,000 between different types of framing, a good deal of which was saved in the footings because shallow footings in the new building eliminated underpinning the adjacent structures.

On an ordinary factory type of structure approximately 150 by 400 ft, 3 stories and basement, designed for a live load of 300 psf and on which highly competitive designs of every possible type of framing were considered, a saving of $46,000 was accomplished by careful studies.

Such studies can only be made by one who is in touch with current costs and familiar with the relative costs of all items, including mechanical trades and finishing operations as well as the structural skeleton itself. Close cooperation with general contractors is necessary in order to keep in touch with changing prices. As will be noted, from the simple examples presented, a great deal of labor is involved, since all different reasonable types of framing must be studied completely. Designs must be roughed through sufficiently to establish quantities and these must be scheduled and priced with a fair degree of accuracy. All that can be hoped for in an elementary text of this character is to direct the student's attention to the necessity of making comparisons and showing some of the steps in so doing.

22-5. War Economies. The diversion of steel to war purposes has curtailed its civilian use in many ways. For one thing the number of bar sizes available has been reduced by the elimination of the ½ in.

ECONOMIC STUDY: FLOOR SYSTEMS
Example 22-6

Sheet ES5

Solid Slab Beam & Girder	Clay Tile and Joists	Ribbed Slab	Two-way Slab	Flat Slab (Two-way)
		Concrete		
20x20x$\frac{1}{3}$ = 133 3 2x0.8x1 17x19 = 35 6 1x0.96x1.67x19 = 30.5 27)199.4 7.38 cy	20x20x0.21 = 84 0 15x0 83x0.33x16$\frac{1}{3}$ = 67 2 15x2x0.83x1x0 67 = 16.7 133x1 46x19 = 36 9 2x0.17x0 83x19 = 5 4 27)210 2 7 8 cy	20x20x1 04 = 416 -9$\frac{1}{2}$x0.83x1 58x19.04 = -238 10x0 83x3x1 = 25 0 96x1 04x19 = 19 27)199 5 7.4 cy	20x20x0.46 = 184 2x1 21x0 8x19 = 36 8 27)220 8 8 2 cy	20x20x0 54 = 216 6.67x6.67x$\frac{1}{4}$ = 17 8 27)233.8 8.7 cy Column Cap would be taken with Column
		Forms		
19x18.4 = 350 2x3.14x19 = 119.2 1x4 3x19 = 81 2 201.0	20x18.67 = 373 2.91x19 = 55.3	20x18.83 = 376 3.20x19 = 60 9	18 4x18 4 = 338 2x3 02x19 = 114 8	20x20 = 400 4x6 67x$\frac{1}{4}$ = 6 7 406 7 None
		Reinforcing Steel		
12-$\frac{1}{2}$ϕx20'-6" = 164 26-$\frac{5}{8}$ϕx15'-3" = 414 18-$\frac{3}{8}$ϕx21'-0" = 142 4-$\frac{7}{8}$ϕx23'-1" = 188 4-1ϕx31'-0" = 331 36-$\frac{3}{8}$ϕx3'-9" = 51 2-1□x23'-0" = 156 2-1$\frac{1}{8}$□x31'-9" = 275 30-$\frac{3}{8}$ϕx3'-10" = 43 1764	15-$\frac{1}{2}$□x20'-3" = 258 15-$\frac{5}{8}$ϕx30'-9" = 480 30-$\frac{1}{4}$ϕx20'-10" = 105 2-1$\frac{1}{8}$□x23'0" = 198 2-1$\frac{1}{8}$□x31'-9" = 275 20-$\frac{1}{2}$ϕx4'-9" = 63 1379	10 $\frac{5}{8}$ϕx20'3" = 212 10-$\frac{3}{4}$ϕx30'-9" = 463 30-$\frac{1}{4}$ϕx20'10" = 105 2-1□x20'-3" = 139 2-1□x31'-9" = 218 22-$\frac{3}{8}$ϕx4'-10" = 40 1177	20-$\frac{1}{2}$ϕx20'-0" = 267 18-$\frac{1}{2}$ϕx30'-3" = 364 12-$\frac{1}{2}$ϕx30'-3" = 242 8-$\frac{1}{2}$ϕx20 0" = 107 76-$\frac{3}{8}$ϕx4'-6" = 129 4-1□x23'0" = 313 4-1□x31'-9" = 432 40-$\frac{3}{8}$ϕx3'-11" = 63 1917	8-$\frac{1}{2}$□x12'-6" = 85 18-$\frac{1}{2}$□x33'-0" = 505 8-$\frac{1}{2}$□x15'-0" = 102 10-$\frac{1}{2}$□x15'-0" = 128 10-$\frac{1}{2}$□x33-0" = 281 1101
	Clay Tile: 12x12x12 = 240 pc 12x10x8 = 30 "			
		Chairs		
$\frac{3}{4}$" Slab = 38 ft 2$\frac{1}{4}$" Comb = 76 1$\frac{1}{2}$" Beam = 9	1$\frac{1}{2}$" Beam = 5 ft Joist-$\frac{3}{4}$"x4" = 60 pc	1$\frac{1}{2}$" Beam = 4 ft Joist-$\frac{3}{4}$"x5" = 40 pc	$\frac{3}{4}$" Slab = 38 ft 3$\frac{3}{8}$" Comb = 38 1$\frac{1}{2}$" Beam = 7	$\frac{3}{4}$" Slab = 60 ft 4$\frac{1}{4}$" Cont = 20
		Costs		
Conc 7.38 @ \$8 50 = \$62 70 Forms 350 @ 18¢ = 63 00 201 @ 25¢ = 50 25 Rods 1764 @ 5¢ = 88 20 Chairs = 5 00 \$269.15	Conc 7 8 @ \$8 50 = \$66 30 Forms 373 @ 18¢ = 67 14 55 3 @ 25¢ = 13 83 Rods 1379 @ 5¢ = 68 95 Tile 240 @ 11¢ = 26 40 30 @ 10¢ = 3 00 \$245 63 Chairs 2 00 \$247 63	Conc 7 4 @ \$8 50 = 62 90 Forms 376 @ 18¢ = 67 68 60 9 @ 25¢ = 15 23 Rods 1177 @ 5¢ = 58.85 Chairs = 1 40 \$206 06	Conc 8 2 @ \$8 50 = \$69 70 Forms 338 @ 18¢ = 60 84 114 8 @ 25¢ = 28 70 Rods 1917 @ 5¢ = 95.85 Chairs = 4.00 259 09	Conc 8 7 @ \$8 50 = \$73 95 Forms 406 7 @ 18¢ = 73.21 Rods 1101 @ 5¢ = 55.05 Chairs = 4 00 \$206.21

Sheet ES5

square and its equivalents; this may prove to be permanent. Of an entirely different order is the necessity imposed upon the concrete designer by war regulations to cut down on the amount of steel used in any structure even though the total expense may be increased. This necessity results in the use of plain concrete instead of reinforced concrete in many situations, for example, in footings and in the replacement of cantilever retaining walls by massive gravity walls.

The saving of steel in reinforced concrete design may be made in several ways. Mr. A. H. Brewer, in the *Engineering News-Record* of November 6, 1941, presents the result of a study of a typical concrete frame (20 × 20 ft panel, beam and girder construction) and reports a steel saving of some 25 per cent through the following expedients: making slab 1 in. thicker than required, beams and girders 20 to 40 per cent deeper than usual, columns designed on a 1 per cent minimum basis for the vertical steel, footings designed with about 0.2 per cent reinforcement, and the use of a high ultimate strength concrete. In the issue of June 18, 1942, of the same journal Mr. J. J. Polivka continues this study and suggests the use of lightweight fillers in slabs, a variation of the common ribbed floor construction. Again in the same journal, issue of May 21, 1942, Mr. A. J. Boase reports on a comparison of seven types of reinforced concrete floor systems designed according to the A.C.I. code (1936) and concludes that the flat slab uses the least steel. Such studies show what can be accomplished under the current codes.

A much discussed proposal, advocated by many but not favored by the more conservative engineers, urges a flat increase of unit stresses. It has long been accepted practice to use a reduced factor of safety for emergency construction for short time use where the conditions are definitely known and possible to control, but the rush and confusion of wartime building argue strongly against this practice. For the most part factors of safety now are set as low as good judgment permits with present-day materials and the relatively good knowledge of live loading to be expected. To cut into these factors when material quality, especially that of steel, seems certain to be lowered by emergency conditions, and when structures often must be planned, and construction even started, in advance of full knowledge of mechanical and other installations, would seem to be hazardous in the extreme. In one respect, however, present usage regarding safety factors is not logical and the rationalizing of this situation will permit of steel savings with perfect safety. It is customary to use the same stresses, and consequently the same factor of safety, for both dead and live loads, with the result that our structures are unbalanced, members with a high ratio of dead to live load having a disproportionately high margin of safety as compared with members where this ratio is low.

The illogic of employing the same safety factor for dead as for live load appears at once upon consideration that the dead load on any structure is a fixed element which, in general, does not change. Failure due to overloading will come only as the superimposed — the live — load is increased. Since stress may be taken to vary directly with load, the superimposed load (x) per unit area necessary to raise stress to its ultimate value (s_u) is expressed by the proportion $(D + x) : (D + L) = s_u : s_w$, where D and L are the dead and live load intensities used in preparing the design and s_w is the design or working unit stress. For convenience of record call this stress ratio r, and also divide through the left side of the equation by L, giving

$$r = \frac{\frac{D}{L} + \frac{x}{L}}{1 + \frac{D}{L}}$$

which gives for the ratio of superimposed additional load required to cause failure to design live load,

$$\frac{x}{L} = r - (1 - r)\frac{D}{L}$$

With current material and design stresses a common value of $r = s_u/s_w$ is 2, a tensile stress of 20,000 psi being permitted with steel whose yield point should not be under 40,000 psi. This gives

$$\frac{x}{L} = 2 + \frac{D}{L} \qquad [22\text{–}1]$$

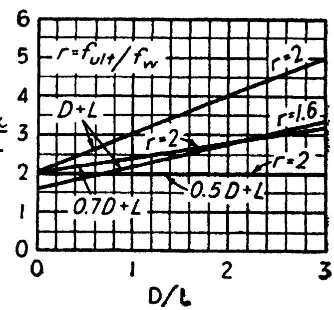


Fig. 22–1

which is plotted as the upper curve in Fig. 22–1. This tells forcibly of the great increase of ratio of superimposed load to design live load which can be carried with increase of the proportionate dead weight of the member.

A common suggestion for increase of working steel stress is to employ 25,000 instead of 20,000 psi, a ratio r of 1.6, giving the equation

$$\frac{x}{L} = 1.6 + 0.6\frac{D}{L} \qquad [22\text{–}2]$$

The slope of this curve is less than that representing ordinary practice but it is evident that the disparity of safety margin largely remains. A common value of the ratio D/L for a slab is ½ (see page 235), giving by equation 22–1, 2.5 for x/L, 1.9 by equation 22–2, a rather sharp difference. Many engineers would hold that a structure which would fail upon a 90 per cent increase of live load is unsafely designed.

The remedy for this situation now being generally discussed is one made by Mr. A. J. Boase,* namely, that for design purposes we use only 70 per cent of the actual dead load. This means, essentially, using a reduced safety factor for the dead load and leaving present factors unchanged as regards live load. Employing this procedure equation 22–1 becomes

$$\frac{x}{L} = 2 + 0.4\frac{D}{L} \qquad [22\text{–}3]$$

which gives a much flatter curve than hitherto, thus tending toward an equalization of reserve capacity of all members. To make this curve horizontal (equal reserve capacity all around), we are obliged to make the reduced dead load one-half of the actual, a practice suggested many years ago in the bridge specifications of Theodore Cooper.† (*Query:* Is it good logic to insist that main carrying members have a larger safety factor than a local part, like a slab? What is the bearing upon this situation of the practice of reducing live load for girders and columns?) The procedure suggested by Mr. Boase is equivalent to making a variable increase of allowable steel stress proportional to the ratio of dead to live load in the member; for example, 20 ksi where that ratio is zero, 23.5 where it is unity, and 28.6 ksi where it is infinite. From another point of view, this method leaves the safety factor at 2 for live loading and decreases that for dead loading by 30 per cent.

This reduction of dead load for design is being used in this country for emergency construction and it seems possible that it may continue as standard practice when peace comes. It is a matter of considerable practical convenience that this rationalizing of the safety factor is accomplished with no change in design procedure beyond using 70 per cent of the dead load instead of 100 per cent: the same unit stresses, design constants, coefficients, and charts are used as at present.

* "Saving Reinforcing Steel by Rationalizing Safety Factors," A. J. Boase, Manager, Structural Bureau, Portland Cement Association, *Engineering News-Record*, May 7, 1942, p. 81.

† See letter by Professor Witmer in *Engineering News-Record* of June 18, 1942, p. 74.

APPENDIX

DESIGN TABLES AND DIAGRAMS

Figs. A-1 to A-9, A-12 to A-17*a*, and A-18 to A-21 are reproduced by permission from Turneaure and Maurer, Principles of Reinforced Concrete Construction, John Wiley & Sons, Inc., 1935.

Fig. A-26 was contributed by Professor J. R. Shank of Ohio State University.

Table A-1 is from " Reinforced Concrete Design Handbook," of the A.C.I., 1928.

Table A-2 is from A. R. Lord, " Handbook of Reinforced Concrete Building Design," of the A.C.I., 1928.

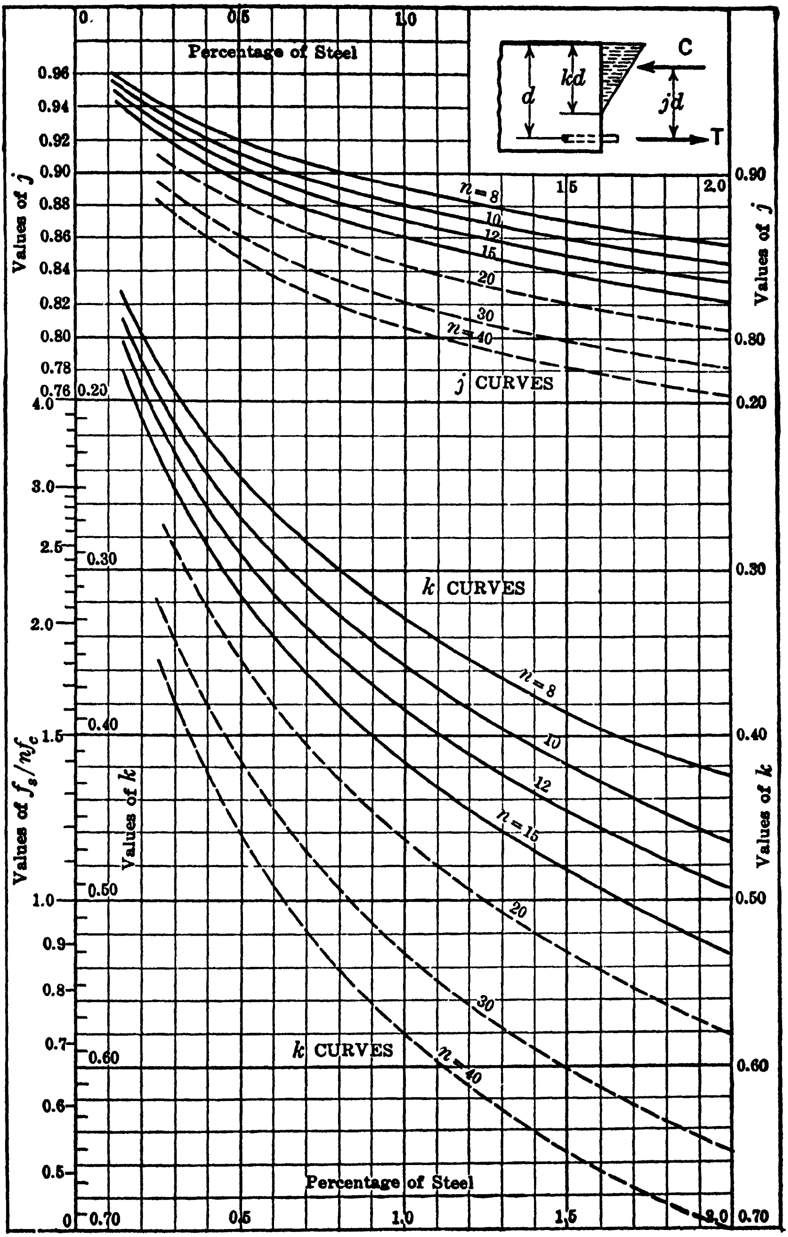


FIG. A-1. Values of k and j for Rectangular Beams.

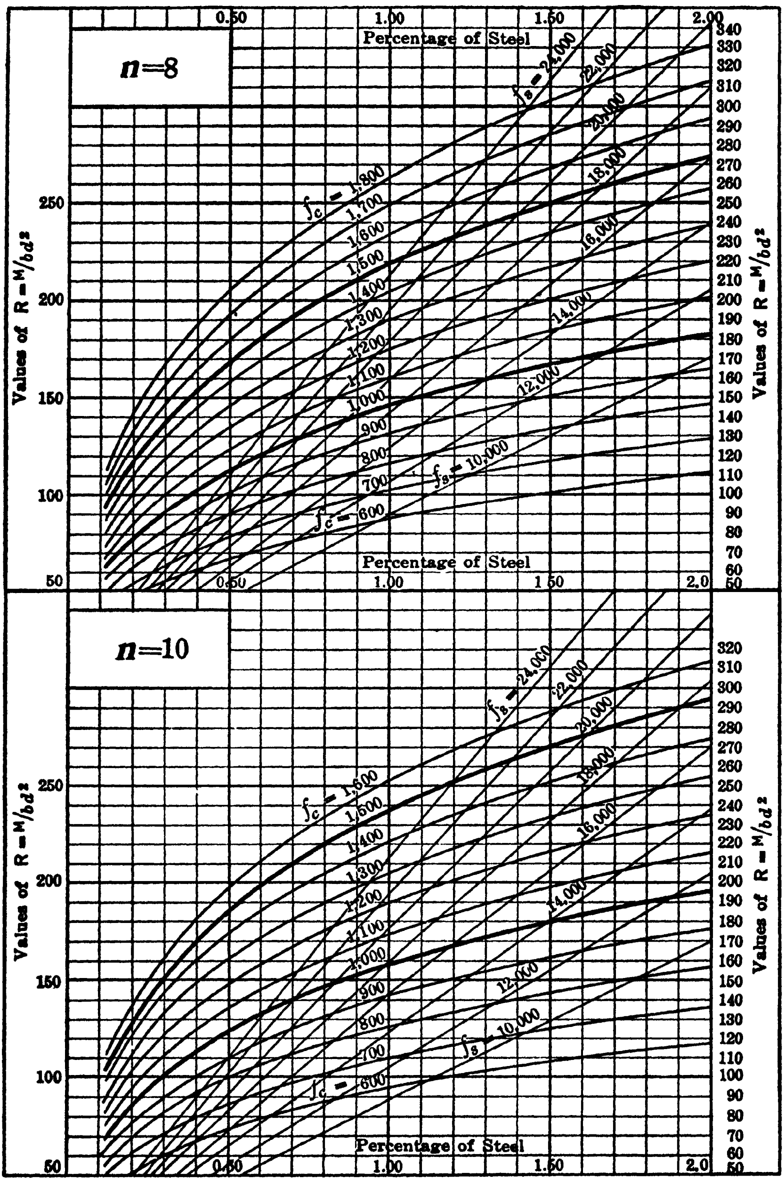


Fig. A–2. Coefficients of Resistance of Rectangular Beams. $M = Rbd^2$.

Design Values of k, j, p, and R for f'_c

		3000 psi	3750 psi
	n	10	8
	f_c	1350	1688
$f_s = 20{,}000$ psi	k	0.403	0.403
	j	0.866	0.866
$f_c = 0.45 f'_c$	p	0.0136	0.0170
	R	236	294

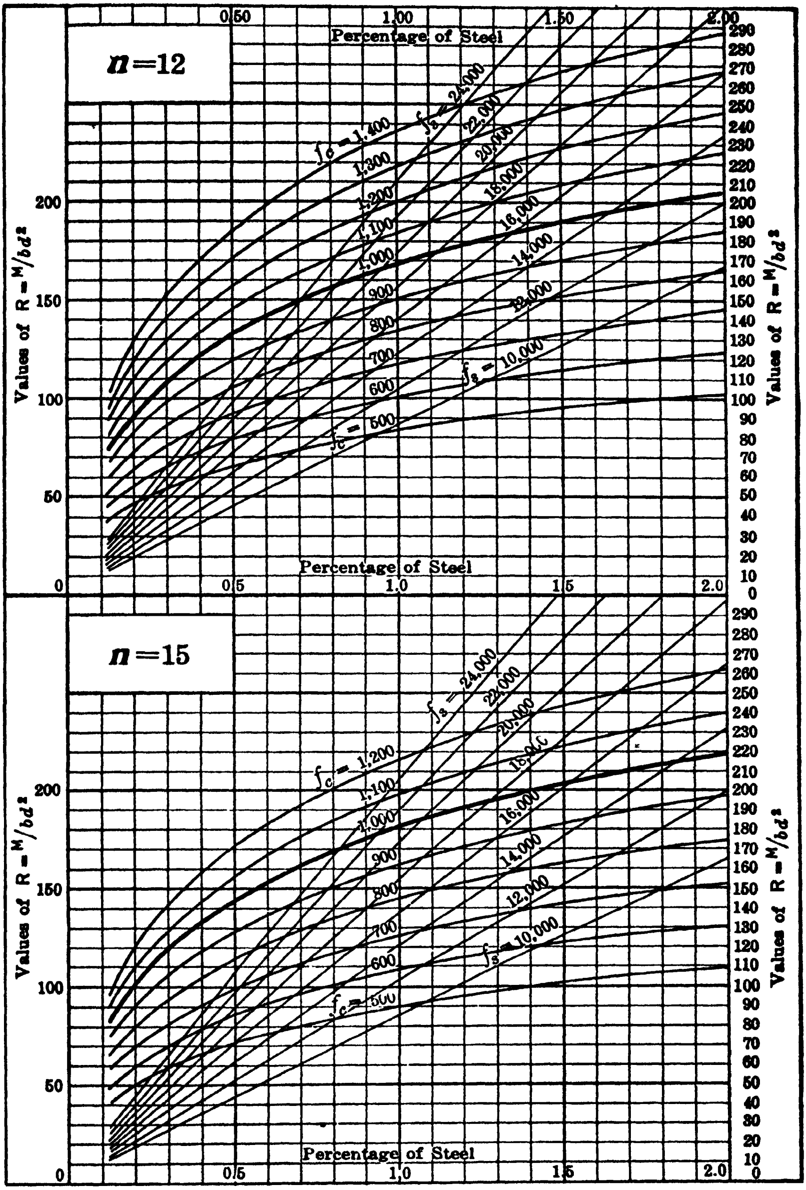


FIG. A-3. Coefficients of Resistance of Rectangular Beams. $M = Rbd^2$.

Design Values of k, j, p, and R for f'_c

		2000 psi	2500 psi
	n	15	12
	f_c	900	1125
f_s = 20,000 psi	k	0.403	0.403
	j	0.866	0.866
$f_c = 0.45 f'_c$	p	0.0091	0.0113
	R	157	196

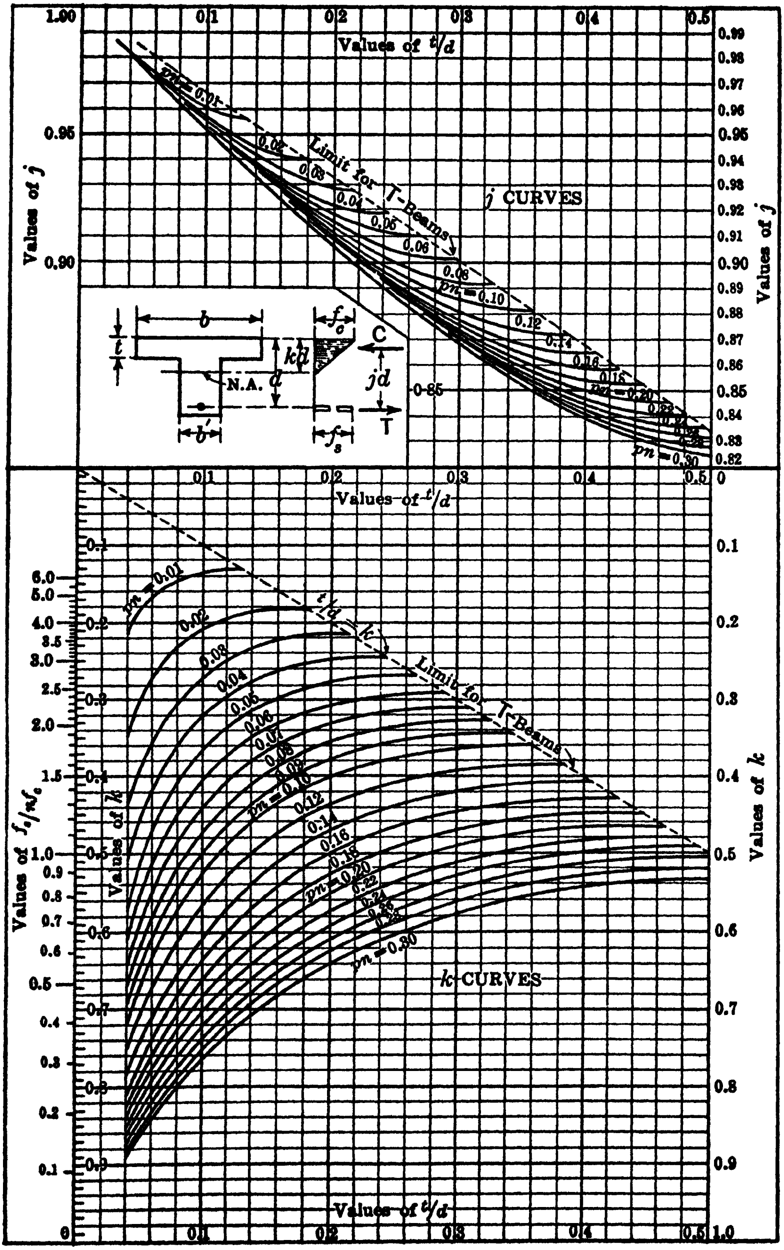


Fig. A-4. Values of k and j for Tee-Beams.

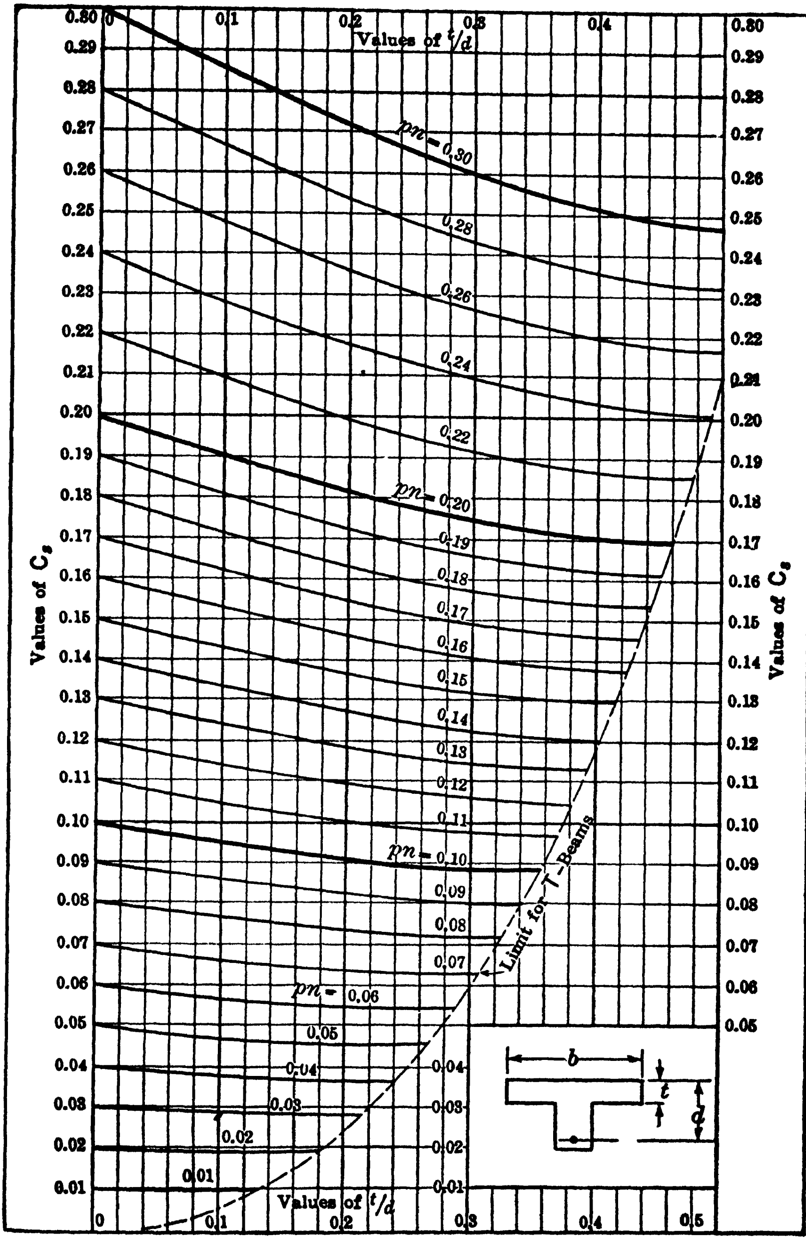


FIG. A-5. Coefficients of Resistance of Tee-Beams with Respect to Steel.

$$M_s = C_s \frac{f_s}{n} bd^2$$

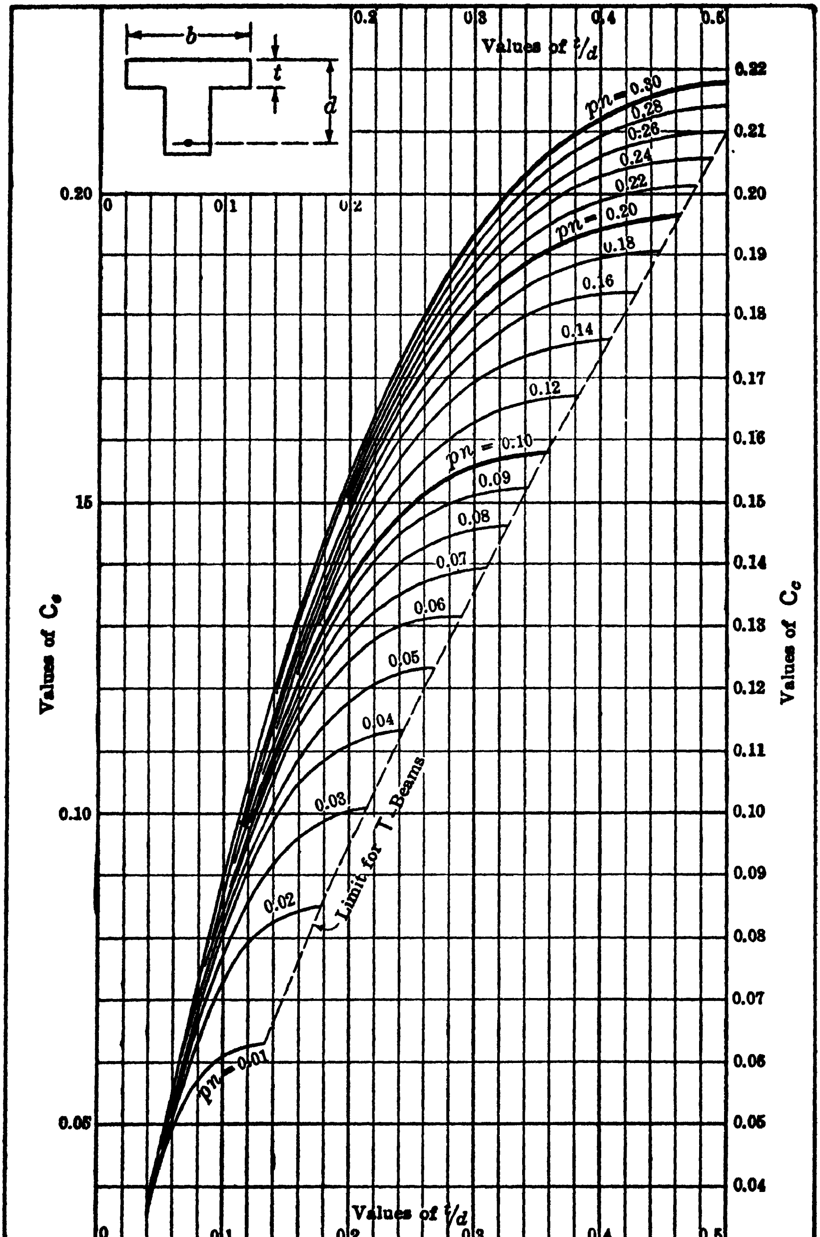


FIG. A-6. Coefficients of Resistance of Tee-Beams with Respect to Concrete.

$$M_c = C_c f_c b d^2$$

TABLE A-1

Values of R for Tee-Beams

$f_s = 20{,}000$ psi

f'_c and n	f_c	t/d = 0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20	0.24	0.28	0.32	0.36	0.40
	650	35	44	53	60	67	73	78	83	89	93	95	95	95
	700	37	48	57	66	73	80	86	91	99	104	106	107	107
2000	750	40	51	62	71	79	87	93	99	108	114	118	119	119
	800	43	55	66	76	85	93	101	107	117	125	129	131	131
	900	49	62	75	87	97	107	116	123	136	145	152	156	157
15	1000	54	70	84	97	110	121	131	140	155	166	175	180	183
	1200	65	84	102	119	134	148	160	172	192	208	221	229	235
	1350	74	96	116	134	152	168	183	196	220	240	255	266	275
	800	43	54	65	74	82	90	96	101	109	114	116	116	116
	875	47	60	71	82	91	100	107	113	123	130	133	133	133
2500	950	51	65	78	90	100	110	118	126	137	145	150	152	152
	1000	54	69	83	95	107	117	126	134	147	156	161	164	164
	1125	61	78	94	108	122	134	145	154	170	181	190	195	196
12	1250	68	87	105	122	137	151	163	174	193	208	219	225	229
	1500	82	106	128	148	167	185	201	215	240	260	276	287	294
	1700	93	120	146	169	191	212	231	247	278	302	321	336	346
	975	52	66	79	90	101	109	117	124	134	140	142	142	142
	1050	56	71	86	98	110	120	128	136	148	156	159	160	160
3000	1125	60	77	92	106	119	130	140	148	162	171	176	178	178
	1200	65	83	99	114	128	140	151	161	176	187	194	197	197
	1350	73	94	113	130	146	160	173	185	204	218	228	234	236
10	1500	81	105	126	146	164	181	196	209	232	250	262	271	275
	1800	98	126	153	178	201	222	241	258	288	312	331	344	353
	2025	111	143	174	202	228	252	274	295	330	359	382	399	411
	1200	64	81	97	111	123	134	144	152	164	171	173	173	173
	1300	69	88	106	121	136	148	159	168	183	192	196	197	197
3750	1400	75	96	115	132	148	162	174	184	201	213	219	221	221
	1500	81	103	124	143	160	175	189	201	220	234	242	246	246
	1700	92	118	142	164	184	202	219	233	257	276	288	295	298
8	1875	102	131	158	183	205	226	245	262	290	312	328	338	344
	2250	123	158	191	222	251	277	301	323	359	390	414	431	441
	2525	138	179	216	252	284	314	342	367	412	448	477	498	514

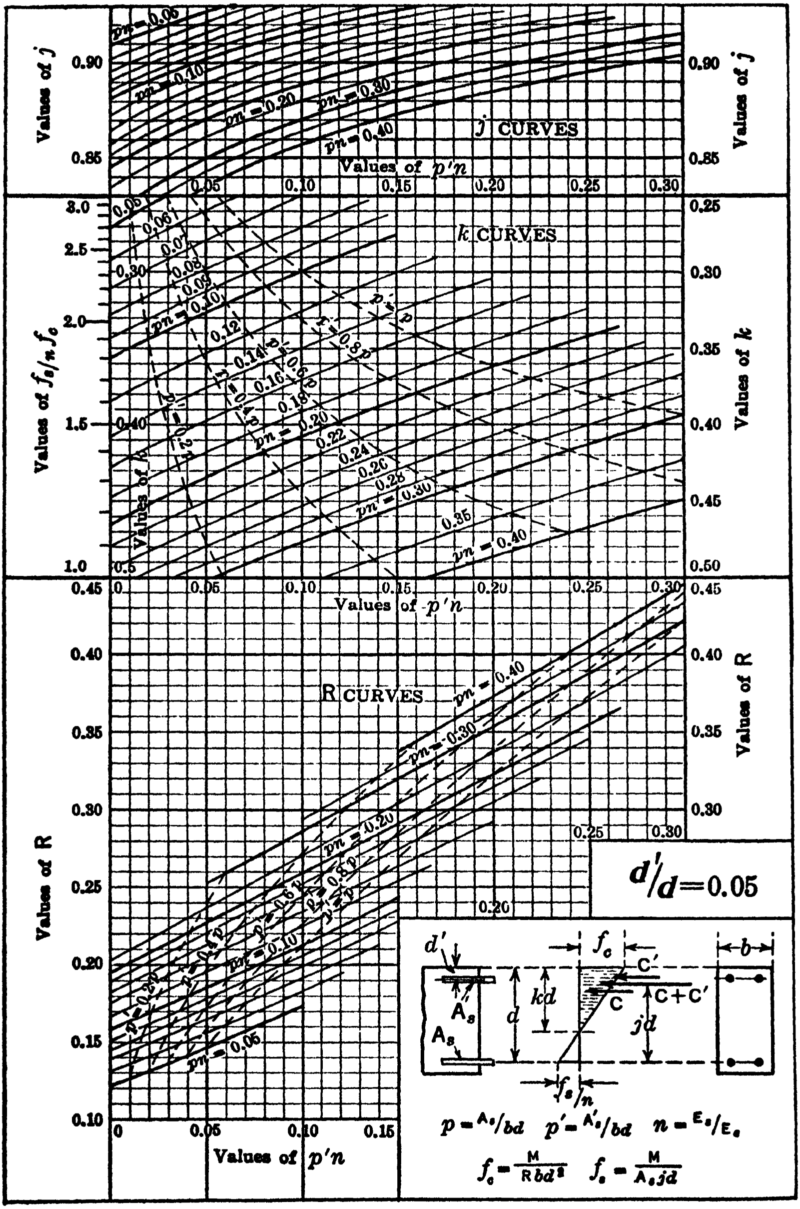


Fig. A-7. Rectangular Beams Reinforced for Compression.

$$M = f_cRbd^2$$

Note. Figs. A-7, A-8, and A-9 are based on formulas which make no allowance for the reduction of concrete compression area due to the presence of the steel. To correct for this small approximation the quantities p' and A'_s in the equations and plates should be taken equal to $(n - 1)/n$ times the actual quantity used; in design the values of p' and A'_s determined from the diagrams should be multiplied by $n/(n - 1)$ to get the correct value to use.

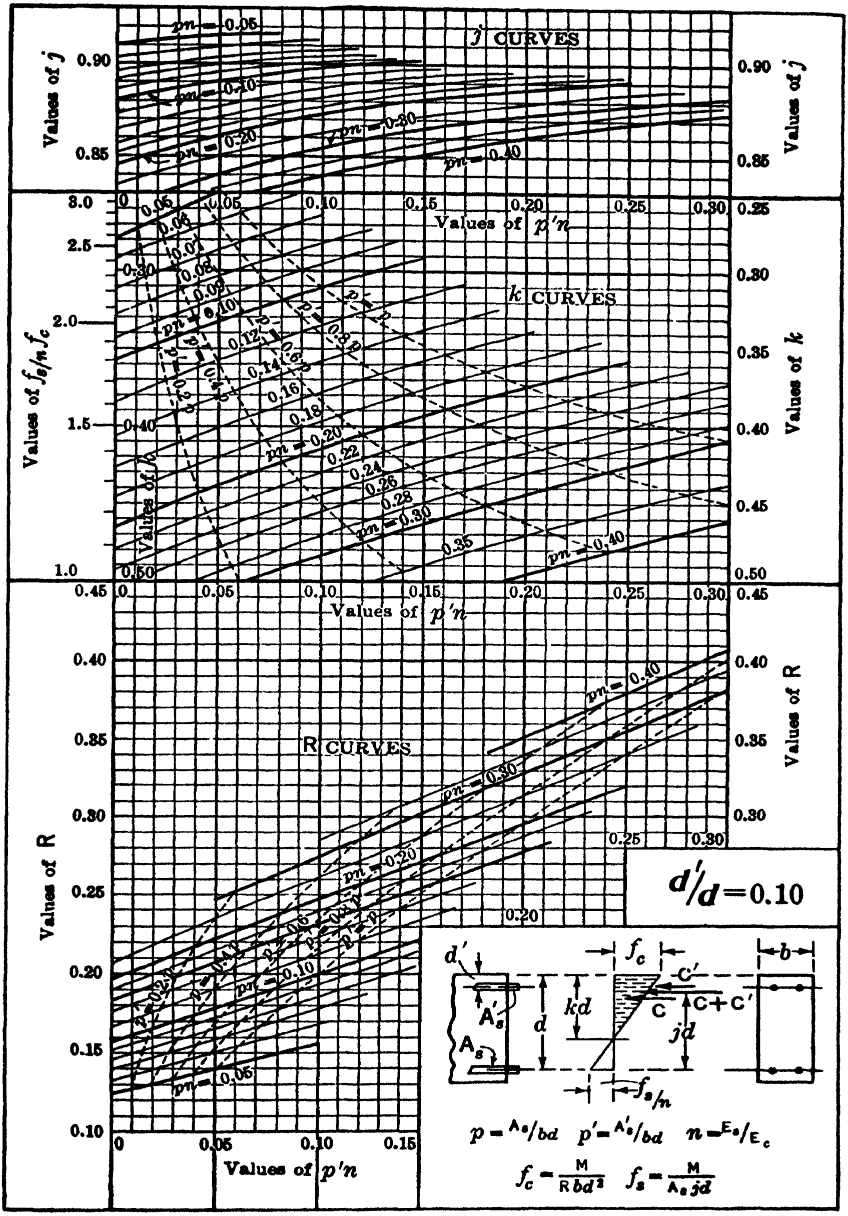


FIG. A-8. Rectangular Beams Reinforced for Compression. $M = f_c R b d^2$. (See note under Fig. A-7.)

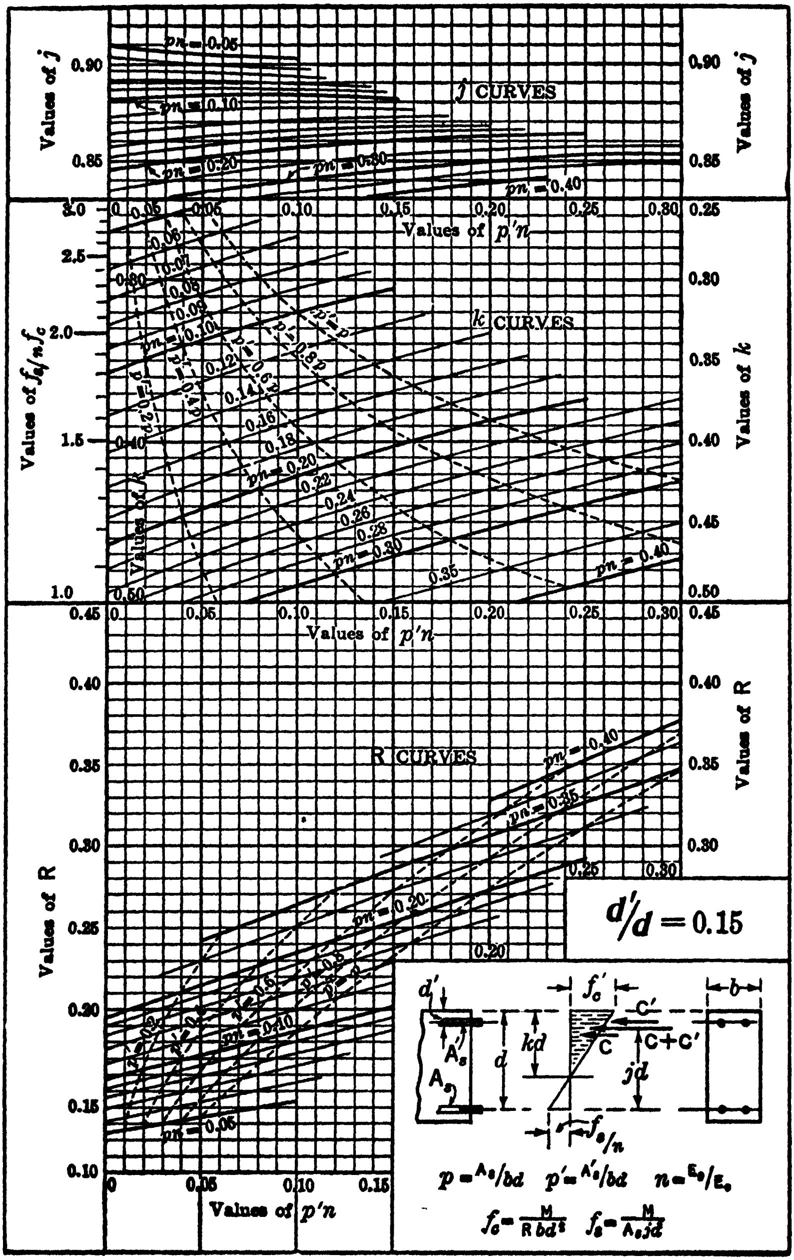


FIG. A-9. Rectangular Beams Reinforced for Compression. $M = f_c R b d^2$.

Note. See note under Fig. A-7. Also note that the lower dotted curves above should be designated $p' = 0.2p$, $0.4p$, etc. Note that the heavy curve marked $pn = 0.35$ should be $pn = 0.30$.

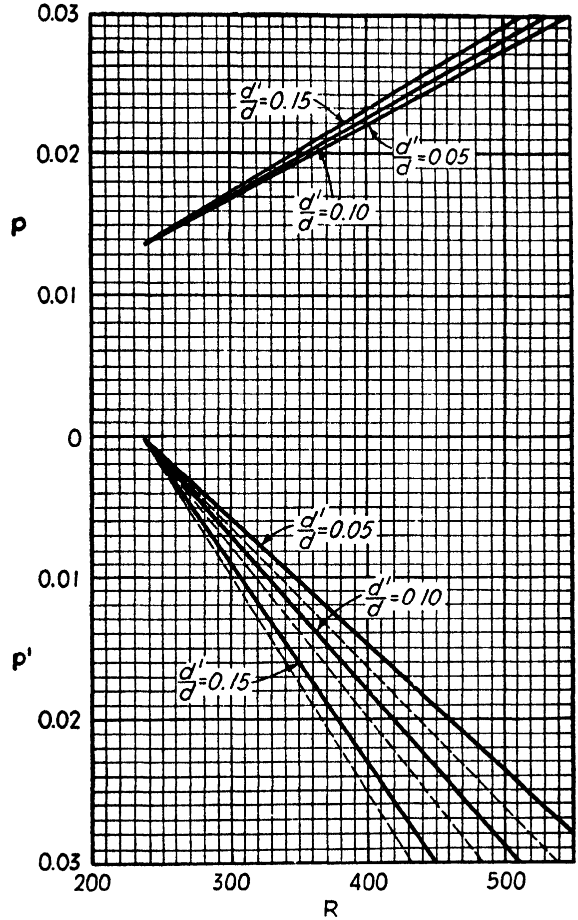


FIG. A-10. Rectangular Beams Reinforced for Compression. $M = Rbd^2$.

$f'_c = 3000$ psi $f_c = 1350$ psi $f_s = 20{,}000$ psi $n = 10$

Note. The lower dotted lines employ $(n - 1)$ instead of n in determining the equivalent homogeneous section.

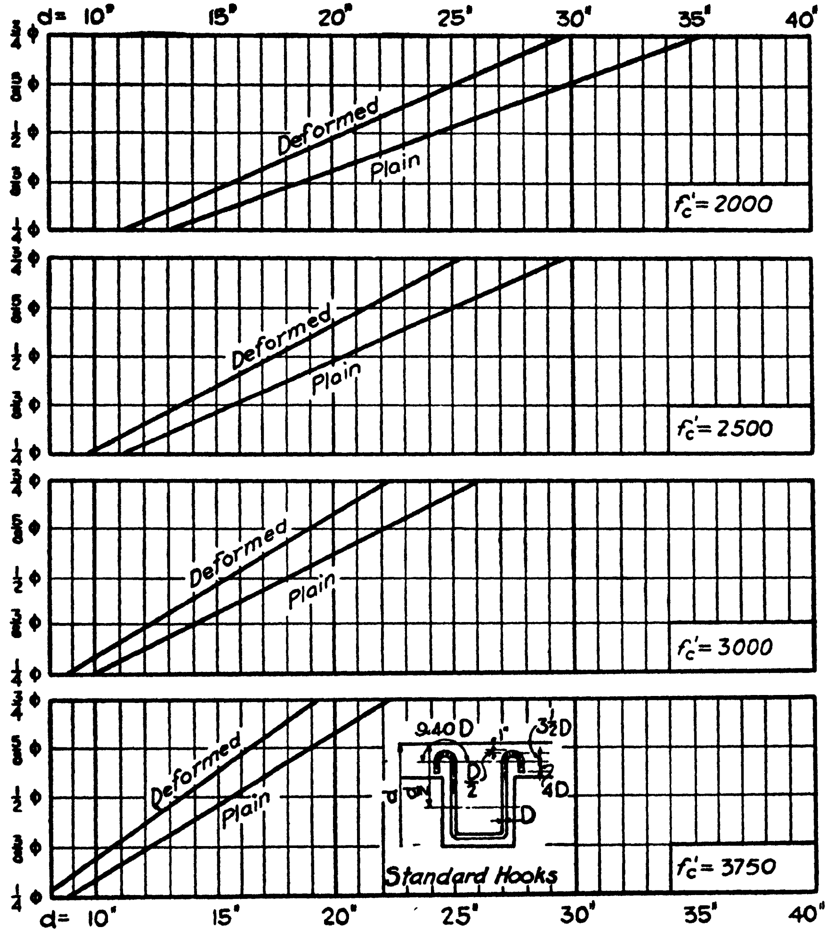


FIG. A–11. Beam Depths to Develop Stirrups. (Based on 1940 A.C.I. Code with standard hook developing 10,000 psi of stirrup stress and bond from center of hook to middle of effective depth developing 6000 psi: f_v = 16,000 psi.)

Number of Stirrups at One End	Distance First Stirrup to Face of Support	Spacing, Center to Center, of Stirrups, in Terms of a									
		1st Group		2nd Group		3rd Group		4th Group		5th Group	
		No.	Spacing	No	Spacing	No.	Spacing	No.	Spacing	No.	Spacing
20	0.013 a	8	0.03a	7	0 04a	2	0.06a	1	0 08a	1	0 11a
19	0.013a	7	0 03a	6	0 04a	3	0 06a	1	0 08a	1	0 12a
18	0.014 a	6	0.03a	5	0.04a	4	0 06a	1	0 08a	1	0 12a
17	0.015 a	5	0.03a	5	0 04a	4	0.06a	1	0.09a	1	0.13a
16	0.016 a	3	0 03a	5	0 04a	5	0.06a	1	0 09a	1	0 13a
15	0017a	2	0.03a	5	0 04a	4	0.06a	2	0 08a	1	0 14a
14	0 018 a	5	0 04a	4	0 05a	2	0 08a	1	0 09a	1	0.14a
13	0.019 a	4	0 04 a	3	0.05a	3	0 08a	1	0 09a	1	0 14a
12	0.021 a	6	0.05a	3	0 07a	1	0 12a	1	0 15a		
11	0023 a	5	0.05a	3	0 08a	1	0 12a	1	0.15a		
10	0.025 a	3	0 05a	4	0 08a	1	0 12a	1	0 16a		
9	0.028a	3	0 06a	3	0 09a	1	0 12a	1	0 17a		
8	0.032a	2	0 07a	3	0 09a	1	0.13a	1	0.18a		
7	0.036a	3	0.08a	2	0 13a	1	0.20a				
6	0.04 a	3	0 10a	1	0 15a	1	0.22a				
5	0.05 a	2	0 12a	1	0 16a	1	0.23a				
4	0 07 a	2	0.16a	1	0.26a						
3	0.09 a	1	0.21a	1	0 30a						
2	0.13 a	1	0.37a								
1	0.29 a										

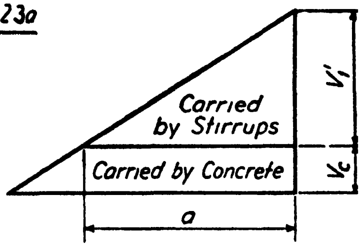


Table A-2. Stirrup Spacing with Triangular Shear Variation.

COL. SIZE		10x10	11x11	12x12	13x13	14x14	15x15	16x16	17x17	18x18	19x19	20x20	21x21	22x22	23x23	24x24	25x25	26x26	27x27	28x28	29x29	30x30	31x31	32x32	33x33	34x34	
		675	81.7	972	1141	1323	151.9	1728	195.1	2187	2437	2700	2977	3267	3571	388.8	4219	4563	4921	5292	5677	6075	6487	6912	7351	7803	
4-5/8"	75	750	892	104.7	1216	1398	1594																				
4-3/4"	107	78.2	924	1079	124.8	1430	1626	183.5	2058	2294																	
4-7/8"	14.6		96.3	111.8	1287	146.9	166.5	1874	2097	233.3	258.3	2846	3123														
6-3/4"	160			113.2	1301	148.3	1679	1888	2111	2347	2597	2860	3137	3427													
4-1"	192				1333	151.5	171.1	1920	2143	237.9	262.9	2892	3169	3459	3763	408.0	4411										
8-3/4"	214					1537	1733	1942	2165	240.1	265.1	2914	3191	3481	378.5	410.2	443.3	4777									
4-1"	243						1762	197.1	2194	2430	2680	2943	3220	3510	381.4	4131	4462	4806	5164	5535							
6-1"	28.8							2016	2239	2475	2725	298.8	326.5	355.5	385.9	4176	450.7	4851	5209	558.0	596.5	6363					
4-1 1/8"	309							2037	2260	2496	2746	300.9	3286	3576	3880	4197	4528	4872	5230	5601	598.6	6384	6796				
6-1"	36.5									2552	2802	3065	3342	3632	3936	4253	4584	4928	528.6	5657	6042	6440	685.2	7277	771.6	816.8	1/2%
4-1 1/4"	379									2566	281.6	3079	3356	3646	3950	4267	459.8	4942	5300	5671	6056	6454	6866	729.1	7730	818.2	
6-1 1/8"	46.3											3163	3440	3730	403.4	435.1	4682	5026	538.4	5755	6140	653.8	695.0	737.5	7814	826.6	
8-1"	486												3463	3753	405.7	4374	470.5	5049	5407	5778	6163	6561	6973	7398	783.7	828.9	
6-1 1/4"	569													3836	4140	4457	4788	5132	5490	586.1	6246	6644	7056	748.1	792.0	837.2	
10-1"	60.8														4179	4496	4827	517.1	5529	5900	628.5	668.3	709.5	7520	795.9	841.1	1%
12-1"	729																4948	529.2	5650	6021	6406	6804	7216	7641	808.0	8532	
8-1 1/4"	758																4977	5321	5679	6050	6435	683.3	7245	7670	8109	856.1	
14-1"	85.1																		5772	6143	652.8	6926	7338	7763	820.2	8654	
12-1 1/8"	926																			6218	660.3	7001	741.3	783.8	827.7	8729	
10-1 1/4"	948																			6240	6625	7023	7435	786.0	829.9	8751	1 1/2%
14-1 1/8"	1080																					715.5	7567	799.2	843.1	888.3	
12-1 1/4"	113.7																						7624	8049	848.8	8940	
16-1 1/8"	1234																							814.6	858.5	903.7	
14-1 1/4"	132.7																									9130	
18-1 1/8"	138.9																									919.2	

2%

Table A–3. Carrying Capacity of Square Tied Columns, 1928 A.C.I. Code.

$P = 675\,A_c + (9 \times 675\,A_s)$ $f'_c = 3000$ psi $n = 10$.

COL. SIZE		10x10	11x11	12x12	13x13	14x14	15x15	16x16	17x17	18x18	19x19	20x20	21x21	22x22	23x23	24x24	25x25	26x26	27x27	28x28	29x29	30x30	31x31	32x32	33x33	34x34
		54.0	65.3	77.8	91.3	105.8	121.5	138.2	156.1	175.0	194.9	216.0	238.1	261.4	285.7	311.0	337.5	365.0	393.7	423.4	454.1	486.0	518.9	553.0	588.1	624.2
4-⅝"φ	15.9	69.9	81.2																							
4-¾"φ	22.5	76.5	87.8	100.3	113.8																					
4-⅞"φ	30.7	84.7	96.0	108.5	122.0	136.5	152.2																			
6-¾"φ	33.8	87.8	99.1	111.6	125.1	139.6	155.3	172.0																		
4-1"φ	40.4	94.4	105.7	118.2	131.7	146.2	161.9	178.6	196.5																	
8-¾"φ	45.1	99.1	110.4	122.9	136.4	150.9	166.6	183.3	201.2	220.1																
4-1"□	51.2	105.2	116.5	129.0	142.5	157.0	172.7	189.4	207.3	226.2	246.1	267.2														
6-1"φ	60.7		126.0	138.5	152.0	166.5	182.2	198.9	216.8	235.7	255.6	267.7	298.8													
4-1⅛"□	65.0			142.8	156.3	170.8	186.5	203.2	221.1	240.0	259.9	281.0	303.1	326.4												
6-1"□	76.8				168.1	182.6	198.3	215.0	232.9	251.8	271.7	292.8	314.9	338.2	362.5	387.8										
4-1¼"□	79.9				171.2	185.7	201.4	218.1	236.0	254.9	274.8	295.9	318.0	341.3	365.6	390.9										
6-1⅛"□	97.5					203.3	219.0	235.7	253.6	272.5	292.4	313.5	335.6	358.9	383.2	408.5	435.0	462.5	491.2							
8-1"□	102.4						223.9	240.6	258.5	277.4	297.3	318.4	340.5	363.8	388.1	413.4	439.9	467.4	496.1	525.8						
6-1¼"□	119.8							258.0	275.9	294.8	314.7	335.8	357.9	381.2	405.5	430.8	457.3	484.8	513.5	543.2	573.9	605.8				
10-1"□	128.0							266.2	284.1	303.0	322.9	344.0	366.1	389.4	413.7	439.0	465.5	493.0	521.7	551.4	582.1	614.0	646.9			
12-1"□	153.6									328.6	348.5	369.6	391.7	415.0	439.3	464.6	491.1	518.6	547.3	577.0	607.7	639.6	672.5	706.6	741.7	777.8
8-1¼"□	159.7									334.7	354.6	375.7	397.8	421.1	445.4	470.7	497.2	524.7	553.4	583.1	613.8	645.7	678.6	712.7	747.8	783.9
14-1"□	179.2										374.1	395.2	417.3	440.6	464.9	490.2	516.7	544.2	572.9	602.6	633.3	665.2	698.1	732.2	767.3	803.4
12-1⅛"□	195.1											411.1	433.2	456.5	480.8	506.1	532.6	560.1	588.8	618.5	649.2	681.1	714.0	748.1	783.2	819.3
10-1¼"□	199.7											415.7	437.8	461.1	485.4	510.7	537.2	564.7	593.4	623.1	653.8	685.7	718.6	752.7	787.8	823.9
14-1⅛"□	227.6													489.0	513.3	538.6	565.1	592.6	621.3	651.0	681.7	713.6	746.5	780.6	815.7	851.8
12-1¼"□	239.6													501.0	525.3	550.6	577.1	604.6	633.3	663.0	693.7	725.6	758.5	792.6	827.7	863.8
16-1⅛"□	260.1														545.8	571.1	597.6	625.1	653.8	683.5	714.2	746.1	779.0	813.1	848.2	884.3
14-1¼"□	279.6															590.6	617.1	644.6	673.3	703.0	733.7	765.6	798.5	832.6	867.7	903.6
18-1⅛"□	292.6															603.6	630.1	657.6	686.3	716.0	746.7	778.6	811.5	845.6	880.7	916.8

1% 4% 3% 2%

TABLE A-4. Carrying Capacity of Square Tied Columns, 1941 A.C.I. and 1940 J.C. Codes.

$$P = 540\,A_g + 12{,}800\,A_s : f'_c = 3000 \text{ psi.}$$

VERTICAL STEEL	AREA OF STEEL	CORE DIAMETER 13"	15"	17"	19"	21"	23"	25"	27"	29"	31"	33"	35"	37"	
6-½"φ	1.20														
6-½"φ	1.50	107.6 1/4"φ-3"													
6-⅝"φ	1.86	114.8 1/4"φ-3"	140.5 1/4"φ-3"												
6-¾"φ	2.64	131.0 1/4"φ-3"	156.0 1/4"φ-3"	185.3 1/4"φ-3"											
6-⅞"φ	3.60	152.8 1/4"φ-2¼"	176.3 1/4"φ-2½"	204.7 1/4"φ-2⅞"	237.5 1/4"φ-3"	273.9 1/4"φ-3"									
6-1"φ	4.74	180.4 1/4"φ-1¾"	202.2 1/4"φ-2"	228.9 1/4"φ-2⅜"	261.0 1/4"φ-2½"	296.7 1/4"φ-2⅝"	337.5 1/4"φ-3"								
6-1"φ 8-1"φ	6.00	213.3 3/8"φ-3"	232.2 1/4"φ-1½"	257.5 1/4"φ-1¾"	287.8 1/4"φ-1⅞"	323.3 1/4"φ-2⅛"	362.7 1/4"φ-2⅜"	406.6 1/4"φ-2⅝"	453.8 1/4"φ-2¾"						
6-1⅛"φ	7.62	259.4 3/8"φ-2⅜"	274.0 3/8"φ-2¾"	296.0 3/8"φ-3"	324.3 1/4"φ-1½"	358.5 1/4"φ-1⅝"	396.6 1/4"φ-1⅞"	439.6 1/4"φ-2"	487.0 1/4"φ-2¼"	537.9 1/4"φ-2¼"	592.3 1/4"φ-2½"				
8-1"φ	8.00		284.0 3/8"φ-2⅜"	305.6 3/8"φ-2⅝"	333.6 3/8"φ-3"	366.2 1/4"φ-1⅜"	404.6 1/4"φ-1¾"	446.8 1/4"φ-1⅞"	494.0 1/4"φ-2⅛"	545.8 1/4"φ-2¼"	599.7 1/4"φ-2¼"				
6-1¼"φ	9.36		322.1 3/8"φ-2¼"	340.5 3/8"φ-2½"	366.3 3/8"φ-2⅞"	397.8 3/8"φ-3"	434.7 1/4"φ-1½"	475.9 1/4"φ-1⅝"	522.4 1/4"φ-1¾"	572.5 1/4"φ-2"	628.2 1/4"φ-2"	686.4 1/4"φ-2¼"	749.4 1/4"φ-2⅜"		
8-1⅛"φ	10.16		345.5 3/8"φ-2"	361.7 3/8"φ-2¼"	386.7 3/8"φ-2⅝"	416.5 3/8"φ-3"	452.4 3/8"φ-3"	493.0 1/4"φ-1½"	539.2 1/4"φ-1⅝"	588.9 1/4"φ-1⅞"	643.5 1/4"φ-1⅞"	701.7 1/4"φ-2"	764.5 1/4"φ-2⅛"		
8-1¼"φ 10-1⅛"φ	12.48			426.9 3/8"φ-1⅞"	446.5 3/8"φ-2⅛"	473.3 3/8"φ-2⅜"	506.5 3/8"φ-2⅝"	545.7 3/8"φ-2¾"	588.8 3/8"φ-3"	637.6 1/4"φ-1½"	691.7 1/4"φ-1½"	748.4 1/4"φ-1½"	810.5 1/4"φ-1¾"	877.8 1/4"φ-1⅞"	1%
10-1¼"φ	15.60			523.2 1/2"φ-2¾"	534.1 1/2"φ-3	554.8 3/8"φ-1⅞"	583.5 3/8"φ-2"	618.6 3/8"φ-2⅛"	660.3 3/8"φ-2⅜"	707.3 3/8"φ-2⅝"	758.0 3/8"φ-2⅞"	814.6 3/8"φ-3"	875.7 1/4"φ-1½"	940.8 1/4"φ-1½"	
12-1¼"φ	18.72					642.3 1/2"φ-2¼"	665.5 1/2"φ-3	696.9 3/8"φ-1⅞"	734.0 3/8"φ-2"	778.7 3/8"φ-2⅛"	828.7 3/8"φ-2¼"	881.5 3/8"φ-2½"	941.1 3/8"φ-2⅝"	1005.8 3/8"φ-2¾"	
14-1¼"φ	21.84					736.0 1/2"φ-2⅜"	752.5 1/2"φ-2⅝"	778.3 1/2"φ-2⅞"	812.9 1/2"φ-3"	853.3 3/8"φ-1⅞"	900.5 3/8"φ-2"	952.7 3/8"φ-2⅛"	1010.8 3/8"φ-2¼"	1072.8 3/8"φ-2⅜"	
16-1¼"φ	24.96						844.6 1/2"φ-2¼"	865.2 1/2"φ-2½"	893.9 1/2"φ-2⅝"	931.0 1/2"φ-2⅞"	975.0 3/8"φ-1¾"	1024.5 3/8"φ-1⅞"	1080.6 3/8"φ-2"	1141.7 3/8"φ-2⅛"	
18-1¼"φ	28.08							956.5 1/2"φ-2¼"	980.2 1/2"φ-2⅜"	1013.6 1/2"φ-2½"	1054.2 1/2"φ-2¾"	1100.7 1/2"φ-3"	1152.4 3/8"φ-1¾"	1212.5 3/8"φ-1⅞"	
20-1¼"φ	31.20							1050.7 1/2"φ-2"	1068.5 1/2"φ-2⅛"	1097.7 1/2"φ-2⅜"	1134.3 1/2"φ-2½"	1177.3 1/2"φ-2⅝"	1128.7 1/2"φ-2¾"	1285.3 1/2"φ-2⅞"	
22-1¼"φ	34.32								1162.2 1/2"φ-1⅞"	1185.0 1/2"φ-2⅛"	1217.1 1/2"φ-2¼"	1258.5 1/2"φ-2⅜"	1304.9 1/2"φ-2½"	1360.1 1/2"φ-2¾"	
24-1¼"φ	37.44									1275.3 1/2"φ-1⅞"	1304.8 1/2"φ-1⅞"	1340.0 1/2"φ-2⅛"	1385.7 1/2"φ-2⅜"	1436.8 1/2"φ-2½"	
26-1¼"φ	40.56									1370.9 5/8"φ-2⅞"	1393.2 1/2"φ-1¾"	1426.1 1/2"φ-2"	1466.2 1/2"φ-2⅛"	1515.5 1/2"φ-2¼"	
28-1¼"φ	43.68										1484.3 5/8"φ-2⅞"	1512.4 5/8"φ-2⅞"	1551.5 1/2"φ-2"	1596.1 1/2"φ-2⅛"	
30-1¼"φ	46.80										1580.4 5/8"φ-2⅝"	1603.6 5/8"φ-2¾"	1636.5 5/8"φ-2⅞"	1678.7 1/2"φ-2"	
32-1¼"φ	49.92											1694.8 5/8"φ-2⅝"	1723.3 5/8"φ-2½"	1763.2 5/8"φ-2⅞"	
34-1¼"φ	53.04												1815.2 5/8"φ-2⅝"	1849.7 5/8"φ-2¾"	
36-1¼"φ	56.16											6%	1906.4 5/8"φ-2½"	1938.1 5/8"φ-2⅜"	

TABLE A-5. Carrying Capacity of Round Spiral Reinforced Columns, 1928 A.C.I. Code.

$$P = f_c A[1 + (n - 1)p] : f_c = [300 + (0.10 + 4p)f'_c]$$

A = core area (diameter equals outside diameter of spiral)

f'_c = 3000 psi $\qquad$ n = 10

Outside Col Dia		16"	18"	20"	22"	24"	26"	28"	30"	32"	34"	36"	38"	40"
Spiral		3/8"φ @ 1 3/4" p	3/8"φ @ 1 3/4" p	3/8"φ @ 2" p	3/8"φ @ 2" p	1/2"φ @ 3 1/4" p	1/2"φ @ 3" p	1/2"φ @ 2 3/4" p	1/2"φ @ 2 1/2" p	1/2"φ @ 2 1/4" p	1/2"φ @ 2 1/4" p	1/2"φ @ 1 7/8" p	1/2"φ @ 1 7/8" p	5/8"φ @ 2 3/4" p
		135.7	171.8	212.1	256.6	305.4	358.4	415.6	477.1	542.9	612.8	687.1	765.5	848.2
6-1/2"♦	19.2													
6-1/2"°	24.0													
6-5/8"♦	29.8													
6-3/4"♦	42.2	177.9	214.0											
6-7/8"♦	57.6	193.3	229.4	269.7										
6-1"♦	75.8	211.5	247.6	287.9	332.4	381.2								
6-1"° (8-1"♦)	96.0	231.7	267.8	308.1	352.6	401.4	454.4							
6-1 1/8"°	121.9	257.6	293.7	334.0	378.5	427.3	480.3	537.5	599.0					
8-1"°	128.0	263.7	299.8	340.1	384.6	433.4	486.4	543.6	605.1					
6-1 1/4"°	149.8	285.5	321.6	361.9	406.4	455.2	508.2	565.4	626.9	692.7	762.6			
8-1 1/8"°	162.6	298.3	334.4	374.7	419.2	468.0	521.0	578.2	639.7	705.5	775.4	849.7		
8-1 1/4"° (10-1 1/8"°)	199.7	335.4	371.5	411.8	456.3	505.1	558.1	615.3	676.8	742.6	812.5	886.8	965.2	1%
10-1 1/4"°	249.6	385.3	421.4	461.7	506.2	555.0	608.0	665.2	726.7	792.5	862.4	936.7	1015.1	1097.8
12-1 1/4"°	299.5		471.3	511.6	556.1	604.9	657.9	715.1	776.6	842.4	912.3	986.6	1065.0	1147.7
14-1 1/4"°	349.4			561.5	606.0	654.8	707.8	765.0	826.5	892.3	962.2	1036.5	1114.9	1197.6
16-1 1/4"°	399.4			611.5	656.0	704.8	757.8	815.0	876.5	942.3	1012.2	1086.5	1164.9	1247.6
18-1 1/4"°	449.3				705.9	754.7	807.7	864.9	926.4	992.2	1062.1	1136.4	1214.8	1297.5
20-1 1/4"°	499.2					804.6	857.6	914.8	976.3	1042.1	1112.0	1186.3	1264.7	1347.4
22-1 1/4"°	549.1					854.5	907.5	964.7	1026.2	1092.0	1161.9	1236.2	1314.6	1397.3
24-1 1/4"°	599.0						957.4	1014.6	1076.1	1141.9	1211.8	1286.1	1364.5	1447.2
26-1 1/4"°	649.0						1007.4	1064.6	1126.1	1191.9	1261.8	1336.1	1414.5	1497.2
28-1 1/4"°	698.9							1114.5	1176.0	1241.8	1311.7	1386.0	1464.4	1547.1
30-1 1/4"°	748.7							1164.4	1225.9	1291.7	1361.6	1435.1	1514.3	1597.0
32-1 1/4"°	798.7								1275.8	1341.6	1411.5	1485.8	1564.2	1646.9
34-1 1/4"°	848.6								1325.7	1391.5	1461.4	1535.7	1614.1	1696.8
36-1 1/4"°	898.6							8%	1375.7	1441.5	1511.4	1585.7	1664.1	1746.8

Table A-6. Carrying Capacity of Round Spiral Reinforced Columns, 1941 A.C.I. and 1940 J.C. Codes.

$P = 675\,A_g + 16{,}000\,A_s : f'_c = 3000$ psi.

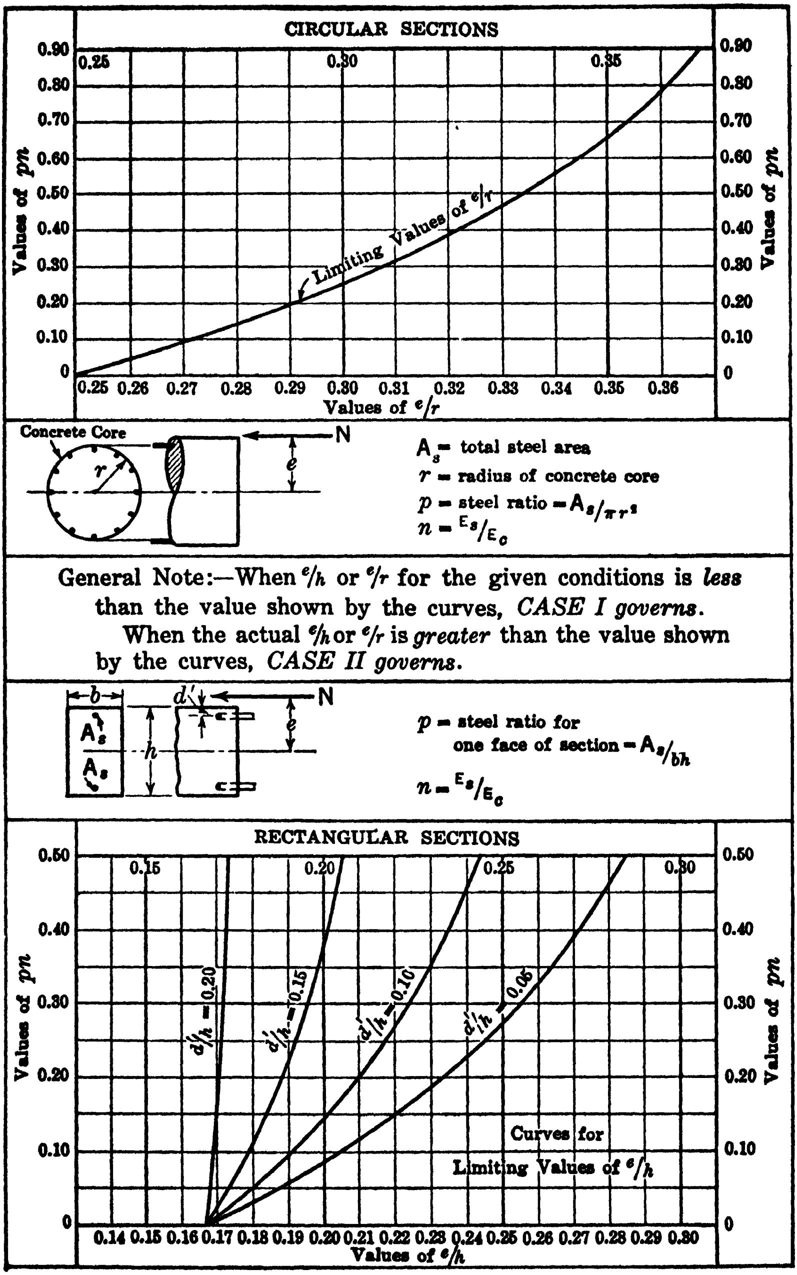
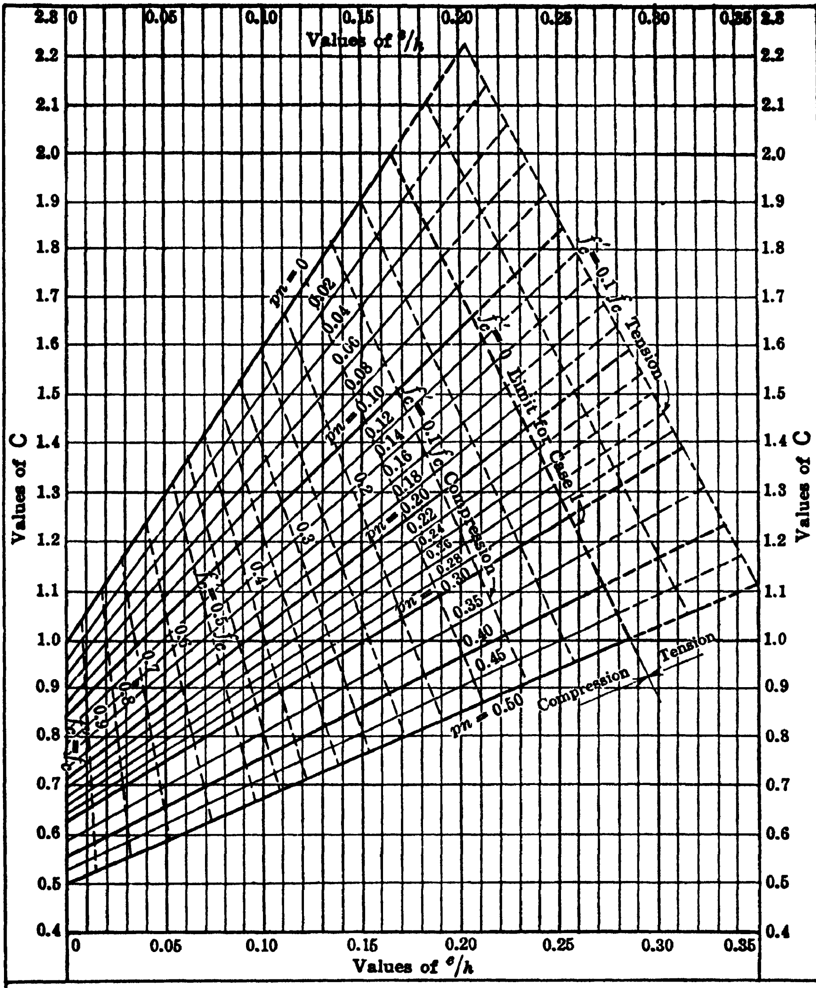
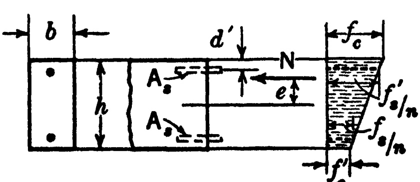


FIG. A-12. Bending and Direct Stress. Limiting Conditions for Cases I and II.





p = steel ratio for one face of section $= A_s/bh$

$n = E_s/E_c$

$$f_c = C \cdot N/bh$$

$$f_s = nf_c\left[1 - d'/h\left(1 - f_c'/f_c\right)\right]$$

$$f_s' = nf_c\left[f_c'/f_c + d'/h\left(1 - f_c'/f_c\right)\right]$$

CASE I

$d'/h = 0.05$

FIG. A–13. Bending and Direct Stress. Case I, Rectangular Sections.

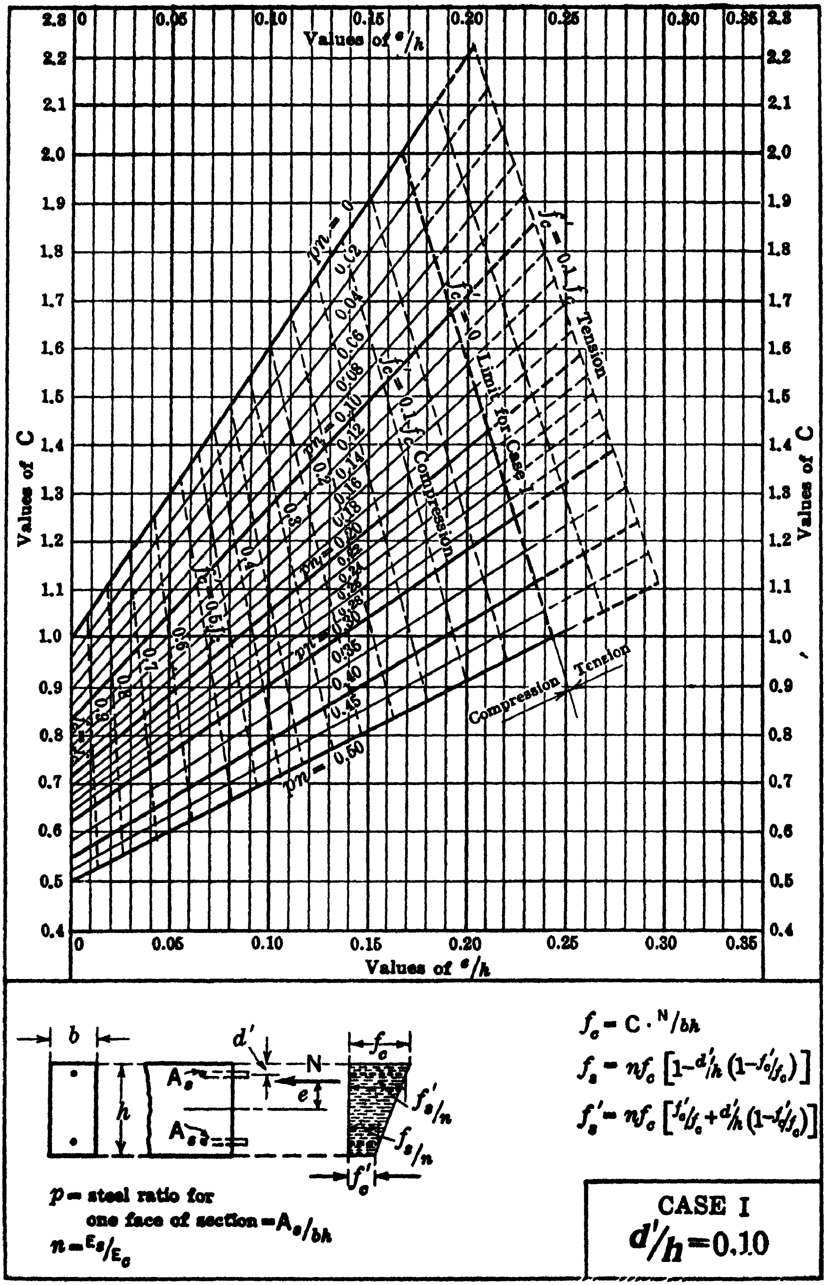


FIG. A-14. Bending and Direct Stress. Case I, Rectangular Sections.

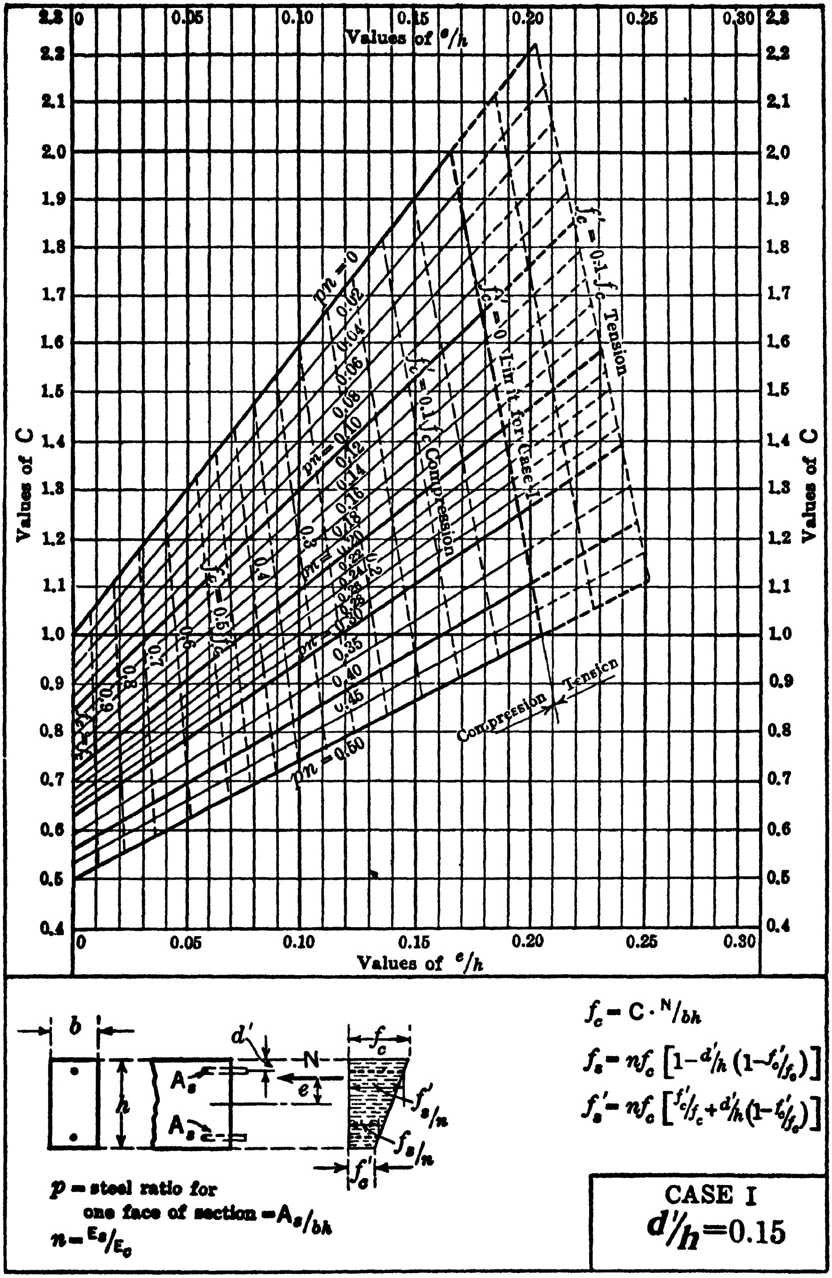


FIG. A-15. Bending and Direct Stress. Case I, Rectangular Sections.

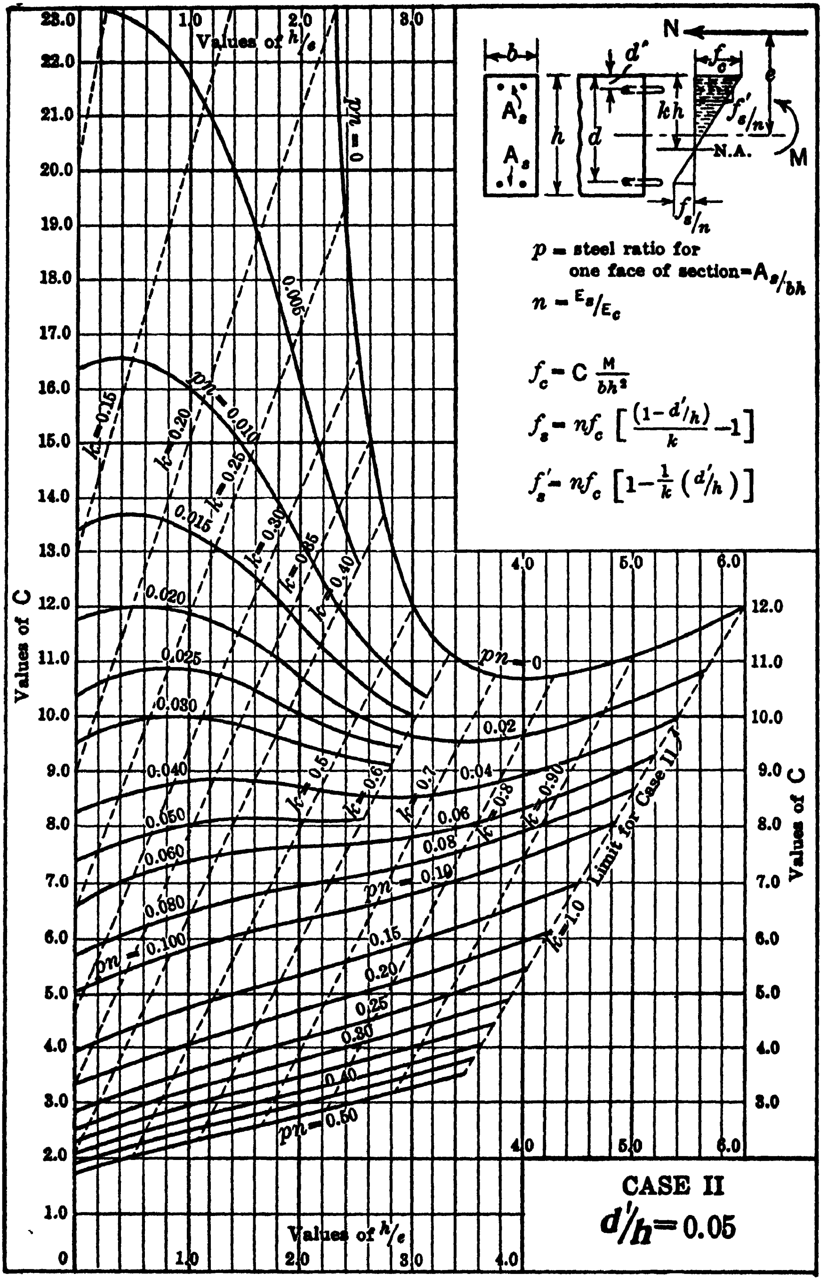


FIG. A-16. Bending and Direct Stress. Case II, Rectangular Sections.

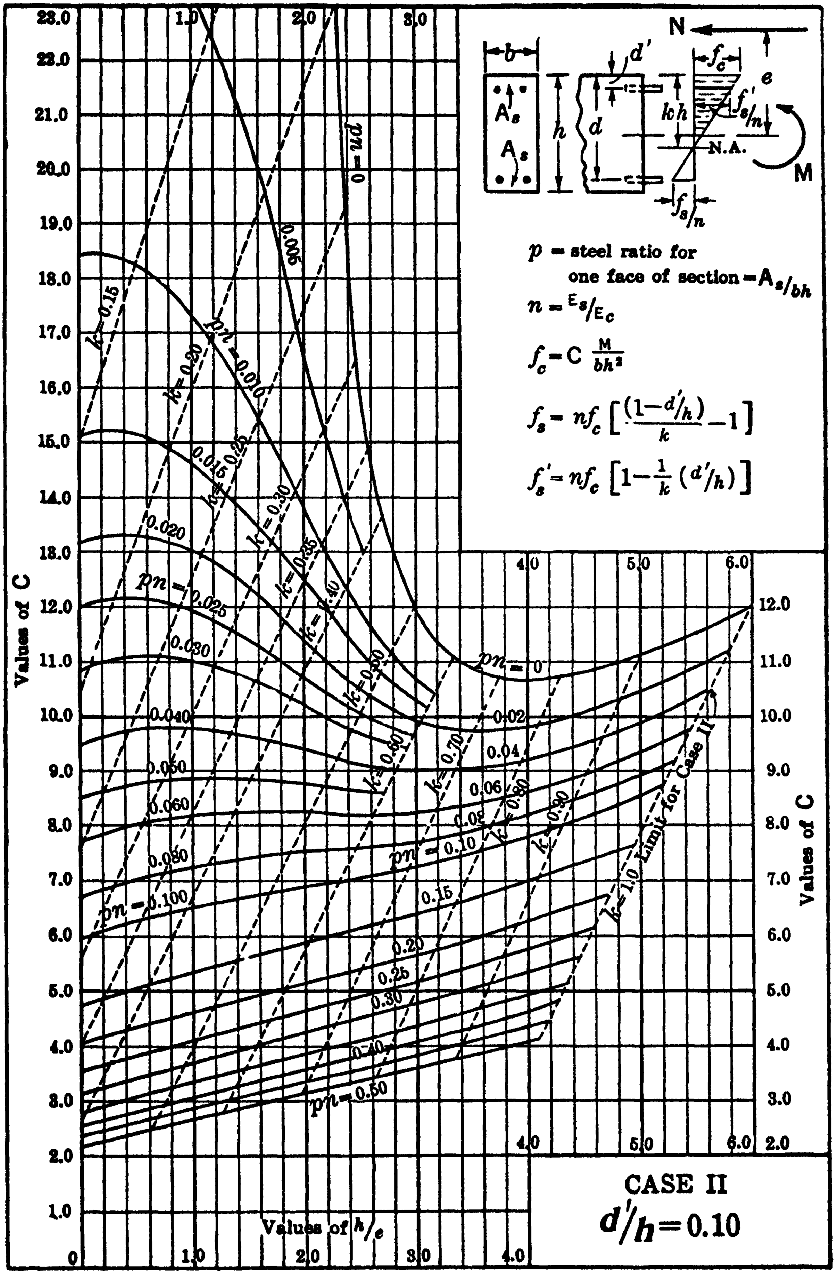


FIG. A-17a. Bending and Direct Stress. Case II, Rectangular Sections.

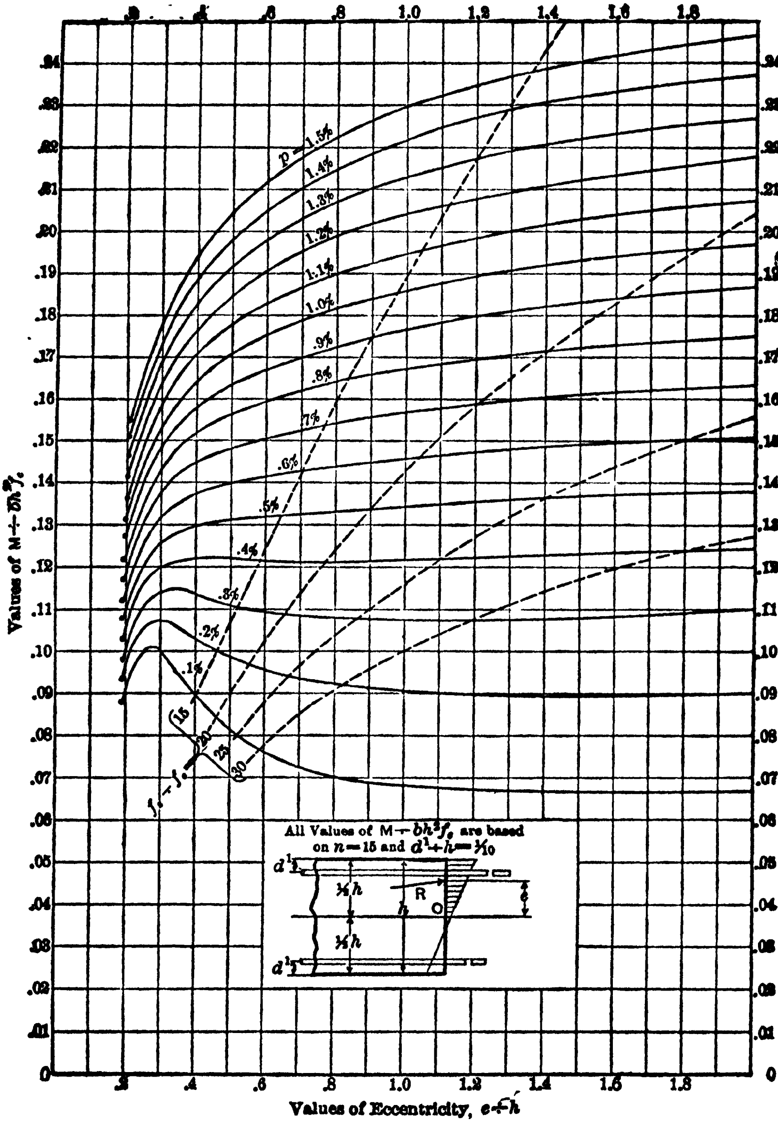


FIG. A-17b. Bending and Direct Stress. Case II, Rectangular Sections.

$p = p'$ $n = 15$

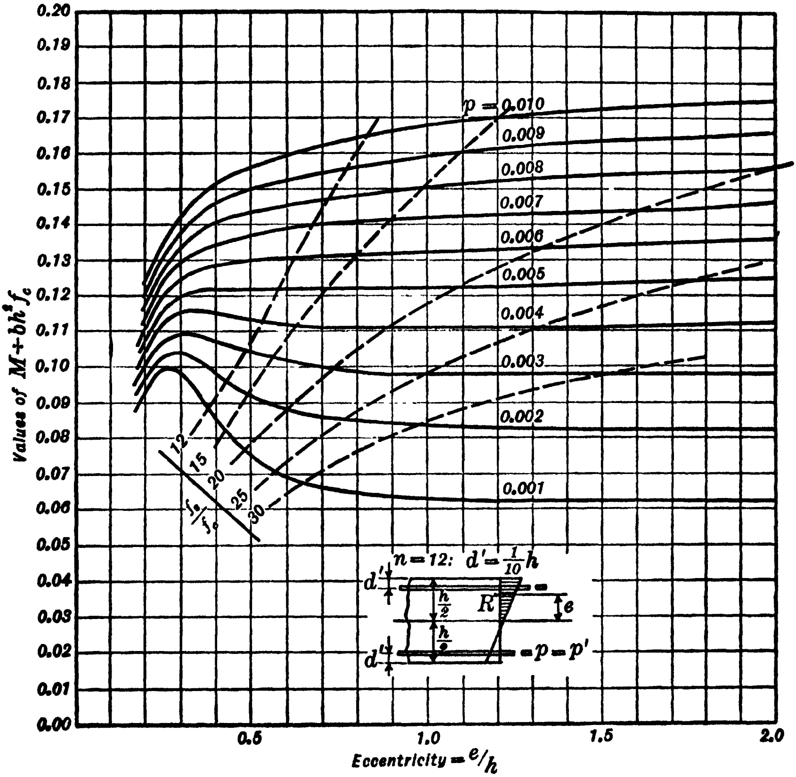


FIG. A–17c. Bending and Direct Stress. Case II, Rectangular Sections.

$p = p'$ $n = 12$

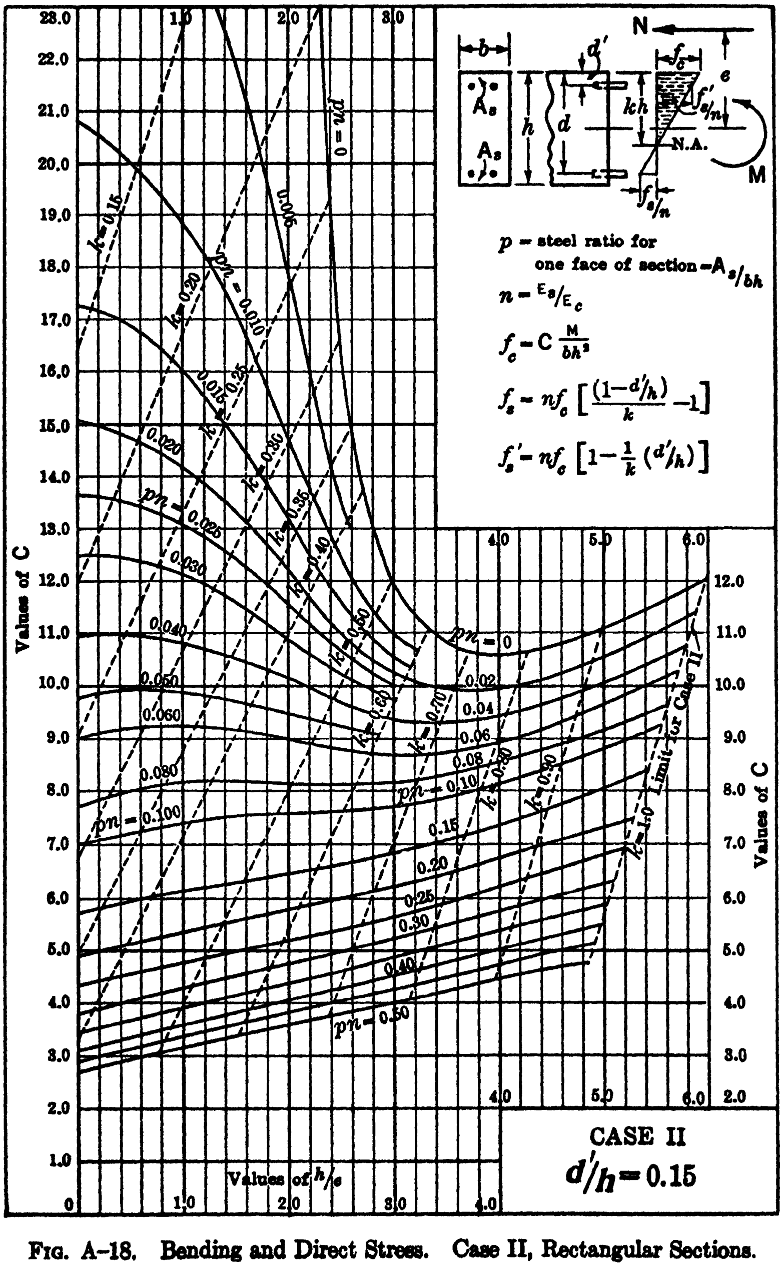


FIG. A-18. Bending and Direct Stress. Case II, Rectangular Sections.

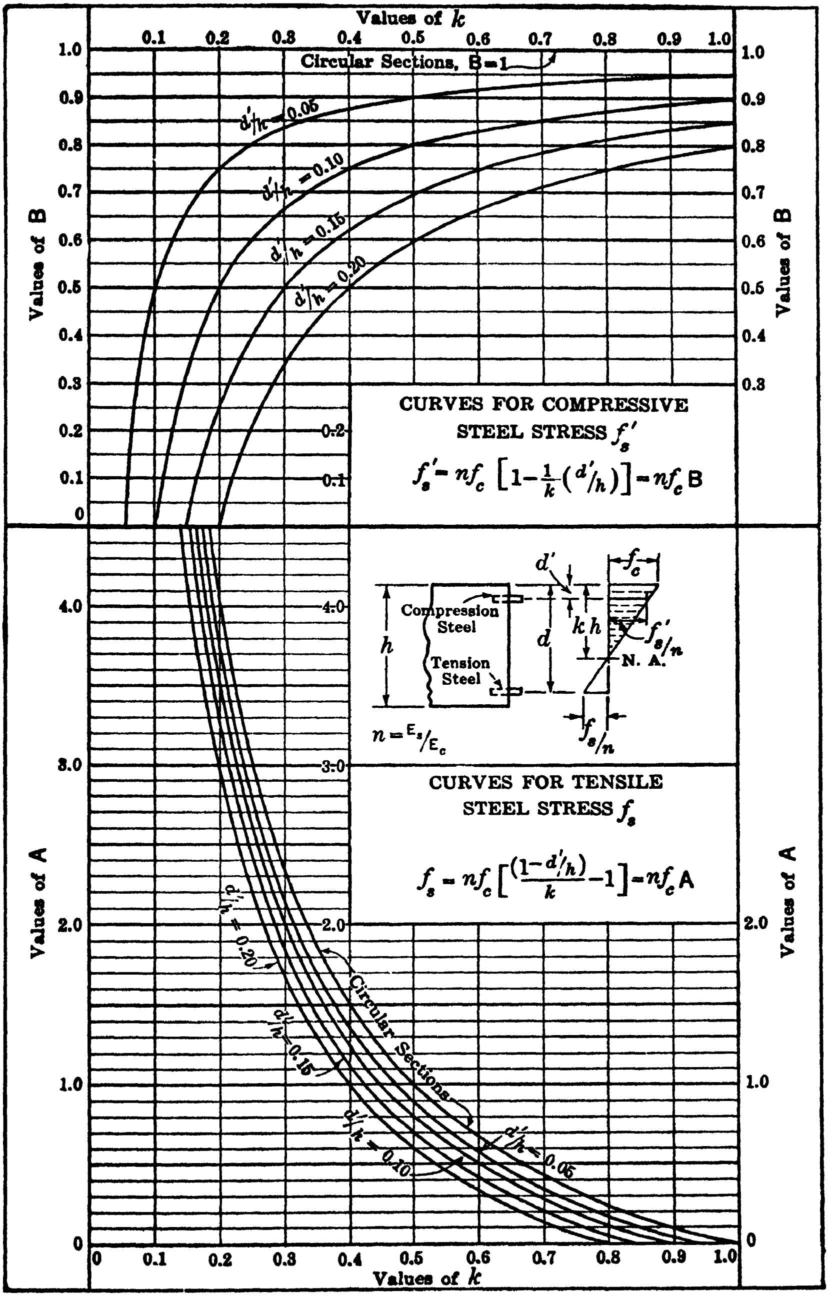


FIG. A-19. Bending and Direct Stress. Case II. Steel Stresses.

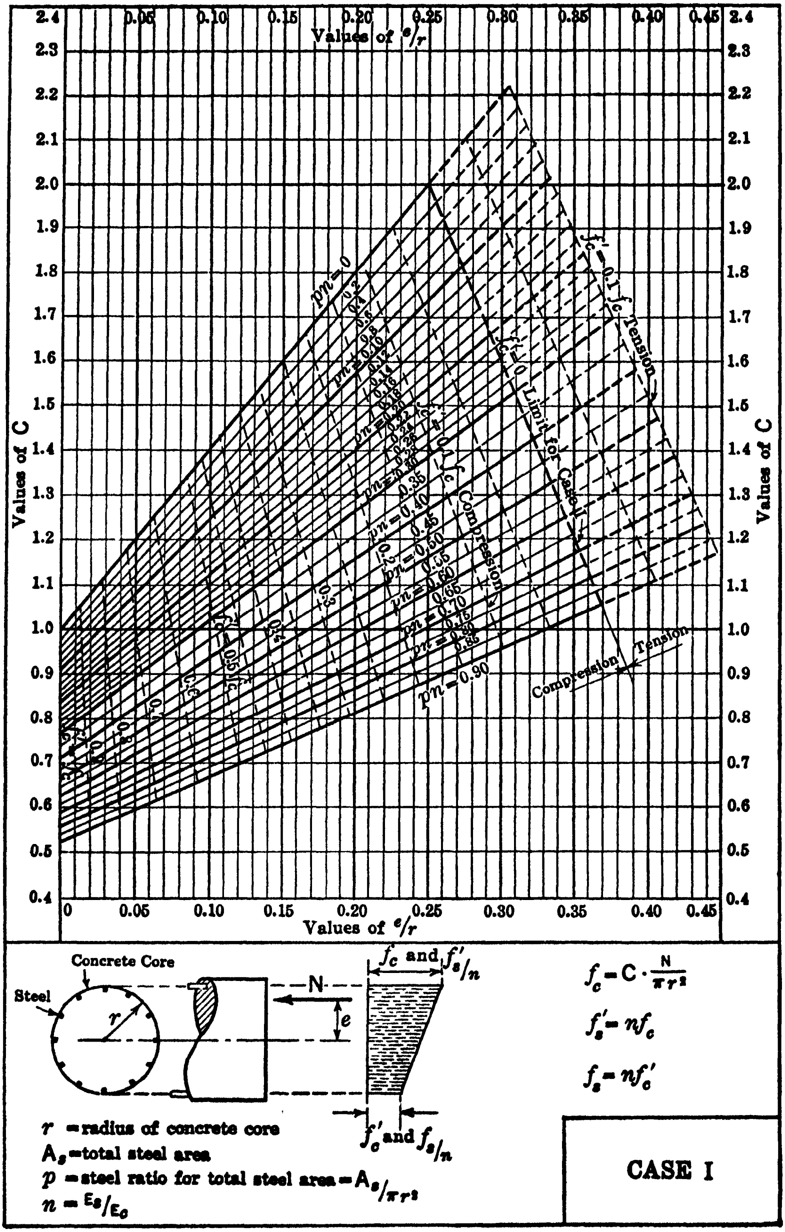


FIG. A-20. Bending and Direct Stress. Case I, Circular Sections.

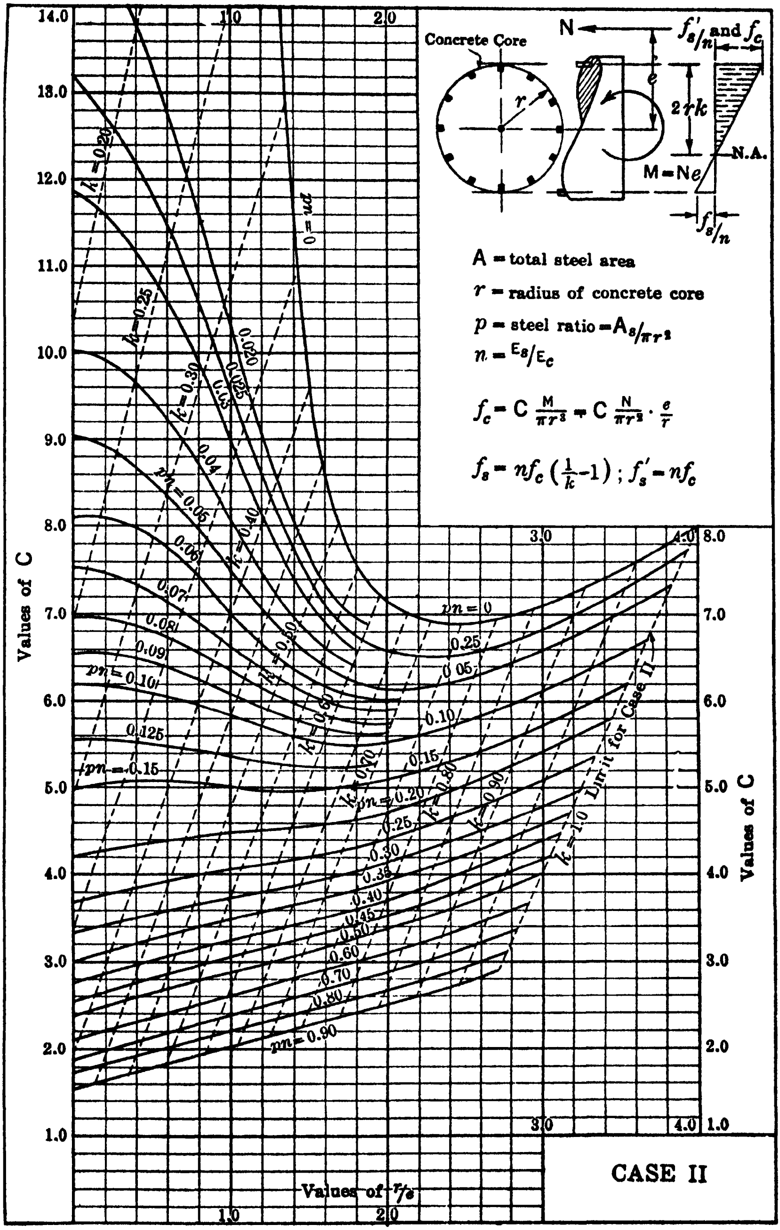


Fig. A-21. Bending and Direct Stress. Case II, Circular Sections.

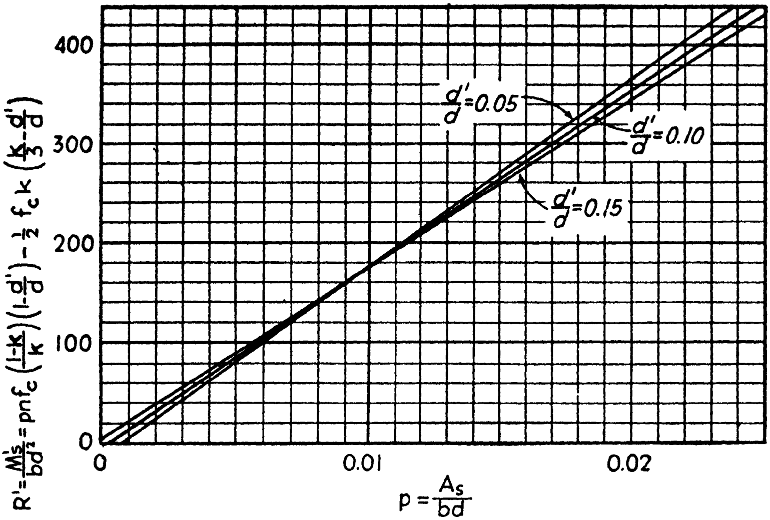


FIG. A-22. Bending and Direct Stress. Tension Steel.
M_s' = moment about compression steel

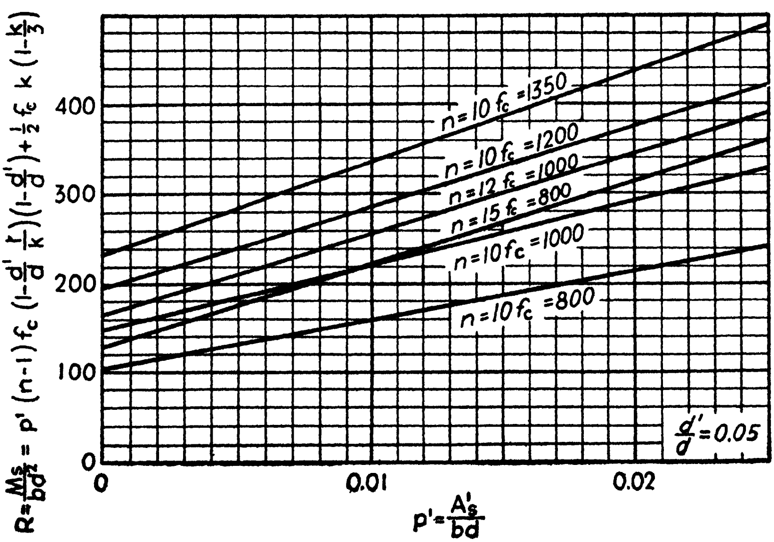


FIG. A-23. Bending and Direct Stress. Compression Steel.
M_s = moment about tension steel

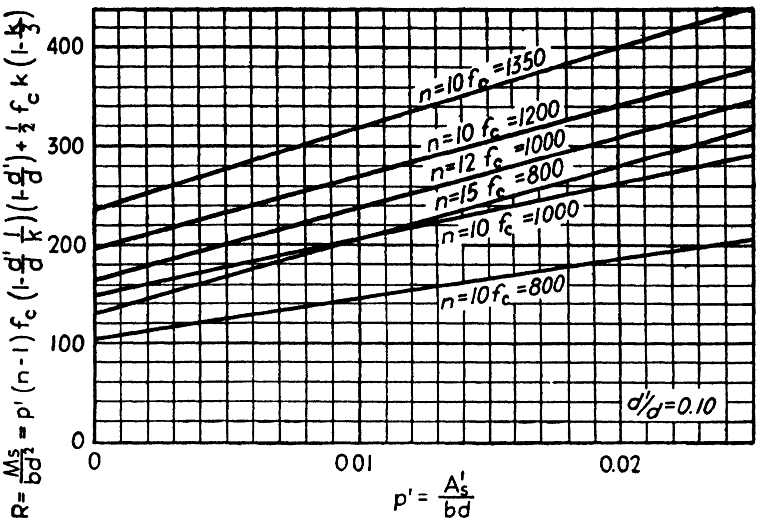


FIG. A-24. Bending and Direct Stress. Compression Steel.
M_s = moment about tension steel

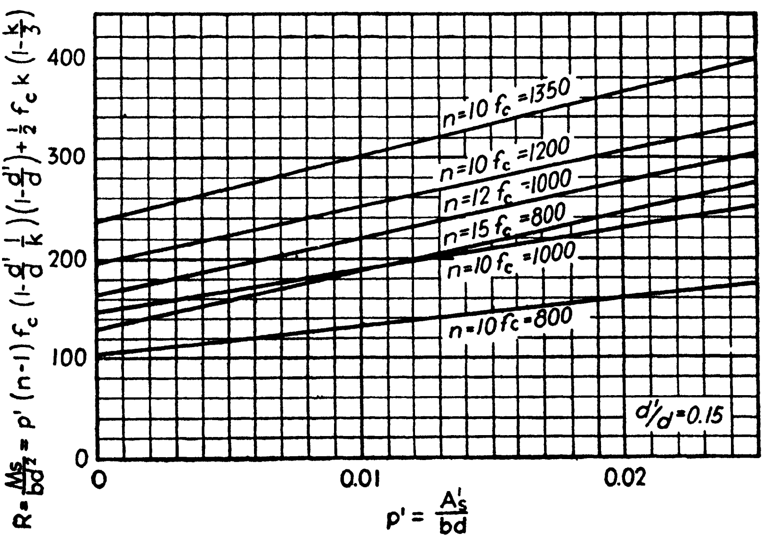


FIG. A-25. Bending and Direct Stress. Compression Steel.
M_s = moment about tension steel

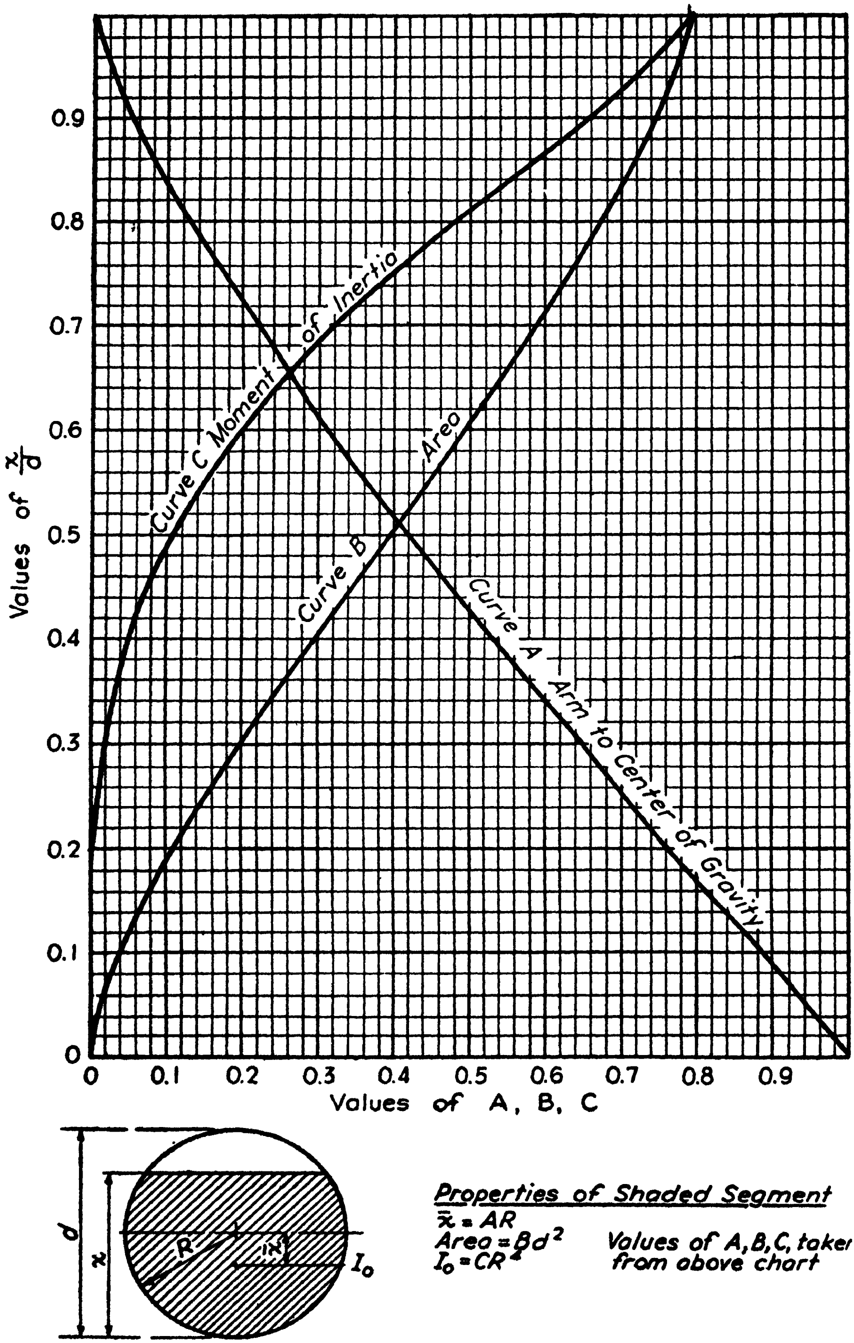


FIG. A-26. Properties of Segment of Circular Section.

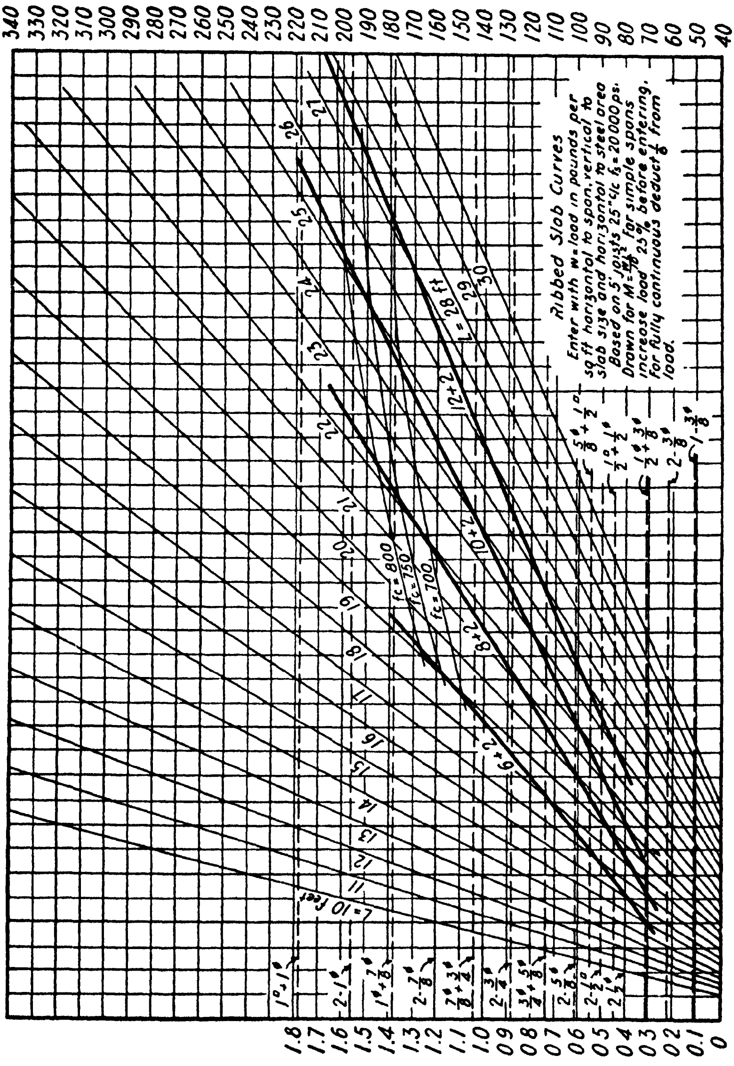

Ribbed Slab Curves
Enter with w = load in pounds per sq ft horizontal to span, vertical to slab size and horizontal to steel area
Based on 5' joists 25" c/c fs = 20000 psi
Drawn for M = wl²/10 for simple spans increase load 25% before entering. For fully continuous deduct 1/6 from load.

FIG. A-27

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Sutherland & others

Introduction to Reinforced Concrete Design.